Frames of continuous functions

Wouter Van Den Haute
Joint work with Wendy Lowen and Mark Sioen

Workshop on Algebra, Logic and Topology

Universiteit Antwerpen

Motivation

- Functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}^{\mathrm{op}}$ represents spaces as frames
- Left adjoint $\Sigma : \mathsf{Frm}^{\mathrm{op}} \to \mathsf{Top}$ 'reconstructs' space: $\Sigma\mathcal{O}X \cong X$ whenever X is sober

Motivation

- Functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}^{\mathrm{op}}$ represents spaces as frames
- Left adjoint $\Sigma : \mathsf{Frm}^{\mathrm{op}} \to \mathsf{Top}$ 'reconstructs' space: $\Sigma\mathcal{O}X \cong X$ whenever X is sober
- There are many other ways to construct frames induced by topological spaces, e.g.

$$\mathsf{Top}(X, \mathbb{P}) = \{f : (X, \mathcal{T}) \to ([0, \infty], \mathcal{T}_{\mathrm{Scott}}) \mid f \text{ continuous}\}$$

Motivation

- Functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}^{\mathrm{op}}$ represents spaces as frames
- Left adjoint $\Sigma : \mathsf{Frm}^{\mathrm{op}} \to \mathsf{Top}$ 'reconstructs' space: $\Sigma\mathcal{O}X \cong X$ whenever X is sober
- There are many other ways to construct frames induced by topological spaces, e.g.

$$\mathsf{Top}(X, \mathbb{P}) = \{f : (X, \mathcal{T}) \to ([0, \infty], \mathcal{T}_{\mathrm{Scott}}) \mid f \text{ continuous}\}$$

- But the spectrum of $\mathsf{Top}(X, \mathbb{P})$ is $X \times]0, \infty]$, not X!
- Can we 'mod out' $\mathbb{P}$?

Preliminaries

- $\mathbb{S} := (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$ and $0 < 1$
- For any space X, $\mathcal{O}X \cong \mathsf{Top}(X, \mathbb{S})$
- For any frame L,
  $\Sigma L = \mathsf{Frm}(L, \mathbb{S}) \cong \mathsf{Spec}_{\wedge}(L) = \{a \in L \mid a \text{ is meet-irreducible}\}$.

Topological frames I

Definition

Let $(\mathbb{F}, \leq)$ be a frame endowed with a topology $\mathcal{T}_{\mathbb{F}}$. We call $(\mathbb{F}, \leq, \mathcal{T}_{\mathbb{F}})$ a *topological frame* provided that the operations

$$\wedge : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a, b) \mapsto a \wedge b$$

and

$$\sup_{i \in I} : \mathbb{F}^I \to \mathbb{F} : (a_i)_{i \in I} \mapsto \sup_{i \in I} a_i$$

are continuous.

Any chain endowed with the Scott topology is a topological frame.

Topological frames II

Definition

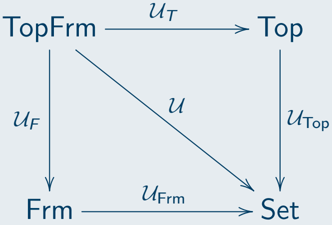
Let $(\mathbb{F}_1, \leq_1, \mathcal{T}_1)$ and $(\mathbb{F}_2, \leq_2, \mathcal{T}_2)$ be topological frames. A map $f : \mathbb{F}_1 \to \mathbb{F}_2$ is called a *topological frame morphism* if $f : (\mathbb{F}_1, \mathcal{T}_1) \to (\mathbb{F}_2, \mathcal{T}_2)$ is continuous and $f : (\mathbb{F}_1, \leq_1) \to (\mathbb{F}_2, \leq_2)$ is a frame homomorphism.

We call TopFrm the category with topological frames as objects and topological frame morphisms as morphisms.

Topological frames III

Proposition

The diagram



commutes, the functors $\mathcal{U}_{\mathsf{Top}}$ and $\mathcal{U}_F$ are topological, $\mathcal{U}_{\mathsf{Frm}}$ is monadic, $\mathcal{U}_T$ is adjoint and $\mathcal{U}$ is faithful and adjoint.

