Enriched Topologies and Topological Representation of Semi-Unital Quantales

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Coimbra, September 2018

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- A monoid in $\mathtt{Sup}$ is a quantale with unit or a unital quantale.
- Let $\top$ be the universal upper bound of a quantale $\mathfrak{Q}$. Then $\mathfrak{Q}$ is
  (1) semi-unital if $\alpha \leq \alpha * \top$ and $\alpha \leq \top * \alpha$ for $\alpha \in \mathfrak{Q}$,
  (2) semi-integral if $\alpha * \top * \beta \leq \alpha * \beta$ for $\alpha, \beta \in \mathfrak{Q}$.
  (3) Let $\mathfrak{Q}$ be a semi-unital quantale. Then an element $p \in \mathfrak{Q}$ is prime, if $p \neq \top$ and the relation $\alpha * \beta \leq p$ implies $\alpha * \top \leq p$ or $\top * \beta \leq p$.
  (4) A semi-unital quantale is spatial if prime elements are order generating — i.e. every element is a meet of prime elements.

Presentation of the Problem.

- Let A be a non-commutative and unital C^*-algebra. Then the ideal lattice $\mathbb{L}(A)$ of all closed left ideals of A provided with the ideal multiplication $*$ is a quantale. It is well known that $(\mathbb{L}(A), *)$ is idempotent, non-commutative and semi-integral. Hence:

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- The replacement of the quantale **2** by a non-commutative and unital quantale opens the door to enriched category theory.

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  Essentially equivalent means the existence of a quantale monomorphism $\mathbb{L}(A) \xrightarrow{\varphi} \mathcal{T}$ such that the range $\varphi(\mathbb{L}(A))$ of φ and the universal upper bound $\top$ of $\mathcal{T}$ generate $\mathcal{T}$.

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The aim of this talk is to present a positive answer to this question by proving the following more general result:

Theorem. There exists a unital quantale $\mathfrak{Q}$ such that for any semi-unital and spatial quantale $\mathbb{X}$ there exists a $\mathfrak{Q}$-enriched sober space $(Z, \mathcal{T})$ satisfying the condition that the quantale $\mathbb{X}$ is essentially equivalent to $\mathfrak{Q}$-enriched topology $\mathcal{T}$.

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- The previous theorem covers the case of the quantale $\mathbb{X} = (\mathbb{L}(A), *)$.

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Theorem 1. (A. Joyal and M. Tierney 1984) Let X be a set. The free right $\mathfrak{Q}$-module generated by X in the sense of $\mathtt{Mod}_r(\mathfrak{Q})$ is the complete lattice $\mathfrak{Q}^X$ of all maps $X \xrightarrow{f} \mathfrak{Q}$ provided with the right action which is determined by

$$(f \boxdot \alpha)(x) = f(x) * \alpha, \qquad \alpha \in \mathfrak{Q},\ f \in \mathfrak{Q}^X.$$

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- The pair (L, p) is skeletal $\mathfrak{Q}$-enriched category where L is a set of objects and $L \times L \xrightarrow{p} \mathfrak{Q}$ is a hom-object assignment satisfying the axioms:

$$e \leq p(t,t),$$

$$p(r,s) * p(s,t) \leq p(r,t),$$

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- A skeletal $\mathfrak{Q}$-enriched category (L, p) is join-complete, if the Yoneda embedding

$$(L,p) \longrightarrow \mathbb{P}(L,p) = \{f \in \mathfrak{Q}^L \mid p(t_2,t_1) * f(t_1) \le f(t_2)\}$$

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  Theorem 2. (I. Stubbe 2006) $\mathtt{Mod}_r(\mathfrak{Q}) \cong \mathtt{Sup}(\mathfrak{Q})$.

Axioms of $\mathfrak{Q}$-enriched Topologies

Theorem 1 and Theorem 2 imply that the right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ is the $\mathfrak{Q}$-enriched power set of X with the hom-object assignment p and the formation of $\mathfrak{Q}$-enriched joins $\sup_{(\mathfrak{Q}^X,p)}$ given as follows:

$$p(f,g) = \bigwedge_{x\in X} f(x) \searrow g(x), \quad \sup_{(\mathfrak{Q}^X,p)}(F)(x) = \bigvee_{f\in\mathfrak{Q}^X} f(x) * F(f).$$

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- A $\mathfrak{Q}$-enriched topology $\mathcal{T}$ on a set X is a right $\mathfrak{Q}$-submodule of free right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ satisfying the following topological axioms:

(RT1) $\underline{\top} \in \mathcal{T}$,

(RT2) if $f_1, f_2 \in \mathcal{T}$, then $f_1 * f_2 \in \mathcal{T}$,

where $\underline{\top}$ is the constant map determined by the universal upper bound $\top$ of $\mathfrak{Q}$ and $(f_1 * f_2)(x) = f_1(x) * f_2(x)$ for all $x \in X$.

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- A pair $(X, \mathcal{T})$ is a $\mathfrak{Q}$-enriched topological space, if X is a set and $\mathcal{T}$ is a $\mathfrak{Q}$-enriched topology on X.

