The pointfree Daniell integral

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An important paper

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Definition

An *abstract Boolean σ-algebra* is a Boolean algebra in which countable joins exist. A morphism of abstract Boolean σ-algebras is a Boolean homomorphism preserving countable joins.

Definition

Let $\mathcal{B}$ be the quotient of the free abstract Boolean algebra on generators $[0,t]$, $0 \leq t \leq \infty$, subject to the relations $[0,s] \leq [0,t]$ for $s \leq t$ and $\bigwedge_n [0,s_n] = [0,t]$ whenever $s_n \searrow t$.

Definition

Let Σ be an abstract Boolean σ-algebra. A *non-negative measurable function on* Σ is a morphism of abstract σ-algebras $f\colon \mathcal{B} \to \Sigma$.

The classical Daniell integral, 1918

Setting

Let X be an index set, and let G be a sub-vector lattice of $\mathbb{R}^X$, the vector lattice of real functions on X.

Daniell's inspiration is to base the development on an axiomatization of the integral.

Definition of the integral

An *integral on G* is a mapping $\mathcal{I}\colon G \to \mathbb{R}$ which is

- a *linear functional*, meaning $\mathcal{I}(uf + vg) = u\mathcal{I}f + v\mathcal{I}g$ for $f, g \in G$ and $u, v \in \mathbb{R}$,
- *positive*, meaning $\mathcal{I}g \geq 0$ whenever $g \geq 0$,
- *continuous*, meaning $\mathcal{I}f_n \to 0$ whenever $f_n \searrow 0$, i.e., $\mathcal{I}f_n \to 0$ whenever $\{f_n\}$ is a decreasing sequence which converges pointwise downwards to 0.

Lebesgue: measure, then integration. Daniell: integration, then measure.

A (very) brief lesson on the Daniell integral

- Start with a given integral $\mathcal{I}$ on a G, a sub-vector lattice of $\mathbb{R}^X$.
- Define $G^{\uparrow}$ to be is the family of all functions $f \in \mathbb{R}^X$ for which there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \nearrow f$ (pointwise upwards convergence) and $\{\mathcal{I}g_n\}$ is bounded.
- Extend $\mathcal{I}$ to $G^{\uparrow}$ by defining $\mathcal{I}f \equiv \bigvee_n \mathcal{I}g_n$. It is easy to show that $\mathcal{I}f$ is well defined.
- For an arbitrary $h \in \mathbb{R}^X$, define $\mathcal{I}^+h \equiv \bigwedge \left\{ \mathcal{I}f : h \leq f \in G^{\uparrow} \right\}$ if $\left\{ f \in G^{\uparrow} : h \leq f \right\} \neq \emptyset$, and $\mathcal{I}^+h = \infty$ otherwise. Define $\mathcal{I}^-h \equiv -\mathcal{I}^+(-h)$.
- For an arbitrary $h \in \mathbb{R}^X$ such that $\mathcal{I}^+h$ and $\mathcal{I}^-h$ are finite and equal, define $\mathcal{I}h \equiv \mathcal{I}^+h = \mathcal{I}^-h$. This selects the family of *integrable functions*, aka *measurable functions*, $X \to \mathbb{R}$.
- The measure theory can be recovered by defining the measure of a subset $A \subseteq X$ to be $\mu A \equiv \mathcal{I}\chi_A$, where χ_A is the characteristic function of A.

A (very) brief lesson on the Daniell integral

- The construction is direct and economical.
- For example, if one starts with piecewise continuous finite-valued step functions on $\mathbb{R}$, the above procedure constructs the Lebesgue measurable functions, and then Lebesgue measure.
- The procedure admits considerable generalization. The first litmus test is the Lebesgue Bounded Convergence Theorem: integrals commute with pointwise limits of bounded sequences of integrable functions.
- The second litmus test is to use the integral to define a measure on X. If all goes well, the measure generates the integral in the familiar fashion.

