Normality for approach spaces and contractive realvalued maps

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Overview of the talk

- The category **App**
- Lower and upper regular functions
- Normality and separation by Urysohn maps
- Katětov-Tong's insertion condition
- Tietze's extension condition
- Links to other normality notions in **App**

The category **App**

Definition (Lowen)

A *distance* is a function

$$\delta : X \times 2^X \to [0, \infty]$$

that satisfies:

(1) $\forall x \in X, \forall A \in 2^X : x \in A \Rightarrow \delta(x, A) = 0$

(2) $\delta(x, \emptyset) = \infty$

(3) $\forall x \in X, \forall A \in 2^X : \delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$

(4) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon \in [0, \infty] : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon$ with

$$A^{(\varepsilon)} = \{x \in X \mid \delta(x, A) \leq \varepsilon\}.$$

The pair (X, δ) is called an *approach space*.

The category **App**

Definition (Lowen)

For X, Y approach spaces, a map $f : X \to Y$ is called a *contraction* if

$$\forall x \in X, \forall A \in 2^X : \delta_Y(f(x), f(A)) \leq \delta_X(x, A).$$

let **App** be the category of approach spaces and contractions

Facts:

- **App** is a topological category
- **Top** $\hookrightarrow$ **Ap** fully + reflectively + coreflectively via

$$\mathcal{T} \mapsto \delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & \text{if } x \in \mathrm{cl}_{\mathcal{T}}(A) \\ \infty & \text{if } x \notin \mathrm{cl}_{\mathcal{T}}(A) \end{cases}$$

- **(q)Met** $\hookrightarrow$ **Ap** fully +coreflectively via

$$d \mapsto \delta_d(x, A) = \inf_{a \in A} d(x, a)$$

Lower and upper regular functions

- on $[0,\infty]$, define the distance

$$\delta_{\mathbb{P}}(x,A) = \begin{cases}(x - \sup A) \vee 0 & A \neq \emptyset \\ \infty & A = \emptyset.\end{cases}.$$

  Then $\mathbb{P} = ([0,\infty], \delta_{\mathbb{P}})$ is initially dense in **App**.
- on $[0,\infty]$, define the quasi-metric

$$d_{\mathrm{P}}(x,y) = (x-y) \vee 0$$

  and its dual

$$d_{\mathrm{P}}^{-}(x,y) = (y-x) \vee 0$$

- note that $d_{\mathbb{E}} = d_{\mathrm{P}} \vee d_{\mathrm{P}}^{-}$: the Euclidean metric

Lower and upper regular functions

- for an approach space X, put

$$\mathfrak{L}_b = \{f : (X, \delta) \to ([0, \infty], \delta_{d_{\mathrm{P}}}) \mid \text{bounded, contractive}\}.$$

$$\mathfrak{U} = \{f : (X, \delta) \to ([0, \infty], \delta_{d_{\mathrm{P}}^-}) \mid \text{bounded, contractive}\}.$$

and

$$\mathcal{K}_b = \{f : (X, \delta) \to ([0, \infty], \delta_{d_{\mathbb{E}}}) \mid \text{bounded, contractive}\}.$$

- observe that

$$\mathfrak{U} \cap \mathfrak{L}_b = \mathcal{K}_b$$

Lower and upper regular functions

- we have *lower* and *upper hull operators* $\mathfrak{l}_b : [0,\infty]_b^X \to [0,\infty]_b^X$, resp. $\mathfrak{u} : [0,\infty]_b^X \to [0,\infty]_b^X$, defined by
$$\mathfrak{l}_b(\mu) := \bigvee\{\nu \in \mathfrak{L}_b | \nu \leq \mu\},$$
resp.
$$\mathfrak{u}(\mu) := \bigwedge\{\nu \in \mathfrak{U} | \mu \leq \nu\}$$
- $\mathfrak{L}_b$ is generated by
$$\{\delta_A^\omega = \delta(\cdot, A) \wedge \omega \mid A \in 2^X, \omega < \infty\}$$
- $\mathfrak{U}$ is generated by
$$\{\iota_A^\omega = (\omega - \delta(\cdot, A^c)) \vee 0 \mid A \in 2^X, \omega < \infty\}$$

Normality and separation by Urysohn maps

Definition

Let X an approach space and $\gamma > 0$. Two sets $A, B \subseteq X$ are called *γ-separated* if $A^{(\alpha)} \cap B^{(\beta)} = \emptyset$, whenever $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta < \gamma$.

