A singular-degenerate parabolic problem: regularity up to the Dirichlet boundary

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Abstract

We show that weak solutions of a free boundary problem, modeling a water-ice phase transition in the case of nonlinear heat diffusion, are continuous up to the lateral boundary. We consider homogeneous Dirichlet boundary conditions and assume that the lateral boundary of the space-time domain satisfies the property of positive geometric density. The results are a follow up from recent results by the author concerning the interior regularity.

1. Introduction

The two phase Stefan problem is a mathematical model for a water-ice phase transition and is probably the most well studied free boundary problem in the literature. An extension of the classical case consists in considering a nonlinear law of Fourier, relating the heat flux $\mathbf{q}$ and the gradient of the renormalized temperature θ by

$$\mathbf{q} = -|\nabla\theta|^{p-2}\nabla\theta \ , \qquad 2 < p < \infty \ . \tag{1}$$

With such a constitutive relation, the equation of conservation of energy leads to the nonlinear degenerate parabolic (heat) equation

$$\partial_t \, \theta - \Delta_p \theta = 0 \tag{2}$$

that is to hold in the solid and liquid regions. The operator $\Delta_p\theta = \nabla \cdot (|\nabla\theta|^{p-2}\nabla\theta)$ is the p-Laplacian and the equation degenerates since its modulus of ellipticity $|\nabla\theta|^{p-2}$ vanishes

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at points where $|\nabla\theta| = 0$. Moreover, the incorporation of the free boundary conditions into the equation leads to a singularity in the time derivative since $\partial_t\,\theta$ has to be replaced in the weak formulation by $\partial_t\,\gamma(\theta)$, for an appropriate maximal monotone graph γ.

The question of the existence of a weak solution for this problem was successfully tackled in [9] by means of a regularization method and the use of an extended weak maximum principle. The uniqueness was recently obtained in [5] and the proof employs a technique based in showing that weak solutions are also entropy solutions, for which uniqueness follows from a straightforward adaptation of known results. Concerning the regularity, we have shown in [10] that the weak solution is locally continuous in the interior of the space-time domain, extending the well known results of [1] and [3] to the nonlinear case. Our aim here is to further extend the result showing that it is valid up to the lateral boundary. Results on the continuity of solutions at a given boundary point consist basically in constructing a sequence of nested and shrinking cylinders with vertex at that point, such that the essential oscillation of the function in those cylinders converges to zero when the cylinders shrink to zero. At the basis of the proof is an iteration technique, that is a refinement of the technique by DeGiorgi and Moser (cf. [2], [8] and [7]), based on energy (and logarithmic) estimates for the solution, that in the degenerate case are not homogeneous in the sense that they involve integral norms corresponding to different powers, namely the powers 2 and p. The key idea, due to DiBenedetto (cf. [4]) is to look at the equation in its own geometry, i.e., in a geometry dictated by its degenerate structure. This amounts to rescale the standard parabolic cylinders by a factor like

$$\left(\frac{\omega}{A}\right)^{p-2} \quad , \quad A = 2^s$$

where ω is an upper bound for the oscillation of the solution in the rescalled cylinder. This procedure, which can be called accommodation of the degeneracy, allows one to recover the homogeneity in the energy estimates written over these rescalled cylinders and carry on with the proof. We can say heuristically that the equation behaves in its own geometry like the heat equation.

Although we are in presence of a singular-degenerate equation, we are able to obtain an estimate of Hölder type near the boundary, unlike in the result on the interior regularity. The reason is twofold. On the one hand, the fact that we assume a property of positive geometric density on the boundary $\partial\Omega$ gives automatically an intermediate lemma in the regularity analysis that in the interior case was only obtained via the analysis of an alternative. On the other hand, we manage to use the energy inequalities (which are the key ingredient in the proof) for truncated functions $(\theta - k)_\pm$, choosing constants k that avoid the singularity at 0, i.e., for $k < 0$ in the negative case and $k > 0$ in the positive case, which results in the absence from the estimates of an integral power of the type

$$\iint (\theta - k)_\pm \zeta^{p-1} \zeta_t \ .$$

Both facts have the consequence of producing an independence on the oscillation for the various constants involved in the proof and allow the establishment of an explicit modulus of continuity.