An obstacle, resolved by truncation

- Unfortunately, all does not always go well. There are examples in the literature of an integral on a vector sublattice $G \subseteq \mathbb{R}^X$ which corresponds to no measure on X.
- The presence of the constant functions in G prevents this pathology, but it is far too strong an assumption for most purposes. The key attribute necessary to make things work is known as Stone's axiom: □

$$\forall g \in G^+ \; (g \wedge 1 \in G).$$

  Notice that the constant function 1 itself is not required to be present in G.
- The function $g \mapsto g \wedge 1$ thus serves as a unary operation on G^+. It has the following properties for all $f, g \in G^+$. □

($\mathfrak{T}1$) $f \wedge \overline{g} \leq \overline{f} \leq f$.
($\mathfrak{T}2$) $\overline{g} = 0$ implies $g = 0$.
($\mathfrak{T}3$) $ng = \overline{ng}$ for all n implies $g = 0$.

Truncated vector lattices

- ## Definition of trunc

  A *truncated vector lattice*, or *trunc* for short, is an archimedean vector lattice G endowed with a unary operation $G^+ \to G^+ = (g \mapsto \overline{g})$ satisfying $(\mathfrak{T}1)$, $(\mathfrak{T}2)$, and $(\mathfrak{T}3)$. A *truncation homomorphism* is a vector lattice homomorphism $\theta\colon G \to H$ such that $\theta(\overline{g}) = \overline{\theta(g)}$ for all $g \in G$. The category of truncs and their homomorphisms is designated **T**.

  - **T** is a modest extension of **W**, in the sense that **W** manifests itself as the (non-full) monoreflective subcategory comprised of the *unital* truncs, i.e., the truncs which contain an element $u \in G^+$ such that $\overline{g} = g \wedge u$ for all $g \in G^+$.
  - For a trunc G, we let

$$\overline{G} \equiv \left\{ \overline{g} : g \in G^+ \right\}. \square$$

Familiar examples of truncs

- A good example of a trunc is $\mathcal{C}_0X$, the family of continuous real-valued functions on a compact Hausdorff pointed space X which vanish at the designated point. The truncation operation is

$$\overline{g}(x) \equiv \begin{cases} g(x) & \text{if } g(x) \leq 1 \\ 1 & \text{if } g(x) > 1. \end{cases} \qquad \square$$

  Notice that the constant function 1 is not in $\mathcal{C}_0X$. Notice also that $\mathcal{C}_0X$ separates the points of X.
- Another good example of a trunc is $\mathcal{LC}_0X$, the family of locally constant continuous real-valued functions on a pointed Boolean space X which vanish at the designated point. Notice that $\mathcal{LC}_0X$ separates the points of X.

Unital components

- **Definition**

  For elements $0 \leq f, g \in \overline{G}$ in a trunc G, we say that f *is a component of* g if $f \leq g$ and $f \wedge (g - f) = 0$. □ We say that u is a *unital component of* G if $u \in \overline{G}$ and $u \wedge v$ is a component of v for each $v \in \overline{G}$. We write

  $$\mathcal{UC}(G) \equiv \{\, u : u \text{ is a unital component of } G \,\}.$$

- **Proposition**

  The following are equivalent for an element u in a trunc G.

  1. u is a unital component.
  2. $u = \overline{2u}$.
  3. If G is a subtrunc of $\mathcal{C}_0 X$ for some compact Hausdorff pointed space X then $u = \chi_C$ for a clopen subset $C \subseteq X$ not containing the designated point $*$.

Simple truncs

- **Definition**
  A *simple element* in a trunc G is a linear combination of unital components. The set of simple elements forms a subtrunc of G, called the *simple part of* G, and written σG. A trunc is *simple* if $G = \sigma G$, i.e., if every element of the trunc is simple.

- **Theorem**
  The following are equivalent for a trunc G.
  1. G is isomorphic to $\mathcal{L}\mathcal{C}_0 X$ for a Boolean pointed space X.
  2. G is isomorphic to a subtrunc of $\mathcal{C}_0 X$ which is *bounded away from* 0. That is, for all $0 < g \in G$ there exists a real number $\varepsilon > 0$ such that
  $$\forall x \in X \; (g(x) > 0 \implies g(x) > \varepsilon)$$
  3. G is hyperarchimedean, i.e., $\operatorname{maxSpec} G = \operatorname{minSpec} G$, and G has enough unital components, i.e., for all $g \in G^+$ there exists a unital component u such that $\overline{g} \leq u$. □
  4. G is simple.

Some categorical equivalences

Theorem

These categories are equivalent.

