7th Workshop on Numerical Ranges and Numerica Radii

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Abstracts

On polynomial numerical hulls of matrices

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(Joint work with Abbas Salemi)

Let $A \in M_n(\mathbb{C})$ be a normal matrix. Chandler Davis and A. Salemi (in 2002) characterized polynomial numerical hull of all $n \times n$ normal matrices whose spectrum contains 3 points. Also they determined polynomial numerical hull of order 2 of matrices $A = A_1 \oplus iA_2$, where A_1, A_2 are Hermitian. In this note we determine polynomial numerical hull of order three of matrices $A = A_1 \oplus iA_2$.

On the convexity of Sampflis numerical range

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(Joint work with Sadia Hassan)

This paper investigates a certain type of Numerical Range introduced by Stampfli. In particular we demonstrate the convexity of this set of elements of operators on Hilbert spaces and its relationship to the algebra Numerical Range implemented by elements of a W*-algebra.

To be announced

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When the numerical range goes flat

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The talk is devoted to matrices with flat portions on the boundary of their numerical range. A constructive criterion for such portions to exist is obtained in the case of tridiagonal matrices, and a particular case of continuant matrices is considered. As an application, the cases of (arbitrary) 3-by-3 and 4-by-4 matrices are treated. In particular, the sharp bound for the number of flat portions on the boundary of the numerical range for 4-by-4 matrices is established.

The q-numerical range of normal operators

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(Joint work with Hiroshi Nakazato)

Let T be a bounded linear operator on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For a real number $q \in [0,1]$, the q-numerical range of T is defined by

$$F_q(T) = \{\, \langle Ty, x\rangle \in \mathbf{C} : x, y \in H,\ \langle x, x\rangle = \langle y, y\rangle = 1,\ \langle y, x\rangle = q\}\,.$$

If T is normal then

$$\begin{aligned} \text{closure}(F_q(T)) \quad &= \quad \{q\,z + \sqrt{1-q^2}w\,\sqrt{h(z)-|z|^2} : z \in \ \text{closure}(W(T)), \\ &\qquad w \in \mathbf{C}, |w| \leq 1\}. \end{aligned}$$

We characterize fundamental properties of the function h which give the equation of the boundary of closure$(F_q(T))$.

The numerical ranges of powers of a matrix

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(Joint work with Chi-Kwong Li)

How are the numerical ranges of different integer powers of a single n x n complex matrix related? There arise some sort of intrinsic inequalities in matrix analysis..

Title to be announced

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$A \geq B \geq 0$ ensures $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$ for $p \geq 0$, $q \geq 1$, $r \geq 0$ with $(1+r)q \geq p+r$ and its applications (Survey talk)

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A capital letter means a bounded linear operator on a Hilbert space H. Löwner-Heinz inequality established in 1934 asserts that $A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$ and $A^\alpha \geq B^\alpha$ *does not hold for any* $\alpha > 1$ *even if* $A \geq B \geq 0$. Löwner-Heinz inequality is very useful, but the condition " $\alpha \in [0,1]$ " is too restrictive to calculate operator inequalities, so that the following result has been obtained from this point of view.

Theorem F (Furuta 1987)**.**

If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

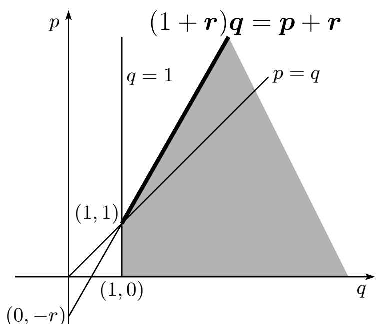

FIGURE 1

Consider two magic boxes: $f(\square) = (B^{\frac{r}{2}}\square B^{\frac{r}{2}})$ and $g(\square) = (A^{\frac{r}{2}}\square A^{\frac{r}{2}})$. Theorem F can be regarded as follows. Although $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general, but Theorem F asserts the following *" two order preserving operator inequalities"*

$f(A^p) \geq f(B^p)$ and $g(A^p) \geq g(B^p)$ hold whenever $A \geq B \geq 0$ under the condition p , q and r in FIGURE 1.

We would like to talk about some of applications of Theorem F.

The equivalence on Kantorovich type inequalities

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We consider Kantorovich type inequalities between $(Tx, x)^q$ and $B(T^p x, x)$ for bounded strictly positive operator on a Hilbert space. We recently extended it to the three cases (a) $p > 1$, $q > 1$, (b) $p < 0$, $q < 0$ and (c) $0 < p < 1$, $0 < q < 1$, respectively. We show the equivalence of Kantorovich type inequalities among these three cases.