2. Statement of the problem and main result

To fix ideas, assume that an incompressible material (say pure water) occupies a bounded domain $\Omega \subset \mathbb{R}^N$, with two phases, a solid phase corresponding to the region $\{\theta < 0\}$ and a liquid phase corresponding to the region $\{\theta > 0\}$, separated by an interface $\Phi = \{\theta = 0\}$, the free boundary. We denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, for some $T > 0$. The problem in its strong formulation reads

$$
(\mathbf{P}) \begin{cases} \partial_t \theta = \Delta_p \theta & \text{in} \quad Q \setminus \Phi = \{\theta < 0\} \cup \{\theta > 0\} \\ [|\nabla\theta|^{p-2}\nabla\theta]_-^+ \cdot \mathbf{n} = \lambda\, \mathbf{w} \cdot \mathbf{n} & \text{on} \quad \Phi = \{\theta = 0\} \\ \theta = 0 & \text{on} \quad \Sigma \\ \theta(0) = \theta_0 & \text{in} \quad \Omega \times \{0\} \end{cases}
$$

where $\mathbf{n}$ the unit normal to Φ, pointing to the solid region, $\mathbf{w}$ the velocity of the free boundary and $\lambda = [e]_-^+ > 0$ the latent heat of phase transition (e is the internal energy), with $[\,.\,]_-^+$ denoting the jump across Φ.

Following the original ideas of [6] we derive a weak formulation, in which all explicit references to the free boundary are absent, considering the maximal monotone graph H associated with the Heaviside function,

$$
H(s) = \begin{cases} 0 & \text{if} \quad s < 0 \\ [0,1] & \text{if} \quad s = 0 \\ 1 & \text{if} \quad s > 0 \end{cases} ,
$$

and introducing a new unknown function, the enthalpy η, such that

$$
\eta \in \gamma(\theta) := \theta + \lambda H(\theta) \ .
$$

A formal integration by parts against appropriate test functions and the replacement of the initial condition for θ by a more adequate initial condition for η, leads to an integral relation that we adopt as definition of weak solution.

Definition 1 *We say that (η, θ) is a* weak solution *of problem* $(\mathbf{P})$*, if*

$$
\theta \in L^p\Big(0,T; W_0^{1,p}(\Omega)\Big) \cap L^\infty(Q) \quad ;
$$

$$
\eta \in L^\infty(Q) \quad \text{and} \quad \eta \in \gamma(\theta) \ , \ \text{a.e. in} \ Q \ ;
$$

$$
-\int_Q \eta \, \partial_t \xi + \int_Q |\nabla\theta|^{p-2}\nabla\theta \cdot \nabla\xi = \int_\Omega \eta_0 \, \xi(0) \ , \quad \forall \xi \in \mathcal{T}(Q) \ .
$$

The space of test functions we are considering is

$$\mathcal{T}(Q) := \left\{ \xi \in L^p\Big(0,T; W_0^{1,p}(\Omega)\Big) \ : \ \partial_t \xi \in L^2(Q) \ , \ \xi(T) = 0 \right\} .$$

We are in presence of a weak form of a singular-degenerate parabolic equation, with principal part in divergence form,

$$\partial_t[\gamma(\theta)] - \mathrm{div}(|\nabla\theta|^{p-2}\nabla\theta) \ \ni \ 0 \ , \qquad p > 2,$$

where γ is a maximal monotone graph with a singularity at the origin. The equation is thus degenerate in the space part and singular in the time part since, roughly speaking, $\gamma'(0) = \infty$.

