A QUANTALE MODEL OF COGNITION

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Some mathematical background

[R 2018b] P. Resende, Quantales and Fell bundles, Adv. Math. 325 (2018) 312–374.

[R 2018a] P. Resende, The many groupoids of a stably Gelfand quantale, J. Algebra 498 (2018) 197–210.

The measurement problem in quantum mechanics

- The 'Copenhagen interpretation' (Bohr, Heisenberg...) assumes that quantum systems evolve in two different ways:
  - *Reversibly* according to the Schrödinger equation

    $$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \hat{H}\psi(\boldsymbol{x},t)$$

    where $|\psi(\boldsymbol{x},t)|^2$ is the probability density of finding the system in position $\boldsymbol{x}$ at time t.
  - Or *irreversibly* upon observation ('collapse' of the wave function ψ).
- But... what is an *observation*?
- Von Neumann's foundations of quantum mechanics rely on Hilbert spaces for describing the 'states' of systems.
- System: H_S , Measuring apparatus: H_A , System+apparatus: $H_S \otimes H_A$
- Example: 'two-state system' with a Hilbert basis $\{|0\rangle, |1\rangle\}$ (a *qubit*).
- Initial state of system: $\alpha|0\rangle + \beta|1\rangle \in H_S$ (with $|\alpha|^2 + |\beta|^2 = 1$)

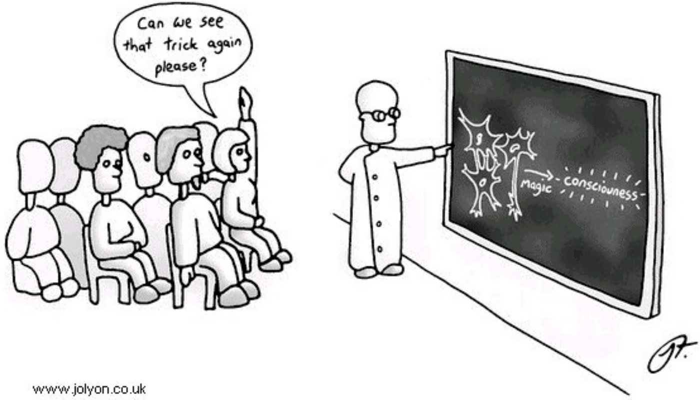
  Initial state of apparatus: $|\mathit{Pointer{=}?}\rangle \in H_A$

  Initial state of both: $\alpha|0\rangle \otimes |\mathit{Pointer{=}?}\rangle + \beta|1\rangle \otimes |\mathit{Pointer{=}?}\rangle \in H_S \otimes H_A$
- Final state of both (after reversible time evolution):
  $\alpha|0\rangle \otimes |\mathit{Pointer{=}0}\rangle + \beta|1\rangle \otimes |\mathit{Pointer{=}1}\rangle \in H_S \otimes H_A$

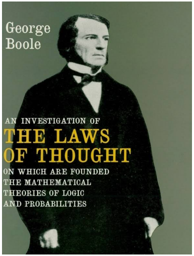
- Impossible to obtain either $|0\rangle \otimes |Pointer{=}0\rangle$ or $|1\rangle \otimes |Pointer{=}1\rangle$, *which are the options that are observed in practice!*
- Whereas Bohr and Heisenberg were cautious in referring to 'observers', in the 1930's Von Neumann proposed that one needs the *subjective experience* of the observer. This brings *consciousness* into the picture (a view endorsed by Wigner, although he later abandoned it).
- The main problem was that one was explaining a supposedly physical phenomenon (wave function 'collapse') in terms of another phenomenon (consciousness), which physics knows nothing about...
- This leads to other complaints, such as that of anthropocentrism (which derives from identifying consciousness with human consciousness), or solipsism (a mind-only view of reality).
- Modifications of Schrödinger dynamics face the barrier of experimental verification, which is hugely successful for orthodox quantum mechanics.
- Interpretations without collapse ('many worlds', or 'many minds') have problems of their own, both mathematical and philosophical.