1. The category **gBa** of *generalized Boolean algebras*, i.e., distributive lattices closed under relative complementation, with bottom element, together with lattice homomorphisms which preserve the bottom element.
2. The category $\mathbf{BSp}_*$ of Boolean pointed spaces with continuous functions which preserve the designated points.
3. The category **sT** of simple truncs.

If $(X, *)$ is a Boolean pointed space and $G = \mathcal{LC}_0 X$ is the corresponding simple trunc, then

$$\mathcal{UC}(G) = \{\, \chi_C : C \text{ is a clopen subset of } X \text{ such that } * \notin C \,\}$$

Integration on simple truncs is straightforward

- Definition

  A *charge* on a generalized Boolean algebra A is a real-valued function $\mu\colon A \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(u \vee v) = \mu(u) + \mu(v)$ whenever $u \wedge v = \bot$. A *measure* is a charge such that $\sum_n \mu(a_n) = \mu(a_0)$ whenever $\{a_n\}$ is a pairwise disjoint subset of A such that $\bigvee_n a_n = a_0$ exists in A.

- Proposition

  In a simple trunc G, every integral restricts to a charge on $\mathcal{UC}(G)$, and every charge on $\mathcal{UC}(G)$ produces an integral on G as follows. Each element $g \in G$ can be uniquely expressed in the form $g = \sum_U r(u)u$ for a pairwise disjoint finite subset $U \subseteq \mathcal{UC}(G)$ and function $r\colon U \to \mathbb{R} \smallsetminus \{0\}$. Define

$$\mathcal{I}g \equiv \sum_U r(u)\mu(u).$$

  These are inverse processes. (This is so because of Dini's Theorem.)

The trunc $\mathcal{R}_0 L$

Another good example of a trunc is $\mathcal{R}_0 L$, the family of frame maps from $\mathcal{O}\mathbb{R}_0$ into a pointed frame L which "vanish at the designated point". □

The truncation operation is

$$\overline{g}(-\infty, r) \equiv \begin{cases} \top & \text{if } r > 1 \\ g(-\infty, r) & \text{if } r \leq 1. \end{cases} \qquad \square$$

A unital component $u \in \mathcal{R}_0 L$ is of the form

$$u(R) = \begin{cases} \top & \text{if } 0, 1 \in R \\ x & \text{if } 0 \notin R \ni 1 \\ x^* & \text{if } 1 \notin R \ni 0 \\ \bot & \text{if } 0, 1 \notin R \end{cases} \qquad R \in \mathbb{R}$$

for a complemented element $x = \operatorname{coz} u$ in L.

The Madden representation of truncs

Theorem (B, 2014)

For every trunc there is a Lindelöf pointed frame L and a trunc injection $\mu_G : G \to \mathcal{R}_0 L = (g \mapsto \hat{g})$ such that

1. L is join-generated by

$$\operatorname{coz} G \cup \operatorname{con} G = \left\{ \operatorname{coz} g : g \in \overline{G} \right\} \cup \left\{ \operatorname{con} g : g \in \overline{G} \right\}$$
$$= \left\{ g(0, \infty) : g \in \overline{G} \right\} \cup \left\{ g(-\infty, 1) : g \in \overline{G} \right\}$$

2. for any subset $G_0 \subseteq \overline{G}$, if $\bigvee_{G_0} \operatorname{con} g = \top$ then G_0 generates G as a truncation kernel.

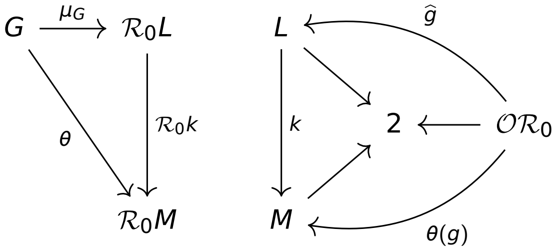
The frame and injection are unique with respect to these properties, and are referred to as the *Madden frame* and *Madden representation*, respectively.

Bumper sticker

Every trunc has a home.

The Madden representation is functorial

For any trunc homomorphism $\theta\colon G \to \mathcal{R}_0 M$ there is a unique pointed frame homomorphism k such that $\mathcal{R}_0 k \circ \mu_G = \theta$, i.e., $\theta(g) = k \circ \hat{g}$ for all $g \in G$.



