Optimization with Differential Equations:

Where many large scale optimization problems come from

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CMUC

Estúdio, Abril 21, 2006

Outline

1. Examples with potatoes
2. The discrete problem
3. Application without potatoes
4. Optimal control – Summary

The Potato Example I

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How to boil a potato by heating its boundary in an optimal way.

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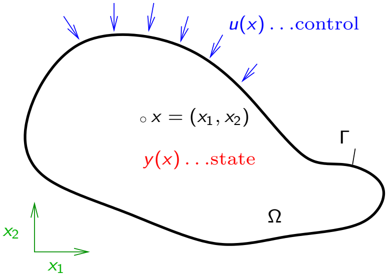
- $\Omega \dots$ potato
- Γ its boundary
- $u(x)$ heat source on Γ
- $y(x)$ temp. in Ω



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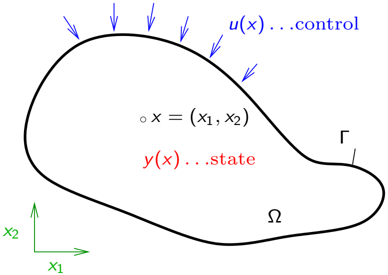
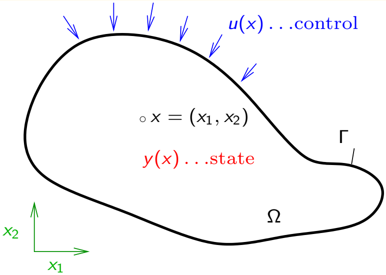


Problem:

Choose $u(x)$ ("control variable") on the boundary Γ such that the heat inside the potato $y(x)$ ("state variable") gets close to a desired function $y_d(x)$.

The Potato Example I (continued)



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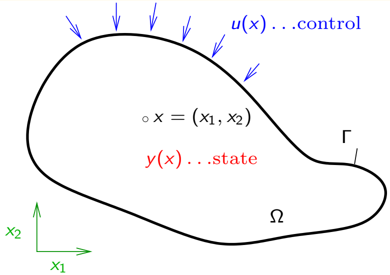

Control u and state y satisfy the stationary heat equation ($\alpha > 0$):

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\begin{aligned}
-\Delta y &= 0 && \text{in } \Omega, \\
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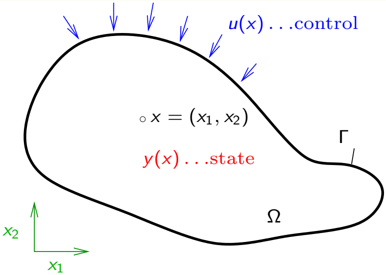

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$$\min_{y,u} J(y,u) = \frac{1}{2}\int_\Omega (y(x) - y_d(x))^2 dx + \frac{\gamma}{2}\int_\Gamma u(x)^2 ds(x)$$

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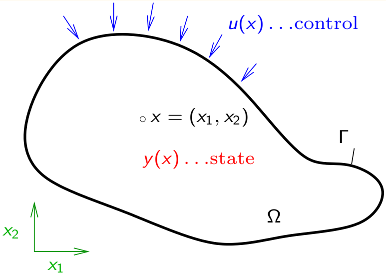

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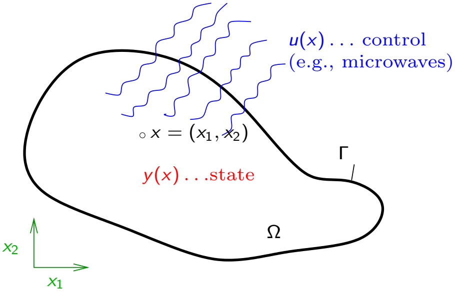

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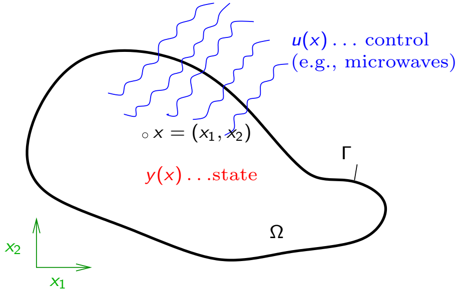

Distributed control u, that is, heating insinde Ω by microwaves or electromagnetic induction ($\beta(x)$ given):

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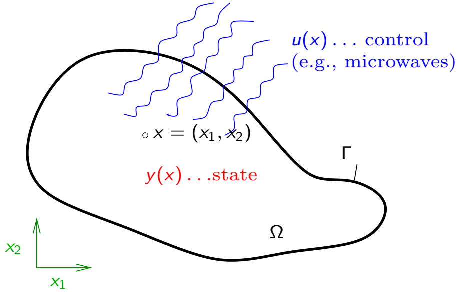