Furuta inequality, operator mean and chaotic order

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Let A, B be positive operators on a Hilbert space. The operator mean $A \ \sharp_\alpha \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $0 \leq \alpha \leq 1$, we call this α-power mean, is introduced by Kubo-Ando. By using this operator mean, we define the relative operator mean and introduce the chaotic order $A \gg B$, ie. $\log A \geq \log B$. We show the Furuta inequality can be described by the form of the α-power mean and show the essential part of the inequality is obtainable under the assumption $A \gg B$.

An eigenvalue criterion for the convexity of the joint numerical range of several hermitian matrices

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(Joint work with Eugene Gutkin and Edmond Jonckheere)

The joint numerical range of a family $A = (A_1, \ldots, A_m)$ of hermitan matrices $A_k \in \mathbb{C}^{n\times n}$ is defined as

$$W(A) = \{\ (x^*A_1x, \ldots, x^*A_mx)\ \mid\ x \in \mathbb{C}^n,\ \ \|x\|_2 = 1\ \}.$$

It is well known that $W(A)$ is convex if $m = 2$ and if $m = 3$ and $n > 2$. We give the following criterion for the convexity of $W(A)$ if $m \geq 4$. Suppose the largest eigenvalue of $A(x) = \sum_{k=1}^{m} x_k A_k$, has constant multiplicity for all $0 \neq x \in \mathbb{R}^m$. Then either $W(A)$ is a smooth convex body or $W(A)$ is a smooth convex surface. Furthermore, if $W(A)$ is convex though the eigenvalue condition fails then the convexity can be destroyed by an arbitrarily small perturbation of A. The results are obtained using methods from differential topology.

Remark: The results are published in Gutkin, Eugene; Jonckheere, Edmond A.; Karow, Michael, Convexity of the joint numerical range: Topological and differential geometric viewpoints, *Linear Algebra Appl.*, 376, 143-171 (2004)

The best of OMC

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Numerical ranges, numerical radii, and multiplicative preservers

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Let $\mathcal{S}$ be a (semi)group of $n \times n$ complex matrices. We discuss some recent results and open problems on the characterizations of multiplicative maps $L : \mathcal{S} \to M_n$ such that

$$\mathcal{W}(L(A)) = \mathcal{W}(A) \qquad \text{for all } A \in \mathcal{S}$$

for the classical and various kinds of generalized numerical ranges and numerical radii $\mathcal{W}(A)$. The talk is based on some recent work with Wai-Shun Cheung, Antonia Duffner, Shaun Fallat, Robert Guralnick, and Leiba Rodman.

Matrices with elliptical c-numerical ranges

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(Joint work with N. Bebiano, J. da Providência and G. Soares)

Classes of tridiagonal matrices with elliptical c-numerical ranges are described, $c \in \mathbf{R}^n$. Some results of Chien, Brown and Spitkovsky for the classical numerical range are generalized.

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k-numerical range and the structure performance of a building

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(Joint work with Kozo Tsumura)

The boundary of the polar set $E^\wedge$ of the sum of k elliptical discs $E_1, \ldots, E_k$ with centers at the origin lies on an algebraic curve C of degree 2^k. We give an algorithm to compute the defining polynomial of C and apply it to k-numerical range and a problem in Architecture.

On numerical approximation of the numerical range of matrix polynomials

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(Joint work with M.-T. Chien, P. Lancaster, J. Maroulas and H. Nakazato.)

Consider a *matrix polynomial* $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$, where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$ and λ is a complex variable. A scalar $\lambda_0 \in \mathbb{C}$ is an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is the *spectrum* of $P(\lambda)$, namely, $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$.

The *numerical range* of $P(\lambda)$ is defined and denoted by

$$W(P) = \{\lambda \in \mathbb{C} : x^*P(\lambda)x = 0,\, x \in \mathbb{C}^n,\, x^*x = 1\}.$$

Evidently, $W(P)$ is always closed and contains the spectrum of $P(\lambda)$, and for $P(\lambda) = I\lambda - A$, $W(P)$ coincides with the numerical range of matrix A, that is, $F(A) = \{x^*Ax : x \in$

$\mathbb{C}^n$, $x^*x = 1\}$. The last decade, the numerical range $W(P)$ has attracted attention, and several results have been obtained. These results are helpful in investigating and understanding matrix polynomials, and lead to interesting applications of the numerical range on the spectral analysis, the factorization and the stability of matrix polynomials. A straightforward procedure for the estimation of $W(P)$ (based on the definition) would be to plot the roots of the polynomial $x^*P(\lambda)x$ for lots and randomly chosen unit vectors $x \in \mathbb{C}^n$. But that would be too costly, and it would probably not accurately depict the boundary of $W(P)$.