Definition

Let X be an approach space. Let $F : \mathbb{Q} \to 2^X$ such that $\bigcup_{q \in \mathbb{Q}} F(q) = X, \bigcap_{q \in \mathbb{Q}} F(q) = \emptyset$. Then F is a *contractive scale* if it satisfies

$$\forall r, s \in \mathbb{Q} : r < s \Rightarrow F(r) \text{ and } (X \setminus F(s)) \text{ are } (s - r)\text{-separated}$$

Normality and separation by Urysohn maps

Definition

An approach space X is said to be *normal* if for all $A, B \subseteq X$, for all $\gamma > 0$ with A and B γ-separated, a contractive scale F exists such that

(i) $\forall q \in \mathbb{Q}^- : F(q) = \emptyset$;

(ii) $A^{(0)} \subseteq \bigcap_{q \in \mathbb{Q}_0^+} F(q)$;

(iii) $B^{(0)} \cap \bigcup_{r \in \mathbb{Q}_0^+ \cap]0,\gamma]} F(r) = \emptyset$.

Normality and separation by Urysohn maps

Proposition

Let X be an approach space. If $F : \mathbb{Q} \to 2^X$ be a contractive scale on X, Then

$$f : (X, \delta) \to (\mathbb{R}, \delta_{d_{\mathbb{E}}}) : x \mapsto \inf\{q \in \mathbb{Q} \mid x \in F(q)\}$$

is a contraction.
Conversely, every contraction $f : (X, \delta) \to (\mathbb{R}, \delta_{d_{\mathbb{E}}})$ can be obtained in this way.

Normality and separation by Urysohn maps

Theorem

For an approach space X, t.f.a.e.:

(1) X is normal,

(2) X satisfies *separation by Urysohn contractive maps* in the following sense:
for every $A, B \in 2^X$ γ-separated ($\gamma > 0$), there exists a contraction

$$f : X \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$$

satisfying $f(a) = \gamma$ for $a \in A^{(0)}$ and $f(b) = 0$ for $b \in B^{(0)}$.

Normality and separation by Urysohn maps

Corollary

For a topological space $(X, \mathcal{T})$, t.f.a.e.

(1) $(X, \mathcal{T})$ is normal in the topological sense,

(2) $(X, \delta_{\mathcal{T}})$ is normal in our sense.

Normality and separation by Urysohn maps

Some examples:

- The approach space $\mathbb{P} = ([0, \infty], \delta_{\mathbb{P}})$ is normal (and not quasi-metric).
- The quasi-metric approach spaces $([0, \infty], \delta_{d_P})$ and $([0, \infty], \delta_{d_P^-})$ are normal.
- The quasi-metric approach space $([0, \infty[, \delta_q)$ defined by

$$q(x, y) = \begin{cases} y - x & x \leq y, \\ \infty & x > y \end{cases}$$

is normal. Note that the underlying topological space is the Sorgenfrey line.

Normality and separation by Urysohn maps

Proof:

- Take $A, B \in 2^X$,γ-separated for δ_q (for some $\gamma > 0$).
- Prove that γ-separated for $\delta_{d_{\mathbb{E}}}$.
- Since $\delta_{d_{\mathbb{E}}}$ is metric, hence (approach) normal, there exists a contraction $f : ([0, \infty[, \delta_{d_{\mathbb{E}}}) \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$ with $f(A^{(0)_{\mathbb{E}}}) \subseteq \{0\}$ and $f(B^{(0)_{\mathbb{E}}}) \subseteq \{\gamma\}$.
- Since $\delta_{\mathbb{E}} \leq \delta_q$, also $f : ([0, \infty[, \delta_q) \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$ with $f(A^{(0)_{\mathbb{E}}}) \subseteq \{0\}$ is a contraction and $A^{(0)_q} \subseteq A^{(0)_{\mathbb{E}}}$ and $B^{(0)_q} \subseteq B^{(0)_{\mathbb{E}}}$. □

Katětov-Tong's insertion condition

Definition

An approach space X satisfies *Katětov-Tong's intsertion condition* if for bounded functions from X to $[0, \infty]$ satisfying $g \leq h$ with g upper regular and h lower regular, there exists a contractive map $f : X \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying $g \leq f \leq h$.