SO WHAT?

- So... physics has a hard time, both regarding consciousness and quantum mechanics!
- Hand consciousness over to the neuroscientists?
- But... still affected by the measurement problem.
- Besides... the *hard problem of consciousness*:



- But subjective experience is *observable* $\Rightarrow$ within the reach of physics (by definition!)
- Look for empirically informed *mathematical* laws of subjective phenomena.
- **Precursors:** Aristotelian logic deals with laws of mental phenomena, albeit centered on notion of truth and provability.
- Algebraic logic:



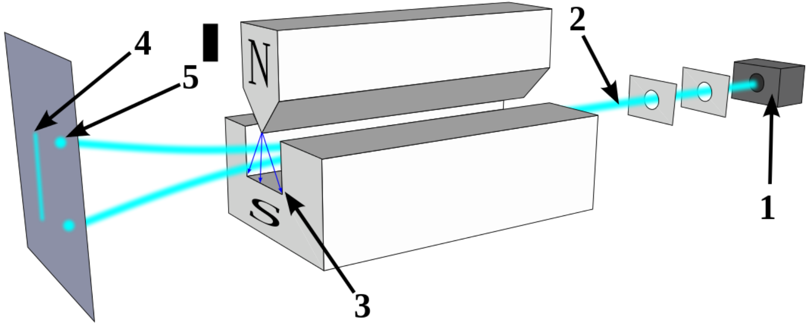
- Boolean algebras (classical logic), Heyting algebras (intuitionistic logic)
- Connection to 'rubber-sheet geometry': spatial representation of mental phenomena? Locales
- Related, in computer science: locales and quantales describe the observable properties of computers running programs.

Fundamental qualities of experience: qualia

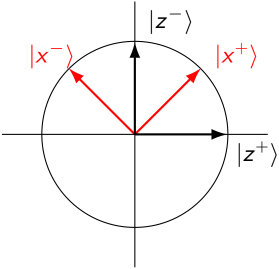
> *"There are recognizable qualitative characters of the given, which may be repeated in different experiences, and are thus a sort of universals; I call these 'qualia'. But although such qualia are universals, in the sense of being recognized from one to another experience, they must be distinguished from the properties of objects."*
>
> – **Clarence Irving Lewis, in: Mind and the World Order (1929)**

- Preliminarily assume there is a set of qualia $\mathcal{Q}$.
- *Principle 1: Concepts are recorded in physical devices (e.g., brains) in response to the interaction with finite numbers of qualia using finite resources.*
- Concepts can be represented as open sets of a topology $\Omega(\mathcal{Q})$.
- *Principle 2: Qualia cannot arise except in relation to concepts.*
- Then $\mathcal{Q}$ is a *sober space*, so the specialization order is a dcpo.
- Add binary joins, so get a complete lattice.
- $a \leq b$ means that b has more *potential properties* than a.

Example: measuring spin with Stern–Gerlach analyzer

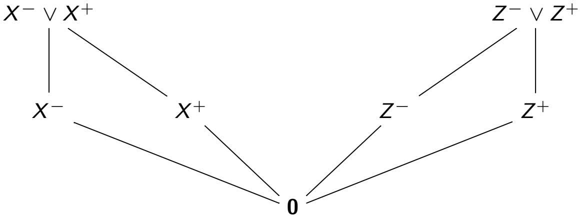


- Different *observables* (e.g., z-spin and x-spin) correspond to different orthonormal bases of the Hilbert space $\mathbb{C}^2$:



- If initial state is $|\psi\rangle = \alpha|z^+\rangle + \beta|z^-\rangle$ the probability of observing $|z^+\rangle$ is $|\alpha|^2$ and the probability of observing $|z^-\rangle$ is $|\beta|^2$.
- After measuring spin along z, a measurement of spin along x will yield each deflection with probability $1/2$.
- Repeated measurements of spin along z yield the same answer.
- In $\mathcal{Q}$ we shall have corresponding qualia Z^+, Z^-, X^+ and X^-, and also $Z^+ \vee Z^-$ and $X^+ \vee X^-$, but $Z^+ \vee Z^- \neq X^+ \vee X^-$.