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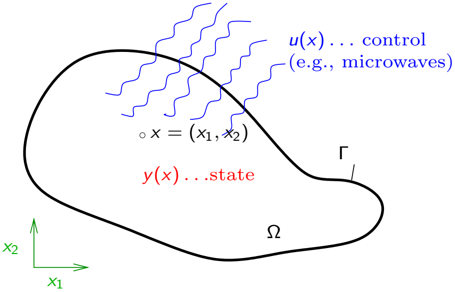

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$$\min_{Y,U} \frac{1}{2}\sum_{i=1}^{N}\{(y_i - y_{d,i})^2 + \gamma u_i^2\}$$
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- A_h is usually invertible, that is, $Y = A_h^{-1} B_h U$.

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- Multigrid methods, preconditioning. . .

Cancer treatment

Application of distributed heating in medicine: Objective: Heat a tumor (up to maybe 42-45 degree) without damaging the tissue around it (compare with Potato II)

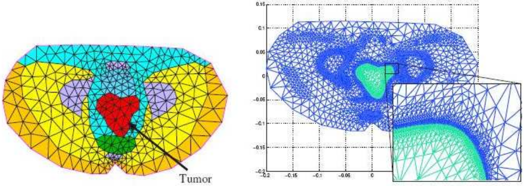
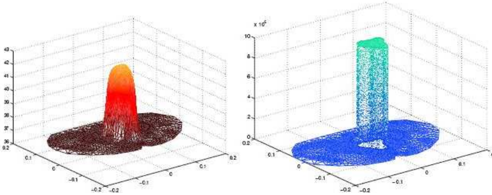
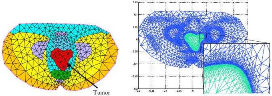


Fig. 2 Cross-section Ω of the pelvic region with different tissue types (left) and adaptively refined mesh (right).



Cancer Treatment (continued)



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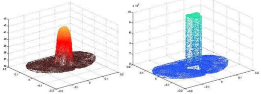

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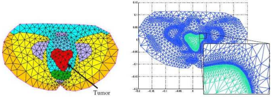

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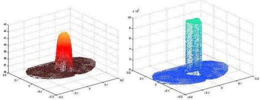

Mathematical formulation (body region Ω, tumor region $\Omega' \subset \Omega$)

$$\min_{y,u} J(y,u) = \frac{1}{2}\int_{\Omega'} (y(x) - 45)^2 \, dx + \frac{\gamma}{2}\int_{\Omega} u(x)^2 \, dx$$

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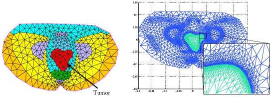

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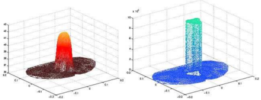

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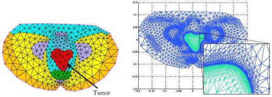

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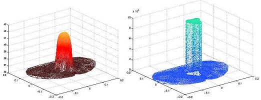

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and the biological constraints $y(x) \leq 40$ on $\Omega \setminus \Omega'$.

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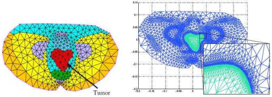

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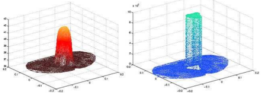

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$$\min_{y,u} J(y,u) = \frac{1}{2}\int_{\Omega'} (y(x) - 45)^2 \, dx + \frac{\gamma}{2}\int_{\Omega} u(x)^2 \, dx$$

with the heat equation

$$\begin{aligned} -\Delta y &= \beta u \quad && \text{in } \Omega, \\ y &= 25 && \text{on } \Gamma. \end{aligned}$$

and the biological constraints $y(x) \leq 40$ on $\Omega \setminus \Omega'$.

Cancer Treatment (continued)

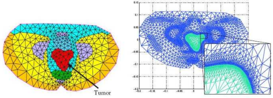
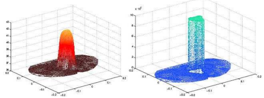


Fig. 2 Cross-section Ω of the pelvic region with different tissue types (left) and adaptively refined mesh (right).



