On the maximal regular ideal of $\mathcal{R}L$

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(in honour of Aleš Pultr)

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All frames in this talk are completely regular, and

1. $\mathcal{R}L$ is the ring of all continuous real-valued functions on L;
2. $\operatorname{Coz} L$ is the cozero part of L;
3. βL is the Stone-Čech compactification of L;
4. $\mathcal{S}(L)$ denotes the lattice of sublocales of L;
5. The supplement of $S \in \mathcal{S}(L)$ is the sublocale

$$L \smallsetminus S = \bigcap \{T \in \mathcal{S}(L) \mid T \vee S = L\};$$

6. For any $a \in L$,

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = \uparrow a = \{x \in L \mid x \geq a\}$$

denote the open and the closed sublocales associated with a.

Let A be a ring, and let $a \in A$. An element $x \in A$ is called a Von Neumann inverse of a in case $axa = a$.

In the article

B. Brown, N. McCoy,
The maximal regular ideal of a ring
Proc. Amer. Math. Soc. **37** (1996), 579–589,

the authors prove that in any ring the sum of regular ideals is a regular ideal, and thus there is a maximal regular ideal, denoted M(A).

Recall that an ideal I of a commutative ring A is called pure if for every $u \in I$ there is a $v \in I$ such that $u = uv$. In the paper

A.R. Aliabad, J. Hashemi, R. Mohamadian, *P-ideals and PMP-ideals in commutative rings*, J. Math. Extension, 4 (2016), 19–33,

there is the following useful characterization.

Theorem

The following are equivalent for an ideal I of a reduced ring A.

1. *I is a regular ideal.*
2. *Every prime ideal of I is a maximal ideal in I.*
3. *Every ideal of A contained in I is pure.*

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Let $r_L : L \to \beta L$ be the localic map that embeds L as a sublocale of βL.

Definition

For any sublocale A of βL, the ideals $\boldsymbol{M}^A$ and $\boldsymbol{O}^A$ of $\mathcal{R}L$ is defined by

$$\boldsymbol{M}^A = \{\alpha \in \mathcal{R}L \mid \mathfrak{c}(r_L(\operatorname{coz}\alpha)) \subseteq A\}$$

and

$$\boldsymbol{O}^A = \{\alpha \in \mathcal{R}L \mid \operatorname{int}_{\beta L} \mathfrak{c}(r_L(\operatorname{coz}\alpha)) \subseteq A\} = \{\alpha \in \mathcal{R}L \mid \mathfrak{o}(r_L(\operatorname{coz}\alpha)^*) \subseteq A\}.$$

Lemma

The following are equivalent for an ideal Q of $\mathcal{R}L$.

1. *Q is a regular ideal.*
2. *For every $\alpha \in Q$, $\operatorname{coz}\alpha$ is complemented.*
3. *$Q = \boldsymbol{O}^{\mathfrak{c}(I)}$, for some $I \in \beta L$ consisting entirely of complemented elements.*

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Theorem

$\mathrm{M}(\mathcal{R}L) = \{\alpha \in \mathcal{R}L \mid \mathfrak{o}(\operatorname{coz}\alpha)$ *is clopen and is a P-frame*$\}$.

Corollary

The following are equivalent for a completely regular frame L.

1. $\mathrm{M}(\mathcal{R}L)$ *is not the zero ideal.*
2. $\operatorname{Coz} L$ *has a nonzero ideal consisting entirely of complemented elements.*
3. L *has a non-void clopen sublocale which is a P-frame.*

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Call a point (= prime element) $\mathfrak{p}$ of βL a *P*-point if $\boldsymbol{M}^{\mathfrak{c}(\mathfrak{p})} = \boldsymbol{O}^{\mathfrak{c}(\mathfrak{p})}$.

(Justification: L is a P-frame iff every point of βL is a P-point)

Definition

We define the sublocale ϱL of βL and the element J_L of βL by

$$\varrho L = \bigvee_{S(\beta L)} \{\mathfrak{c}(\mathfrak{p}) \mid \mathfrak{p} \text{ is a } P\text{-point of } \beta L\}$$

and

$$\mathrm{J}_L = \bigvee_{\beta L} \{I \in \beta L \mid I \subseteq BL\}.$$

Remark

1. $\varrho L = \{I \in \beta L \mid I \text{ is a meet of } P\text{-points}\}$.
2. ϱL is spatial, and its points are precisely the P-points of βL.
3. $\varrho L = \beta L$ if and only if L is a P-frame.

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J. Picado, A. Pultr, A. Tozzi,

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Houst. J. Math. Extension, (to appear),

the authors prove that:

If L is T_1-spatial, S is a join of closed sublocales of L, and $T \in \mathcal{S}(L)$, then

$$S \smallsetminus T = \bigvee \{\mathfrak{c}(x) \mid x \in \mathrm{Max}(L),\ x \in S,\ x \notin T\}.$$

Let $\{A_i \mid i \in I\}$ be collection of closed sublocales of βL, and let $A = \bigcap_i A_i$. Then

$$\sum_i \boldsymbol{o}^{A_i} = \boldsymbol{o}^{A}.$$

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Theorem

$\mathsf{M}(\mathcal{R}L) = \boldsymbol{O}^{\beta L \smallsetminus \varrho L} = \boldsymbol{O}^{\mathfrak{c}(\mathsf{J}_L)}$.

Corollary

For any L, $\mathrm{int}_{\beta L}(\varrho L) = \mathfrak{o}(\mathsf{J}_L)$.

Definition

A cozero element is strongly complemented if every cozero element below it is complemented.

Corollary

The following statements are equivalent for any L.

1. *$\mathsf{M}(\mathcal{R}L)$ is an essential ideal in $\mathcal{R}L$.*
2. *Below every nonzero cozero element of L there is a strongly complemented nonzero cozero element of L.*
3. *The sublocale $\mathrm{int}_{\beta L}(\varrho L)$ of βL is dense in βL.*

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Let $\mathfrak{Z}L$ denote the ℓ-ring of continuous integer-valued functions on L. By a result a proof of which can be found in the paper

B. Banaschewski,
Countable composition closedness and integer-valued continuous functions in pointfree topology,
Cat. Gen. Algebraic Struct. Appl. **1** (2013), 1–10,

there is an ω-updirected collection $\{A_i \mid i \in I\}$ of sub-ℓ-rings of $\mathfrak{Z}L$, each isomorphic to a $C(X, \mathbb{Z})$, such that

$$\mathfrak{Z}L = \bigcup_i A_i.$$

1. In any $C(X,\mathbb{Z})$, any $f \geq \mathbf{0}$ which has a Von Neumann inverse satisfies $f[X] \subseteq \{0,1\}$, and is therefore an idempotent.
2. Consequently, if $g \in \mathrm{M}(C(X,\mathbb{Z}))$, then g^2 has a Von Neumann inverse, and is therefore an idempotent.
3. Thus, if $g \in \mathrm{M}(C(X,\mathbb{Z}))$, then every multiple of g^2 is an idempotent. This forces g^2 to be $\mathbf{0}$, whence $g = \mathbf{0}$.

Proposition

$\mathrm{M}(\mathfrak{Z}(L)) = \{\mathbf{0}\}$.

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Thank you very much

Muito obrigado