A model of $\mathcal{Q}$ in this case is $\mathcal{L}(M_2(\mathbb{C}))$:



$$Z^+ = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \qquad Z^- = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$Z^+ \vee Z^- = D_2(\mathbb{C}) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

$$X^+ = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad X^- = \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle$$

$$X^+ \vee X^- = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

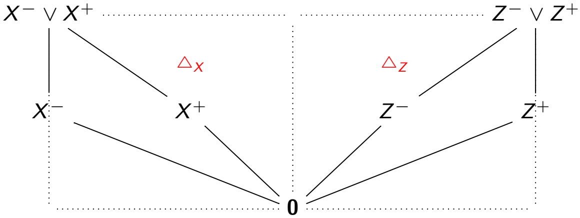
Qualia and (psychological?) time

- *Principle 3: The experience of multiple experiences exists.*
- Time: $a \,\&\, b$ (a *and then* b)
  - $(a \,\&\, b) \,\&\, c = a \,\&\, (b \,\&\, c)$
  - $a \,\&\, \bigvee_\alpha b_\alpha = \bigvee_\alpha a \,\&\, b_\alpha$
  - $\left(\bigvee_\alpha b_\alpha\right) \,\&\, a = \bigvee_\alpha (b_\alpha \,\&\, a)$
- $\mathcal{Q}$ is a (topological) quantale.
- Given any C*-algebra A a model of $\mathcal{Q}$ is the quantale $\mathrm{Max}\, A$ of closed linear subspaces of A with the lower Vietoris topology and multiplication

$$U \& V = \overline{\langle \{ab \mid a \in U,\ \ b \in V\}\rangle}$$

(e.g., $\mathcal{L}(M_2(\mathbb{C}))$ is $\mathrm{Max}\, M_2(\mathbb{C})$)
- Qualia a and b are *mutually consistent* when we can *proceed by approximations*: $a \leq b$ implies $a \,\&\, b = a = b \,\&\, a$.
- Any subset $\triangle \subset \mathcal{Q}$ closed under $\bigvee$ whose elements are mutually consistent is a locale with $\wedge = \&$ — this yields the (intuitionistic) logic of an *emerging observer*, or an *emerging space*, etc.
- Some 'observers' $\triangle_1$ and $\triangle_2$ are *incompatible*: no $\triangle$ exists such that $\triangle_1 \cup \triangle_2 \subset \triangle$.

An example with $\mathcal{Q} = \mathrm{Max}\, M_2(\mathbb{C})$ and observers $\triangle_X$ and $\triangle_Z$:



More generally, let A be a C*-algebra and $B \subset A$ an abelian sub-C*-algebra.

The locale of closed ideals $I(B)$ is an 'observer' in $\mathrm{Max}\, A$:

$$
\begin{aligned}
I(B) = \{\, V \in \mathrm{Max}\, A \mid & V \subset B\,, \\
& V \,\&\, B \subset V\,, \\
& B \,\&\, V \subset V \,\}
\end{aligned}
$$

Diagonals

- More structure: $S(B) = \{\, V \in \mathrm{Max}\, A \mid V \,\&\, V^* \subset B\,,$
  $V^* \,\&\, V \subset B\,,$
  $V \,\&\, B \subset V\,,$
  $B \,\&\, V \subset V \,\}$

  Symmetries of 'observer' $I(B)$

- **Theorem:** [R 2018a] *$S(B)$ is a spatial pseudogroup and $E(S(B)) = I(B)$.*
- **Definition:** Denote by $\mathcal{O}$ the (necessarily involutive and unital) subquantale of $\mathrm{Max}\, A$ generated by $S(B)$.