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathrm{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathrm{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism
- Define $\mathrm{Frm}_{\mathbb{F}}$ as the comma category $\mathbb{F}/\mathrm{Frm}$

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathsf{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism
- Define $\mathsf{Frm}_{\mathbb{F}}$ as the comma category $\mathbb{F}/\mathsf{Frm}$
- $\mathcal{O}_{\mathbb{F}} : \mathsf{Top} \to \mathsf{Frm}_{\mathbb{F}}^{\mathrm{op}}$ with $\mathcal{O}_{\mathbb{F}}(X) = \Gamma_X$ and

$$\mathcal{O}_{\mathbb{F}}(\varphi) : \mathsf{Top}(Y, \mathbb{F}) \to \mathsf{Top}(X, \mathbb{F}) : f \mapsto f\varphi$$

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathrm{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism
- Define $\mathrm{Frm}_{\mathbb{F}}$ as the comma category $\mathbb{F}/\mathrm{Frm}$
- $\mathcal{O}_{\mathbb{F}} : \mathrm{Top} \to \mathrm{Frm}_{\mathbb{F}}^{\mathrm{op}}$ with $\mathcal{O}_{\mathbb{F}}(X) = \Gamma_X$ and

$$\mathcal{O}_{\mathbb{F}}(\varphi) : \mathrm{Top}(Y, \mathbb{F}) \to \mathrm{Top}(X, \mathbb{F}) : f \mapsto f\varphi$$

Let $L = (L, \gamma_L : \mathbb{F} \to L)$ be an $\mathbb{F}$-frame.

- Endow $\mathrm{Spec}_{\mathbb{F}}(L) = \mathrm{Frm}_{\mathbb{F}}(L, \mathbb{F})$ with the initial topology for the source

$$(\mathrm{ev}_l : \mathrm{Frm}_{\mathbb{F}}(L, \mathbb{F}) \to \mathbb{F} : f \mapsto f(l))_{l \in L}$$

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathsf{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism
- Define $\mathsf{Frm}_{\mathbb{F}}$ as the comma category $\mathbb{F}/\mathsf{Frm}$
- $\mathcal{O}_{\mathbb{F}} : \mathsf{Top} \to \mathsf{Frm}_{\mathbb{F}}^{\mathrm{op}}$ with $\mathcal{O}_{\mathbb{F}}(X) = \Gamma_X$ and

$$\mathcal{O}_{\mathbb{F}}(\varphi) : \mathsf{Top}(Y, \mathbb{F}) \to \mathsf{Top}(X, \mathbb{F}) : f \mapsto f\varphi$$

Let $L = (L, \gamma_L : \mathbb{F} \to L)$ be an $\mathbb{F}$-frame.

- Endow $\mathsf{Spec}_{\mathbb{F}}(L) = \mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F})$ with the initial topology for the source

$$(\mathrm{ev}_l : \mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F}) \to \mathbb{F} : f \mapsto f(l))_{l \in L}$$

- We obtain a functor $\mathsf{Spec}_{\mathbb{F}} : \mathsf{Frm}_{\mathbb{F}}^{\mathrm{op}} \to \mathsf{Top}$ which is left adjoint to $\mathcal{O}_{\mathbb{F}}$

$\mathbb{F}$-spatial frames and $\mathbb{F}$-sober spaces

Definition

- L is $\mathbb{F}$-*spatial* if

$$\rho_L : L \to \mathsf{Top}(\mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F}), \mathbb{F}) : l \mapsto (f \mapsto f(l))$$

  is an isomorphism of $\mathbb{F}$-frames.

- X is $\mathbb{F}$-*sober* if

$$\eta_X : X \to \mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) : x \mapsto (f \mapsto f(x))$$

  is a homeomorphism.

$\mathbb{F}$-spatial frames and $\mathbb{F}$-sober spaces

Definition

- L is $\mathbb{F}$-*spatial* if

$$\rho_L : L \to \mathrm{Top}(\mathrm{Frm}_{\mathbb{F}}(L, \mathbb{F}), \mathbb{F}) : l \mapsto (f \mapsto f(l))$$

  is an isomorphism of $\mathbb{F}$-frames.