Simply generated truncs

A trunc G is *simply generated* if every element of G^+ is the join of the simple elements below it. In this case the insertion $\sigma G \to G$ is realized by a compactification k of L.

$$\begin{array}{ccccccc}
\sigma G & \xrightarrow{\mu_{\sigma G}} & \mathcal{L}\mathcal{C}_0 K & \longrightarrow & \mathcal{R}_0 K & & K \\
\downarrow & & & & \downarrow{\scriptstyle \mathcal{R}_0 k} & & \downarrow{\scriptstyle k} \\
G & & \xrightarrow[\mu_G]{\qquad\qquad} & & \mathcal{R}_0 L & & L
\end{array}$$

- Because the insertion $\sigma G \to G$ is one-one, k is dense.
- Because σG join-generates G, k is surjective.
- An integral on G restricts to an integral on the simple trunc σG, which corresponds to a measure on $\mathcal{U}\mathcal{C}(G)$. And conversely, any measure on $\mathcal{U}\mathcal{C}(G)$ provides an integral on σG, which extends easily to G by the rule

$$\mathcal{I}(g) \equiv \bigvee \{ \mathcal{I}f : g \geq f \in \sigma G \} .$$

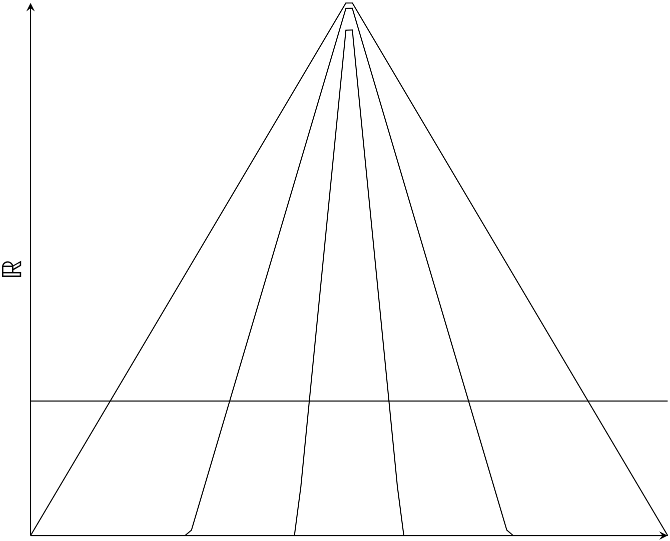
Simply generated truncs are nice but rare

- An integral on a simply generated trunc G reduces to an integral on its simple part σG.
- An integral on σG reduces to a measure on the generalized Boolean algebra $\mathcal{UC}(G)$.
- But, in order to be simply generated, a trunc G must have lots of simple elements, which means lots of unital components, each of which comes from a complemented element in the underlying Madden frame L of the trunc. That is, L must have a base of complemented elements.

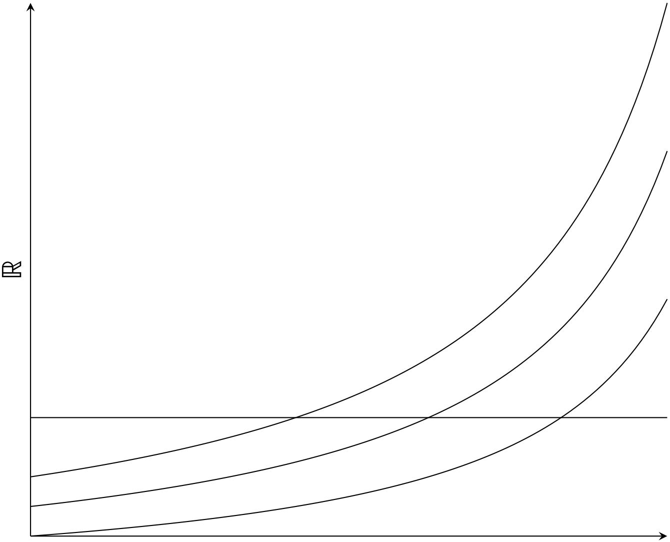
Pointwise convergence in pointfree topology

- The analysis of pointwise convergence reduces to an analysis of pointwise suprema and infima of sequences of real-valued functions.
- For sequences of real-valued functions, the term 'pointwise infimum' defines itself. It can also be fully articulated in the Madden representation, as follows.
- Consider a descending sequence $f_1 \geq f_2 \geq f_3 \ldots$ of nonnegative real-valued functions on $[0, 1]$.