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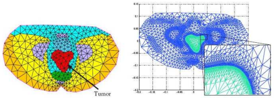

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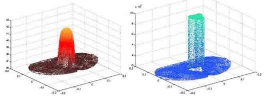

Problem structure:

- Objective functional
- Differential equation

Mathematical formulation (body region Ω, tumor region $\Omega' \subset \Omega$)

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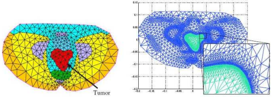

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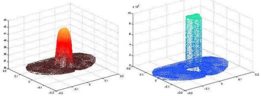

Problem structure:

- Objective functional
- Differential equation
- Inequality constraints

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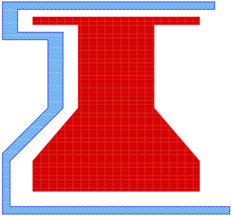
Optimal Cooling

Optimal cooling of a hot tool (compare with Potato I):



- Objective: Cool it down fast and uniformly

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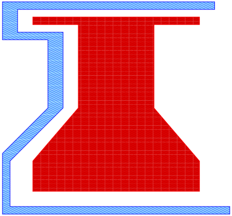

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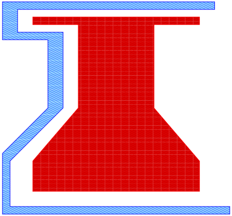

- Objective: Cool it down fast and uniformly
- Differential equation: Heat equation with boundary control
- Constraints: temperature and amount of water,. . .

Control of Fluids

The behavior of fluids can be described by the instationary Navier-Stokes equations:

$$
\begin{aligned}
y_t - \frac{1}{Re}\Delta y + (y \cdot \nabla)y + \nabla p &= u && \text{in } Q := \Omega \times [0, T] \\
\operatorname{div} y &= 0 && \text{in } Q \\
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- Objective functional

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Problem structure:

- Objective functional
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Problem structure:

- Objective functional
- Differential equation
- Inequality constraints

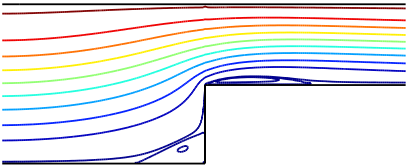
Control of Fluids (continued)

Example (by J.C. de los Reyes), stationary flow.

Control of Fluids (continued)

Example (by J.C. de los Reyes), stationary flow.

- Uncontrolled flow:



Control of Fluids (continued)

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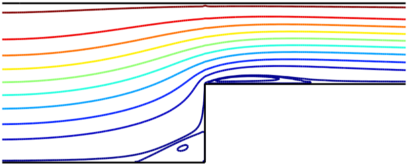
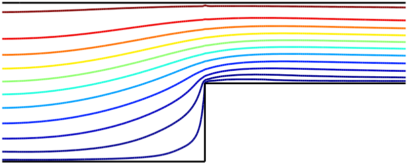
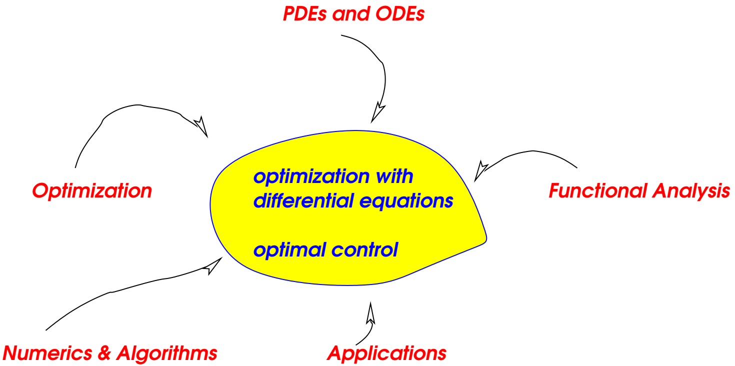


- Controlled flow:



Summary


PDEs and ODEs
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optimization with
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Numerics & Algorithms
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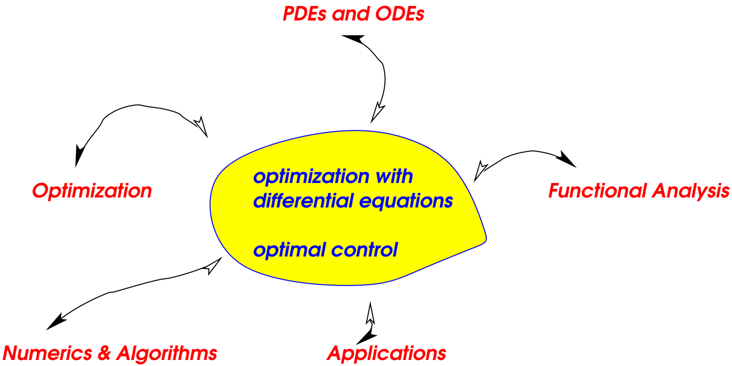
Summary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