- X is $\mathbb{F}$-*sober* if

$$\eta_X : X \to \mathrm{Frm}_{\mathbb{F}}(\mathrm{Top}(X, \mathbb{F}), \mathbb{F}) : x \mapsto (f \mapsto f(x))$$

  is a homeomorphism.

If $\mathbb{F} = \mathbb{S}$, everything reduces to the classical setting.

More on $\mathbb{F}$-soberness

Proposition

X is $\mathbb{F}$-sober if and only if each of the following holds:

1. $(f : X \to \mathbb{F})_{f \in \mathsf{Top}(X,\mathbb{F})}$ *is initial*
2. $(f : X \to \mathbb{F})_{f \in \mathsf{Top}(X,\mathbb{F})}$ *is pointseparating*
3. $\mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X,\mathbb{F}),\mathbb{F}) = \{\mathrm{ev}_x \mid x \in X\}$

More on $\mathbb{F}$-soberness

Proposition

X is $\mathbb{F}$-sober if and only if each of the following holds:

1. $(f : X \to \mathbb{F})_{f \in \mathsf{Top}(X,\mathbb{F})}$ *is initial*
2. $(f : X \to \mathbb{F})_{f \in \mathsf{Top}(X,\mathbb{F})}$ *is pointseparating*
3. $\mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) = \{\mathrm{ev}_x \mid x \in X\}$

Proposition

If X is Hausdorff and ...(conditions on $\mathbb{F}$)..., then X is $\mathbb{F}$-sober

Some spectra I

- $\mathrm{Spec}_{\mathbb{P}}(\mathrm{Top}(\mathbb{S},\mathbb{P})) \cong \mathbb{P}$, so a sober space is not $\mathbb{F}$-sober in general.
  Proof: First we note that

$$\begin{aligned}\mathrm{Top}(\mathbb{S},\mathbb{P}) &= \{f : \mathbf{2} \to [0,\infty] \mid f(0) \leq f(1)\} \\ &\cong \{(x,y) \in \mathbb{P} \times \mathbb{P} \mid x \leq y\}.\end{aligned}$$

  For $\varphi \in \mathrm{Spec}_{\mathbb{P}}(\mathrm{Top}(\mathbb{S},\mathbb{P}))$ and $x, y \in \mathbb{P}$ with $x \leq y$, we have

$$(x,y) = (0,y) \vee (x,x) = ((y,y) \wedge (0,\infty)) \vee (x,x),$$

  so

$$\varphi(x,y) = (\varphi(y,y) \wedge \varphi(0,\infty)) \vee \varphi(x,x) = (y \wedge \varphi(0,\infty)) \vee x.$$

Some spectra II

Then the map

$$\Phi : \mathsf{Spec}_{\mathbb{P}}(\mathsf{Top}(\mathbb{S}, \mathbb{P})) \to \mathbb{P} : \varphi \mapsto \varphi(0, \infty)$$

is a homeomorphism. For $\alpha \in \mathbb{P}$, $\Phi^{-1}(\alpha) = \varphi_\alpha$ with

$$\varphi_\alpha : \mathsf{Top}(\mathbb{S}, \mathbb{P}) \to \mathbb{P} : (x, y) \mapsto (y \wedge \alpha) \vee x.$$

□

Some spectra III

- $\mathrm{Spec}_{\mathbb{P}}(\mathrm{Top}(\mathbf{n}, \mathbb{P})) \cong \mathrm{Top}(\mathbf{n} \setminus \{0\}^{\mathrm{op}}, \mathbb{P})$, where $\mathbf{n} = \{0, \ldots, n-1\}$
  Proof: First we note that

$$\mathrm{Top}(\mathbf{n}, \mathbb{P}) \quad \cong \quad \{(x_0, \ldots, x_{n-1}) \in \mathbb{P}^n \mid \forall n : x_n \leq x_{n+1}\}.$$