$\bigwedge f_n = 0$ means $\neg \exists f_0 \, \forall n \, (0 < f_0 \leq f_n)$



$\bigwedge^{\bullet} f_n = 0$ means $\bigcup_n f_n^{-1}(-\infty, \varepsilon) = X$ for all $\varepsilon > 0$



Pointwise meets and joins

Definition

We say that a subset $K \subseteq \mathcal{R}_0^+ L$ *has pointwise meet* 0, and write $\bigwedge^\bullet K = 0$, if $\bigvee_K k(-\infty, \varepsilon) = \top$ for all $\varepsilon > 0$.
For a general subset $K \subseteq G$ and element k_0 in G, $\bigwedge^\bullet K = k_0$ means that $\bigwedge_K^\bullet (k - k_0) = 0$. And dually for $\bigvee^\bullet K = k_0$.

Pointwise meets and joins are just those which are "context free".

Proposition

For a subset $K \subseteq \mathcal{R}_0^+ L$, $\bigwedge^\bullet K = 0$ iff $\bigwedge_K \theta(k) = 0$ for all trunc homomorphisms $\theta \colon G \to H$.

Definition of directional pointwise convergence in $\mathcal{R}_0 L$

A sequence $\{g_n\} \subseteq \mathcal{R}_0^+ L$ *converges pointwise downwards to* 0, and write $g_n \searrow 0$, if it is decreasing and $\bigwedge_n^\bullet g_n = 0$, i.e., if $\bigvee_n g_n(-\infty, \varepsilon) = \top$ for all $\varepsilon > 0$. Likewise $g_n \searrow g_0$ means $(g_n - g_0) \searrow 0$. $g_n \nearrow g_0$ is defined dually.

Nakano-Stone type theorems

Theorem (Banashewski/Hong)

$\mathcal{R}L$ has the feature that every bounded (countable) subset of positive elements has a supremum iff L is extremally (basically) disconnected.

A space X is called a *P-space* if $\mathcal{C}X = \mathcal{LC}X$, i.e., if every continuous real-valued function on X is locally constant, i.e., if every zero set is clopen. A frame L is called a *P-frame* if every cozero element of L is complemented.

Theorem (B., Hager, Walters-Wayland)

$\mathcal{R}L$ has the feature that every bounded (countable) subset of positive elements has a pointwise supremum iff L is a Boolean algebra (a P-frame).

Theorem (B., Walters-Wayland, Zenk)

Pointed P-frames are monocoreflective in pointed frames. We write $L \to \mathcal{P}_* L$ for the reflector arrow for a pointed frame L.

The reflection $\mathcal{P}_* L$ has the same points as L does. Thus if L is spatial but not a P-frame then $\mathcal{P}_* L$ is not spatial.

The epimorphism theory in **T** mirrors that in **W**

Theorem

A trunc G is epicomplete, i.e., G has no proper extensions epic in $\mathbf{T}$, iff G is of the form $\mathcal{R}_0 L$ for a P-frame L.

Theorem

The epicomplete objects form a monoreflective subcategory of $\mathbf{T}$. A reflector for the trunc G with Madden frame L is the extension $G \to \mu_G(G) \leq \mathcal{R}_0 L \to \mathcal{R}_0 \mathcal{P}_* L$.

Notation for the epicompletion

$G \to \xi G$

Pointwise dense extensions are closely related to epic extensions

Theorem

The epicompletion $G \to \xi G$ can be understood as the pointwise completion of G.

1. Pointwise convergence on G is the restriction of pointwise convergence on ξG.
2. G is pointwise dense in ξG.
3. ξG is *pointwise complete* in the sense that it has no proper extension in which it is pointwise dense.

We shall use ξG to play the role of $\mathbb{R}^X$ in the construction of the Daniell integral.

Pointwise convergence in $\mathcal{R}_0 L$

Definition of pointwise convergence

A sequence $\{g_n\} \subseteq \mathcal{R}_0 L$ *converges pointwise to* 0, written $g_n \xrightarrow{\bullet} 0$, provided that it is bounded and $\bigvee^{\bullet}_{n \geq m} g_n \searrow 0$ in ξG.