Definition 1. A triple $(L, *, \boxdot)$ is a right $\mathfrak{Q}$-algebra if $(L, *)$ is a quantale and $(L, \boxdot)$ is a right $\mathfrak{Q}$-module such that the following compatibility relation holds:

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- **Examples.**
  (a) Because of (RT2) every $\mathfrak{Q}$-enriched topology is a right $\mathfrak{Q}$-algebra.
  (b) Given a unital quantale $(L, *, d)$ and a unital quantale homomorphism $\mathfrak{Q} \xrightarrow{j} L$. Then j induces a right action $\boxdot$ on L

$$t \boxdot \alpha = t * j(\alpha)$$

such that $(L, *, \boxdot)$ is a right $\mathfrak{Q}$-algebra.

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- elements of $\mathcal{T}$ separate elements in $\mathrm{pt}(\mathbb{L})$ – $\mathrm{pt}(\mathbb{L})$ is a T_0-space,
- every strong right $\mathfrak{Q}$-algebra morphism $\mathcal{T} \xrightarrow{\varphi} \mathfrak{Q}$ is induced by an element $h \in \mathrm{pt}(\mathbb{L})$ — i.e. $\varphi(\mathbb{A}_t) = \mathbb{A}_t(h)$.

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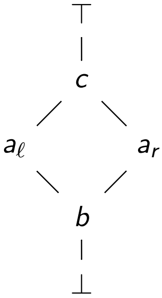
- a is right-sided and not left-sided. This leads to the quantale C_3^r.

Quantization of **2**

For an idempotent and non-commutative quantale on $C_3 = \{\bot, a, \top\}$ the element a strictly located between the top and the bottom can only play two roles:

- a is left-sided and not right-sided. This leads to the quantale C_3^ℓ.
- a is right-sided and not left-sided. This leads to the quantale C_3^r.

- The quantization of **2** is the tensor product $\mathfrak{Q}_\mathbf{2} = C_3^\ell \otimes C_3^r$ and consists of six elements:



$\star$	$\bot$	b	a_ℓ	a_r	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	b	b	a_r	a_r	a_r
a_ℓ	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
c	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

Prime Elements of Semi-unital Quantales and Strong Homomorphisms

Given a semi-unital quantale $\mathfrak{Q}$. Then every prime element p of $\mathfrak{Q}$ can be identified with a strong (quantale) homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q}_2$ satisfying the condition:

$$p = \bigvee\{\alpha \in \mathfrak{Q} \mid h(\alpha) \leq c\}.$$

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- **Construction**:

$$h_p(\alpha) = \begin{cases} \bot, & \top * \alpha * \top \leq p, \\ b, & \top * \alpha * \top \nleq p,\ \alpha * \top \leq p \text{ and } \top * \alpha \leq p, \\ a_\ell, & \alpha * \top \nleq p \text{ and } \top * \alpha \leq p, \\ a_r, & \alpha * \top \leq p \text{ and } \top * \alpha \nleq p, \\ c, & \alpha \leq p,\ \alpha * \top \nleq p \text{ and } \top * \alpha \nleq p, \\ \top, & \alpha \nleq p. \end{cases}$$

Theorem 3. (H. 2015) A semi-unital quantale $\mathfrak{Q}$ is spatial — i.e. prime elements are order generating — if and only if strong homomorphisms $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q}_{\mathbf{2}}$ separate elements of $\mathfrak{Q}$.

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- If $\mathfrak{Q}$ is semi-unital and semi-integral, then there exists a quantale monomorphism $\mathfrak{Q} \xrightarrow{\varphi} \mathfrak{Q} \otimes C_3^\ell$ determined by

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- Finally, let $\overline{\mathfrak{Q} \otimes C_3^\ell}$ be the one point extension of $\mathfrak{Q} \otimes C_3^\ell$ given by the left semi-unitalization.

Results

Let us consider the right $\widehat{\mathfrak{Q}_{\mathbf{2}}}$-subalgebra $\mathbb{L}_{\mathfrak{Q}}$ of the one point extension $\overline{\mathfrak{Q} \otimes C_3^{\ell}}$ of the tensor product $\mathfrak{Q} \otimes C_3^{\ell}$ which is generated by the range $\varphi(\mathfrak{Q})$ of φ and the added point of the one point extension of the tensor product $\mathfrak{Q} \otimes C_3^{\ell}$. Then $\mathbb{L}_{\mathfrak{Q}}$ has the following properties:

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- There exists a bijective map between the spectrum $\mathrm{pt}(\mathbb{L}_{\mathfrak{Q}})$ of $\mathbb{L}_{\mathfrak{Q}}$ and the set of all strong homomorphism $\mathfrak{Q} \to \mathfrak{Q}_2$.
- A semi-unital and semi-integral quantale $\mathfrak{Q}$ is spatial if and only if elements of the spectrum $\mathrm{pt}(\mathbb{L}_{\mathfrak{Q}})$ separate elements in $\mathbb{L}_{\mathfrak{Q}}$.
- If a quantale $\mathfrak{Q}$ is semi-unital and spatial, then the $\widehat{\mathfrak{Q}_2}$-enriched sober space induced by the right $\widehat{\mathfrak{Q}_2}$-algebra $\mathbb{L}_{\mathfrak{Q}}$ is the topological representation of $\mathfrak{Q}$ — i.e. $\mathfrak{Q}$ is essentially equivalent to the $\widehat{\mathfrak{Q}_2}$-enriched topology $\mathcal{T}$ on the spectrum $\mathrm{pt}(\mathbb{L}_{\mathfrak{Q}})$.