For $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathrm{Top}(\mathbf{n}, \mathbb{P})$,

$$\mathbf{x} = \bigvee_{i=0}^{n-1} ((x_i, x_i, \ldots, x_i) \wedge (0, 0, \ldots, 0, \infty, \infty, \ldots, \infty)),$$

so again

$$\varphi(\mathbf{x}) = \bigvee_{i=0}^{n-1} x_i \wedge \varphi(e_i).$$

Some spectra IV

Then the map

$$\Phi : \mathsf{Spec}_{\mathbb{P}}(\mathsf{Top}(\mathbf{n}, \mathbb{P})) \to \mathsf{Top}(\mathbf{n} \setminus \{0\}^{\mathrm{op}}, \mathbb{P}) : \varphi \mapsto (\varphi(e_i))_{i=1}^{n-1}$$

is a homeomorphism. For $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in \mathsf{Top}(\mathbf{n} \setminus \{0\}^{\mathrm{op}}, \mathbb{P})$ and $\alpha_0 := \infty$, the inverse is given by

$$\varphi_\alpha : \mathsf{Top}(\mathbf{n} \setminus \{0\}^{\mathrm{op}}, \mathbb{P}) \to \mathbb{P} : \mathbf{x} \mapsto \bigvee_{i=0}^{n-1} x_i \wedge \alpha_i.$$

□

Some spectra V

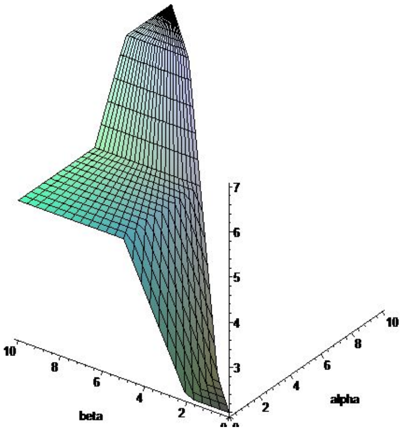


Figure: $(\mathrm{ev}_{(2,5,7)} \circ \Phi^{-1})(\alpha, \beta) = 2 \vee (5 \wedge \beta) \vee (7 \wedge \alpha)$ with $(\beta, \alpha) \in \mathsf{Top}(\mathbf{2}^{\mathrm{op}}, \mathbb{P}) \cap [0, 10]^2$

Some spectra VI

- $\mathsf{Spec}_{\mathbf{3}}(\mathsf{Top}(\mathbb{P},\mathbf{3})) \cong \mathsf{Top}(\mathbf{2},\mathbb{P})$
  Proof: Define a **3**-frame isomorphism

$$\theta : \mathsf{Top}(\mathbb{P},\mathbf{3}) \to \mathsf{Top}(\mathbf{2},\mathbb{P}_\perp)^{\mathrm{op}} : f \mapsto (j_0(f), j_1(f))$$

  where $j_i(f) = \sup\{\alpha \in \mathbb{P} \mid f(\alpha) \leq i\}$. Define

$$\Phi : \mathsf{Top}(\mathbf{2},\mathbb{P}) \to \mathsf{Frm}_{\mathbf{3}}(\mathsf{Top}(\mathbf{2},\mathbb{P}_\perp)^{\mathrm{op}},\mathbf{3}) : (\alpha,\beta) \mapsto \varphi^{\alpha,\beta}$$

  with

$$\varphi^{\alpha,\beta} : \mathsf{Top}(\mathbf{2},\mathbb{P}_\perp)^{\mathrm{op}} \to \mathbf{3} : (x,y) \mapsto \begin{cases} 0 & \beta \leq x \\ 1 & x < \beta \text{ and } \alpha \leq y \\ 2 & y < \alpha \end{cases}$$

Some spectra VII

Then Φ is well-defined and injective. To prove that it is onto, take $\varphi \in \mathsf{Frm}_{\mathbf{3}}(\mathsf{Top}(\mathbf{2}, \mathbb{P}_\perp)^{\mathrm{op}}, \mathbf{3})$ and define

$$\beta := \inf\{x \in \mathbb{P} \mid \varphi(x,x) \leq 0\}, \quad \alpha = \inf\{x \in \mathbb{P} \mid \varphi(x,x) \leq 1\}.$$