A special instance of Tong's Lemma

For an approach space X and $\omega < \infty$, put

$$K = \{f : X \to ([0, \omega], \delta_{d_{\mathbb{E}}}) \mid f \text{ contractive}\}$$

and $M = [0, \omega]^X$, let $s \in K_\delta = \{\bigwedge_{n \geq 1} t_n \mid \forall n : t_n \in K\}$ and $t \in K_\sigma = \{\bigvee_n t_n \mid \forall n : t_n \in K\}$ with $s \leq t$ then a $u \in K_\sigma \cap K_\delta$ exists satisfying $s \leq u \leq t$.

Katětov-Tong's insertion condition

Theorem

For an approach space X, t.f.a.e.

(1) (X, δ) satisfies Katětov-Tong's interpolation condition,

(2) $\forall A, B \in 2^X, \forall \omega < \infty : (\iota_A^\omega \leq \delta_B^\omega \Rightarrow \exists f \in \mathcal{K}_b : \iota_A^\omega \leq f \leq \delta_B^\omega)$,

(3) X satisfies separation by Urysohn contractive maps,

(4) X is normal.

Corollary

(1) We recover the classical Katětov-Tong's interpolation characterization of topological normality

(2) For every metric space (X, d), the corresponding approach space (X, δ_d) is normal.

Tietze's extension condition

- Given a set X and a subset $A \subset X$, we define $\theta_A : X \to [0, \infty]$ by
$$\theta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & x \in X \setminus A. \end{cases}$$
- Given $f \in [0, \infty]_b^X$, a family $(\mu_\varepsilon)_{\varepsilon > 0}$ of functions taking only a finite number of values, written as
$$\left(\mu_\varepsilon := \bigwedge_{i=1}^{n(\varepsilon)} \left(m_i^\varepsilon + \theta_{M_i^\varepsilon} \right) \right)_{\varepsilon > 0} \quad \text{with } (M_i^\varepsilon)_{i=1}^{n(\varepsilon)} \text{ a partitioning of } X$$
and all $m_i^\varepsilon \in \mathbb{R}^+$, for $\varepsilon > 0$, is called a *development of f* if for all $\varepsilon > 0$
$$\mu_\varepsilon \leq f \leq \mu_\varepsilon + \varepsilon.$$

Tietze's extension condition

Definition

We say that an approach space X, satisfies *Tietze's extension condition* if for every $Y \subseteq X$ and $\gamma \in \mathbb{R}^+$, and every contraction

$$f : Y \to ([0,\gamma], \delta_{d_{\mathbb{E}}})$$

which allows a development $\left(\mu_\varepsilon := \bigwedge_{i=1}^{n(\varepsilon)} \left(m_i^\varepsilon + \theta_{M_i^\varepsilon}\right)\right)_{0<\varepsilon<1}$ such that

$$\forall x \notin Y, \forall \varepsilon \in]0,1[, \forall 1 \leq l, k \leq n(\varepsilon) : m_l^\varepsilon - m_k^\varepsilon \leq \delta_{M_k^\varepsilon}(x) + \delta_{M_l^\varepsilon}(x),$$

there exists a contraction

$$g : X \to ([0,\gamma], \delta_{d_{\mathbb{E}}})$$

extending, i.e. $g|_Y = f$.

Tietze's extension condition

Corollary

We recover the classical Tietze extension characterization of topological normality.

Summary

We have shown that for an approach space X t.f.a.e.

(1) X is normal (via *contractive scales*),

(2) X satisfies *separation by Urysohn contractive maps*,

(3) X satisfies *Katětov-Tong's insertion condition,*

(4) X satisfies *Tietze's extension condition*.

Links to other normality notions in **App**: approach frame normality

Proposition

Let X be an approach space. Consider the following properties:

(1) (X, δ) is normal

(2) For $A, B \subseteq X$, γ-separated for some $\gamma > 0$, there exists $C \subseteq X$ such that A and C are $\gamma/2$-separated and $X \setminus C$ and B are $\gamma/2$-separated.

(3) $\mathfrak{L}$ is approach frame normal: For $A, B \subseteq X$, $\varepsilon > 0$ such that $A^{(\varepsilon)} \cap B^{(\varepsilon)} = \emptyset$ there exist $\rho > 0$, $C \subseteq X$ with

$$A^{(\rho)} \cap C^{(\rho)} = \emptyset \text{ and } (X \setminus C)^{(\rho)} \cap B^{(\rho)} = \emptyset.$$

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Note: we have finite counterexamples to the converse implications.