  Call $\mathcal{O}$ a *diagonal* of A if

  1. $\mathcal{O}$ is a *regular locale* under the order of $\mathrm{Max}\, A$ and
  2. $\mathcal{O}$ is *closed under arbitrary intersections* (in particular $1_{\mathcal{O}} = A$), so that a *closure operator* $\sigma : \mathrm{Max}\, A \to \mathcal{O}$ exists.
- These conditions imply that $\mathcal{O}$ is isomorphic to $\Omega(G)$ for a locally compact Hausdorff étale groupoid G, and that $\mathcal{I}(\mathcal{O}) = S(B)$.
- Conversely, **Theorem:** [R 2018a] *If G is a second-countable compact principal Hausdorff groupoid then $C_0(G_0)$ defines a diagonal in $\mathrm{Max}\, C_r^*(G)$.*

Logical complementarity

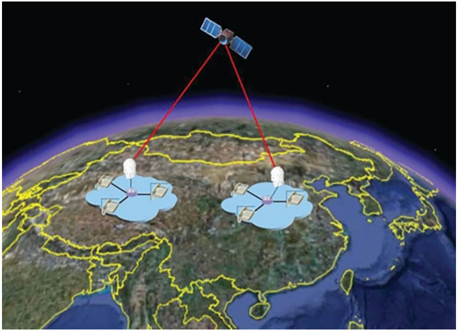
- Generalize for any topological *stably Gelfand quantale* $\mathcal{Q}$ (involutive quantale such that $a\&a^*\&a \leq a \iff a\&a^*\&a = a$).
- Many diagonals $\mathcal{O}$ with closure operators $\sigma : \mathcal{Q} \to \mathcal{O}$.
- Write $\triangle = \downarrow(e) \subset \mathcal{O}$ (the 'observers').
- If $\triangle_1$ and $\triangle_2$ are incompatible we can nevertheless compare them.
- σ_1 restricts to a sup-lattice homomorphism $\sigma_1 : \mathcal{O}_2 \to \mathcal{O}_1$, hence to a a frame homomorphism $P_L(\mathcal{O}_2) \to \mathcal{O}_1$.
- The corresponding map of locales $f : \mathcal{O}_1 \to P_L(\mathcal{O}_2)$ *sends each point of* $\mathcal{O}_1$ *to a closed subspace of the spectrum of* $\mathcal{O}_2$.
- $\sim$ *Bohr complementarity* (points of $\mathcal{O}_1$ can be regarded as 'unfocused points' of $\mathcal{O}_2$).
- E.g., values of z-spin versus x-spin: $f(z^+) = f(z^-) = \{x^-, x^+\}$.

Logical entanglement

- If $\triangle_1$ and $\triangle_2$ are compatible let $\triangle$ be such that $\triangle_1 \cup \triangle_2$ generates $\triangle$.
- The copairing map of the inclusion homomorphisms yields a regular monomorphism of locales

$$g : \triangle \to \triangle_1 \otimes \triangle_2$$

  so each point of $\triangle$ maps to a pair of points (x_1, x_2) of $\triangle_1$ and $\triangle_2$.
- If g is not an isomorphism *some pairs (x_1, x_2) are forbidden from the point of view of $\triangle$.*
- $\sim$ *entanglement*



To conclude...

- *Summary*:
  - Mathematical laws of subjective experience guided by simple principles.
  - An algebraic 'logic of quantum mechanics'.
- *Physical principle? If a physical system is described by a C*-algebra A (either within QM or QFT) then is* $\mathrm{Max}\, A$ *a model of the subjective experience which can ultimately be associated with the system?*
- *Philosophical questions*:
  - How fundamental is the structure of $\mathcal{Q}$? Is it merely relative to a class of organisms, albeit a large one?
  - Can qualia be reduced to currently known physical concepts? (Again the measurement problem!)
- *Mathematical questions*:
  - Can there be a diagonal $\mathcal{O}$ such that $\mathcal{O} \cong \Omega(\mathcal{Q})$ or $\triangle \cong \Omega(\mathcal{Q})$?
  - More perspicuous conditions on diagonals for general $\mathcal{Q}$ and for $\mathrm{Max}\, A$?
  - Comparison with other notions of diagonal for C*-algebras.