The numerical approximation of $W(P)$ is still an open and challenging problem. Here, we review three currently in use techniques, which are based on recent theoretical results. In particular, we describe:

- an algorithm for plotting accurately the boundary of the numerical range of $Q(\lambda) = I\lambda^2 + B\lambda + C$ when the coefficients B and C are hermitian,
- an inclusion-exclusion methodology for the estimation of $W(P)$ when $A_m = I$, i.e., for the monic case, and
- an algorithm for the approximation of an algebraic curve that contains the boundary of $W(P)$.

Finally, illustrative examples are given to demonstrate the feasibility of the methods.

On polynomial numerical hulls of normal matrices

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(Joint work with Chandler Davis)

The notion of polynomial numerical hull was introduced by O. Nevanlinna in 1993. In this note we determine the polynomial numerical hulls of matrices of the form $A = A_1 \oplus iA_2$, where A_1, A_2 are hermitian matrices. Also we study the relationship between rectangular hyperbolas and polynomial numerical hulls of order two for normal matrices. The polynomial numerical hulls of order two for some special matrices is studied.

Norm inequalities involving matrix monotone functions

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Let A, B, X be complex matrices with A, B Hermitian positive definite and let $f : (0,\infty) \to (0,\infty)$ be matrix monotone increasing. We prove

$$(2+t) \mid\mid\mid A^{\frac{1}{2}}(f(A)Xf^{\perp}(B) + f^{\perp}(A)Xf(B))B^{\frac{1}{2}} \mid\mid\mid \leq 2 \ \mid\mid\mid A^2X + tAXB + XB^2 \mid\mid\mid$$

and

$$(2+t) \mid\mid\mid f(A)X + Xf(B) \mid\mid\mid \leq 2 \ \frac{f(\lambda)}{\lambda} \mid\mid\mid A^{\frac{3}{2}}XB^{-\frac{1}{2}} + tA^{\frac{1}{2}}XB^{\frac{1}{2}} + A^{-\frac{1}{2}}XB^{\frac{3}{2}} \mid\mid\mid$$

where $f^{\perp}(x) = x(f(x))^{-1}$, $t \in [-2,2]$ and $\lambda = \min\{\sigma(A), \sigma(B)\}$; $\sigma(A), \sigma(B)$ being the spectrum of A, B respectively and $\mid\mid\mid . \mid\mid\mid$ any unitarily invariant norm. These inequalities generalize Zhan's inequalities.

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On the Geometry of Numerical Ranges in Krein Spaces

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(Joint work with N. Bebiano, R. Lemos and J. da Providência)

Geometric properties of the numerical ranges of operators in Krein spaces are investigated. In particular, classes of matrices are presented such that the boundary generating curves of the J-numerical range are hyperbolical. For A a J-normal matrix, such that $A + A^{[*]}$ has simple eigenvalues, the generalization to an indefinite inner product space of the Davis-Wiedlant shell of A is also investigated. The curvature of the J-numerical range at a boundary point is studied, generalizing results of Fiedler on the classical numerical range.

Inclusion regions for numerical ranges and linear preservers

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Let R be a proper subset of the complex plane, and let $\mathcal{S}_{\mathcal{R}}$ be the set of $n \times n$ complex matrices A such that the numerical range $W(A)$ satisfies $W(A) \subseteq R$. Linear maps ϕ on matrices satisfying $\phi(\mathcal{S}_{\mathcal{R}}) = \mathcal{S}_{\mathcal{R}}$ are characterized. Denote by $\tilde{\mathcal{S}}_R$ the set of $n \times n$ complex matrices A such that the numerical radius $r(A)$ satisfies $r(A) \subseteq R$ for a proper subset R of nonnegative real numbers. Linear maps ϕ on matrices satisfying $\phi(\tilde{\mathcal{S}}_R) = \tilde{\mathcal{S}}_R$ are also characterized. Analogous results on Hermitian matrices are obtained.