Although the result we have in mind could be obtained working directly with an equivalent formulation for the equation in the definition that makes use of Stekolov averages (assuming in addition that the solution is in $C(0,T; L^2(\Omega))$) we adopt here a different approach. Namely, we will consider approximate problems and show that the equibounded sequence of approximate solutions is also equicontinuous by deriving for them a boundary modulus of continuity that is independent of the approximation. To be precise, let $0 < \epsilon \ll 1$ and consider the function

$$\gamma_\epsilon(s) = s + \lambda H_\epsilon(s) \quad ,$$

where H_ϵ is a $\mathcal{C}^\infty$-approximation of the Heaviside function, such that

$$H_\epsilon(s) = 0 \quad \text{if} \quad s \le 0 \ \ , \qquad\quad H_\epsilon(s) = 1 \quad \text{if} \quad s \ge \epsilon \ ,$$

$H'_\epsilon \ge 0$ and $H_\epsilon \to h$ uniformly in the compact subsets of $\mathbb{R} \setminus \{0\}$, as $\epsilon \to 0$. The function γ_ϵ is bilipschitz and satisfies

$$1 \le \gamma'_\epsilon(s) \le 1 + \lambda L_\epsilon \ , \quad s \in \mathbb{R} \ , \tag{3}$$

with $L_\epsilon \equiv \mathcal{O}(\frac{1}{\epsilon})$ being the Lipschitz constant of H_ϵ. Taking also a sequence of functions $\theta_{0\epsilon} \in W^{1,p}(\Omega)$ such that

$$\theta_{0\epsilon} \to \theta_0 \quad , \quad \gamma_\epsilon(\theta_{0\epsilon}) \to \eta_0 \quad \text{in} \quad L^p(\Omega) \quad \text{and} \quad |\theta_{0\epsilon}| \le M \ , \quad \text{a.e. in } \Omega$$

we define the approximated problem as follows

(P_ϵ): For each $0 < \epsilon \ll 1$, find a function

$$\theta_\epsilon \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$$

such that

$$-\int_Q \gamma_\epsilon(\theta_\epsilon)\, \partial_t \xi + \int_Q |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon \cdot \nabla\xi = \int_\Omega \gamma_\epsilon(\theta_{0\epsilon})\, \xi(0) \ , \quad \forall \xi \in \mathcal{T}(Q) \ . \tag{4}$$

In the presence of the regularity required, equation (4) can be shown to be equivalent to the two conditions: $\theta_\epsilon(0) = \theta_{0\epsilon}$ and, for a.e. $t \in (0,T)$,

$$\int_{\Omega\times\{t\}} \partial_t[\gamma_\epsilon(\theta_\epsilon)]\,\varphi + \int_{\Omega\times\{t\}} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla\varphi = 0\ ,\quad \forall\varphi\in W_0^{1,p}(\Omega)\ . \tag{5}$$

We show in [9] that this approximated problem has a unique solution and derive enough *a priori* estimates to pass to the limit and obtain a solution of the original problem with the regularity required in Definition 1, that in addition satisfies an extended weak maximum principle:

$$\|\theta\|_{L^\infty(Q)} \leq M\ .$$

We will show here that there is a uniform, i.e. independent of ϵ, boundary modulus of continuity for θ_ϵ and this will allow us to obtain a continuous temperature at the boundary as a solution to the original problem as a consequence of Ascoli's theorem. The main assumption concerning the boundary of Ω is that it satisfies a property of *positive geometric density*, i.e.,

$$(\mathsf{PGD})\quad \left\{\begin{array}{l} \exists\,\nu^*\in(0,1),\ \exists\,\rho_0>0\ :\ \forall\,x_0\in\partial\Omega,\ \forall\,\rho\leq\rho_0, \\[2ex] \qquad\Big|\Omega\cap K_\rho(x_0)\Big| \leq (1-\nu^*)\,|K_\rho(x_0)| \end{array}\right. \quad ; \tag{6}$$

given a point $x_0\in\mathbb{R}^N$, $K_\rho(x_0)$ denotes the N-dimensional cube with centre at x_0 and wedge 2ρ:

$$K_\rho(x_0) := \Big\{x\in\mathbb{R}^N : \max_{1\leq i\leq N}|x_i - x_{0i}| < \rho\Big\}\ .$$

Under this assumption, we will prove that

Theorem 1 *The sequence $(\theta_\epsilon)_\epsilon$ is equicontinuous at the lateral boundary, i.e., there is an independent of ϵ boundary modulus of continuity for θ_ϵ. As a consequence, the solution of problem* **(P)** *is continuous up to the boundary. Moreover, there exist $\alpha\in(0,1)$ and $C>0$, depending only on the data and independent of ϵ, such that*

$$|\theta_\epsilon(x,t)| \leq C\left(\mathrm{dist}\Big\{(x,t),\partial\Omega\Big\}\right)^\alpha\ ,$$

for $t > t_0 > 0$.