Since $(x, y) = ((y, y) \vee' (\perp, \infty)) \wedge' (x, x)$ and $\varphi(\perp, \infty) = 1$, we have that

$$\varphi(x, y) = (\varphi(y, y) \vee 1) \wedge \varphi(x, x)$$

for all $(x, y) \in \mathsf{Top}(\mathbf{2}, \mathbb{P}_\perp)^{\mathrm{op}}$. It can be easily verified that $\varphi = \varphi^{\alpha,\beta}$. □

Some spectra VIII

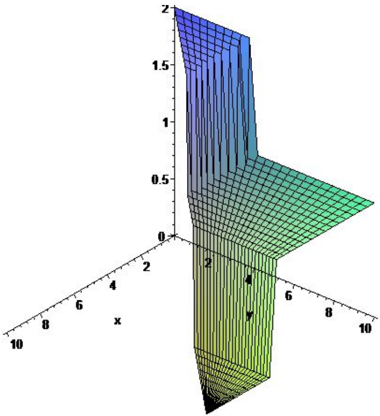


Figure: $\varphi^{4,6}$ with $(x, y) \in \mathrm{Top}(\mathbf{2}, \mathbb{P})^{\mathrm{op}} \cap [0, 10]^2$

Open questions

- For $\mathbb{F}_1, \mathbb{F}_2$ in a class of topological frames with additional properties, can we find a general description for $\mathrm{Spec}_{\mathbb{F}_2}(\mathrm{Top}(X, \mathbb{F}_1))$? Or just for $\mathbb{F}_1 = \mathbb{F}_2$?

Open questions

- For $\mathbb{F}_1, \mathbb{F}_2$ in a class of topological frames with additional properties, can we find a general description for $\mathsf{Spec}_{\mathbb{F}_2}(\mathsf{Top}(X, \mathbb{F}_1))$? Or just for $\mathbb{F}_1 = \mathbb{F}_2$?
- For what conditions on $\mathbb{F}_1, \mathbb{F}_2$ does

$$X \text{ Hausdorff} \Rightarrow X\ \mathbb{F}_1\text{-sober} \Rightarrow X\ \mathbb{F}_2\text{-sober} \Rightarrow X \text{ sober}$$

  hold?

Open questions

- For $\mathbb{F}_1, \mathbb{F}_2$ in a class of topological frames with additional properties, can we find a general description for $\mathrm{Spec}_{\mathbb{F}_2}(\mathrm{Top}(X, \mathbb{F}_1))$? Or just for $\mathbb{F}_1 = \mathbb{F}_2$?
- For what conditions on $\mathbb{F}_1, \mathbb{F}_2$ does

$$X \text{ Hausdorff} \Rightarrow X\ \mathbb{F}_1\text{-sober} \Rightarrow X\ \mathbb{F}_2\text{-sober} \Rightarrow X \text{ sober}$$

  hold?
- What can be said about the forgetful functors $\mathrm{Frm}_{\mathbb{F}} \to \mathrm{Frm}$ and $\mathrm{Frm}_{\mathbb{F}} \to \mathrm{Set}$?

References

- *Category theory at work*, ch. Algebra Union Topology, Heldermann Verlag, 1991.
- *Category theory at work*, ch. On the Existence and Structure of Free Topological Groups, Heldermann Verlag, 1991.
- J. Adamek, H. Herrlich, and G. Strecker, *Abstract and concrete categories: the joy of cats*, katmat.math.uni-bremen.de/acc/acc.pdf, 2005.
- Y. H. Hong, *Studies on categories of universal topological algebras*, Ph.D. thesis, McMaster University, 1974.
- R. Lowen, *Index analysis: Approach theory at work*, Springer Monographs in Mathematics, Springer London, 2015.
- J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in mathematics, Springer Basel AG, 2012.
- A. Pultr, *Frames*, Handbook of algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 791–857. MR 2035108 (2004j:06009)