Note that the following are equivalent.

1. $\{g_n\}$ has an upper bound in some supertrunc of G.
2. $\{g_n\}$ has a pointwise join in ξG.
3. $\left\{ g_i - g_j : i, j \in \mathbb{N} \right\}$ has a pointwise join in ξG.

We say that the sequence $\{g_n\}$ is *bounded somewhere*.

Pointwise convergence has nice properties.

Proposition

The trunc operations are pointwise continuous.

Proposition

Trunc homomorphisms are pointwise continuous.

Proposition

A pointwise dense subtrunc is epically embedded.

Conjecture

A subtrunc is epically embedded iff it is pointwise dense.

Pointwise Cauchy sequences

- Definition of pointwise Cauchy sequence

  A sequence $\{g_n\} \subseteq \mathcal{R}_0 L$ is said to be *pointwise Cauchy* if it is bounded somewhere and $\bigvee^{\bullet}_{i,j\geq m}(g_i - g_j) \searrow 0$.

- Proposition

  For a sequence $\{g_n\} \subseteq G^+$ which is bounded somewhere, the following are equivalent.

  1. There exists $h \in \xi G$ such that $g_n \xrightarrow{\bullet} h$.
  2. $\{g_n\}$ is pointwise Cauchy.
  3. $\bigvee^{\bullet}_m \bigwedge^{\bullet}_{n\geq m} g_n = \bigwedge^{\bullet}_m \bigvee^{\bullet}_{n\geq m} g_n$.

- Caution

  It is not the case that every element of ξG is the pointwise limit of a sequence from G. The pointwise closure operator must be iterated transfinitely to get from G to ξG.

Extending a given integral

Let $\mathcal{I}$ be an integral on G, i.e., $\mathcal{I}\colon G \to \mathbb{R}$ is positive, linear, and continuous.

Definition

Call a sequence $\{g_n\}$ *integrable* if it is pointwise Cauchy and has finite total integral, i.e., $\bigvee_m \mathcal{I} \bigvee_{i,j \le m}(g_i - g_j) < \infty$.
We denote the family of integrable sequences by $\mathcal{IS}$. Let

$$G' \equiv \left\{ h \in \xi G : \exists \{g_n\} \in \mathcal{IS} \ (g_n \xrightarrow{\bullet} h) \right\}.$$

Proposition

- If $\{g_n\}$ is an integrable sequence in a trunc G then $\{\mathcal{I}g_n\}$ is a convergent sequence of real numbers.
- G' is a subtrunc of ξG containing G.
- The extension of $\mathcal{I}$ is an integral on G' which extends $\mathcal{I}$ on G.

Extending a given integral

Theorem

Let $\mathcal{I}$ be an integral on a trunc G. Then there is a subtrunc $G^\circ \subseteq \xi G$ containing G with the following properties.

1. Pointwise convergence on G° restricts to pointwise convergence on G, and G is pointwise dense in G°.
2. G° is *almost pointwise complete*, in the sense that every integrable sequence on G° is convergent to an element of G°.
3. $\mathcal{I}$ extends to G°.
4. G° is simply generated.
5. Thus $\mathcal{I}$ restricts to an integral on σG° which comes from a measure on $\mathcal{UC}(G^\circ)$, and this measure generates $\mathcal{I}$ in the standard fashion.

A few comments

- The Lebesgue Dominated Convergence Theorem takes the following form. Suppose $G = G^\circ$, $\{g_n\} \in \mathcal{IS}$, and $g_0 \in G$. Then
$$g_n \xrightarrow{\bullet} g_0 \implies \mathcal{I}g_n \to \mathcal{I}g_0.$$
- The construction of the extension $G \to G^\circ$ of the integral is done by a transfinite iteration, using pointwise convergence at every step, rather than extending the integral all at once as is done in the classical construction.
- When $\mathcal{I}$ is taken to be the Riemann integral on the trunc G of continuous functions $\mathbb{R} \to \mathbb{R}$ of compact support, the extension $G \to G^\circ$ generates the trunc of Baire measurable functions. We do not get the Lebesgue measurable functions, because G is not pointwise dense in them. Nor are the Lebesgue measurable functions an epic extensions of G in the category **T**.

Happy birthday, Aleš!