Links to other normality notions in **App**: (topological) normality of the underlying topology

Neither of the implications is valid:

- Let $X = \{x, y, z\}$ and put $d(a, a) = 0$ (all $\in X$), $d(x, z) = 1$, $d(y, z) = 2$, $d(x, y) = 4$ and all other distances equal to ∞. Then the metric approach space (X, δ_d) is not (approach) normal but the **Top**-coreflection (X,$\mathcal{T}_d$) is discrete, hence (topologically) normal.
- Define a quasi-metric q_S on $[0, \infty[\times[0, \infty[$ by

$$q_S((a', a''), (b', b'')) = q(a', b') + q(a'', b'').$$

  Then $([0, \infty[\times[0, \infty[, \delta_{q_S})$ can be shown to be (approach normal) but it's underlying topological space is the Sorgenfrey plane which is known to be not normal.

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

- From the work of Clementino-Hofmann-Tholen et al. on *monoidal topology*, it follows that **App** can be isomorphically described as the category (β, P_+)-Cat: an approach space (X, δ) is described via the *convergence* P_+*-relation*

$$a : \beta X \longrightarrow\!\!\!\!\!\!\!+\;\; X$$

given by

$$a(\mathcal{U}, x) = \sup_{U \in \mathcal{U}} \delta(x, U) \quad (\mathcal{U} \in \beta X, x \in X)$$

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

- Given an approach space X with representing convergence P_+-relation $a : \beta X \relbar\!\!\mapsto X$, a P_+-relation $\hat{a} : \beta X \relbar\!\!\mapsto \beta X$ is defined by

$$\hat{a}(\mathcal{U}, \mathcal{A}) = \inf\{\varepsilon \in [0, \infty] \mid \mathcal{U}^{(\varepsilon)} \subseteq \mathcal{A}\},$$

with $\mathcal{U}^{(\varepsilon)}$ the filter generated by $\{U^{(\varepsilon)} \mid U \in \mathcal{U}\}$.

Lemma

$$\hat{a}(\mathcal{U}, \mathcal{A}) = \sup_{U \in \mathcal{U}, A \in \mathcal{A}} \inf_{a \in A} \delta(a, U).$$

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

Definition (Clementino-Hofmann-Tholen et al.)

An approach space X represented as a $(\mathbb{\beta}, \mathsf{P}_+)$-space (X, a) is *monoidally normal* if for ultrafilters $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ on X

$$\hat{a}(\mathcal{U}, \mathcal{A}) + \hat{a}(\mathcal{U}, \mathcal{B}) \geq \inf_{\mathcal{W} \in \beta X} \hat{a}(\mathcal{A}, \mathcal{W}) + \hat{a}(\mathcal{B}, \mathcal{W}). \tag{0.1}$$

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

Proposition

Let X be an approach space and (X, a) its representation as a $(\mathbb{\beta}, \mathsf{P}_+)$-space then t.f.a.e.

(1) X is monoidally normal,

(2) For all $\gamma > 0$ and γ-separated $A, B \subseteq X$ and for all $\mathcal{A}, \mathcal{B}, \mathcal{U} \in \beta X$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\hat{a}(\mathcal{U}, \mathcal{A}) + \hat{a}(\mathcal{U}, \mathcal{B}) \geq \gamma,$$

(3) For all $\gamma > 0$ and γ-separated $A, B \subseteq X$ and for all $\alpha + \beta < \gamma$, there exists $C \subseteq X$ satisfying $A \cap (X \setminus C)^{(\alpha)} = \emptyset$ and $C^{(\beta)} \cap B = \emptyset$.

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

Theorem

Given a quasimetric approach space (X, δ_q) and considering the representing $(\mathbb{\beta}, \mathsf{P}_+)$-space (X, a_q), approach normality of (X, δ_q) is equivalent to monoidal normality of (X, a_q).

Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

Theorem

For an approach space (X, δ) with representing (β, P_+)-space (X, a) and quasimetric coreflection (X, q), we have the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3):

(1) (Approach) normality of (X, δ).

(2) Monoidal normality of (X, a).

(3) (Approach) normality of the quasimetric coreflection (X, δ_q).

- (3) does not imply (2): consider the topological Sorgenfrey plane, considered as **App**-object.
- Whether (1) and (2) are equivalent is still an open problem!

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Happy birthday Ales!