3. The energy estimates near the boundary

In this section we derive certain uniform energy estimates near the boundary that will be the main tool in the proof of the continuity up to the boundary. Given a point $(x_0,t_0)\in\mathbb{R}^{N+1}$, the cylinder of radius ρ and height $\tau>0$ is

$$(x_0,t_0) + Q(\tau,\rho) := K_\rho(x_0)\times(t_0-\tau,t_0)\ .$$

Fix a point $(x_0, t_0) \in \Sigma$ and consider the cylinder $(x_0, t_0) + Q(\tau, \rho) \subset Q$, where τ is small enough so that $t_0 - \tau > 0$. Let $0 \le \zeta \le 1$ be a piecewise smooth cutoff function in $(x_0, t_0) + Q(\tau, \rho)$ such that

$$|\nabla \zeta| < \infty \qquad \text{and} \qquad \zeta(x,t) = 0\ , \quad x \notin K_\rho(x_0)\ . \tag{7}$$

The estimates will follow after taking, in the weak formulation (5), the test functions

$$\varphi^{\pm} = \pm(\theta_\epsilon - k)_{\pm}\zeta^p\ ,$$

which are an admissible choice provided

$$(\theta_\epsilon - k)_{\pm}\zeta^p \ \in\ W_0^{1,p}\big(\Omega \cap K_\rho(x_0)\big)\ . \tag{8}$$

But $\zeta(\cdot, t)$ does not vanish on the boundary of $\Omega \cap K_\rho(x_0)$, so in order to satisfy (8) we must restrict the choice of the levels k to

$$(-k)_{\pm} = 0\ .$$

For these choices of k we obtain the following energy inequalities near Σ, that, for the sake of simplicity and without loss of generality, will be stated for cylinders that are centered at the origin $(0,0)$, the changes being obvious in the case the centre is a point $(x_0, t_0) \in \Sigma$.

Proposition 1 *Let θ_ϵ be a solution of (P_ϵ) and $k > \epsilon$. There exists a constant $C > 0$, that is independent of ϵ, such that for every cylinder $Q(\tau, \rho) \subset Q$, such that $-\tau > 0$,*

$$\begin{aligned}
\sup_{-\tau<t<0} \int_{[K_\rho \cap \Omega]\times\{t\}} (\theta_\epsilon - k)_+^2 \zeta^p + \int_{-\tau}^0 \int_{K_\rho \cap \Omega} |\nabla(\theta_\epsilon - k)_+ \zeta|^p \\
\le C \int_{-\tau}^0 \int_{K_\rho \cap \Omega} (\theta_\epsilon - k)_+^p |\nabla \zeta|^p + C \int_{[K_\rho \cap \Omega]\times\{-\tau\}} (\theta_\epsilon - k)_+^2 \zeta^p \\
+ C \int_{-\tau}^0 \int_{K_\rho \cap \Omega} (\theta_\epsilon - k)_+^2 \zeta^{p-1} \partial_t \zeta\ .
\end{aligned} \tag{9}$$

Proof. Let $\varphi = (\theta_\epsilon - k)_+ \zeta^p$ in (5) and integrate in time over $(-\tau, t)$ for $t \in (-\tau, 0)$. Due to the choice of k, we are above the singularity so $\gamma'_\epsilon \equiv 1$ and the first term gives

$$\begin{aligned}
\int_{-\tau}^t \int_{K_\rho} \partial_t[\gamma_\epsilon(\theta_\epsilon)]\, \big((\theta_\epsilon - k)_+ \zeta^p\big) = \frac{1}{2} \int_{K_\rho} \int_{-\tau}^t \partial_t (\theta_\epsilon - k)_+^2 \zeta^p \\
= \frac{1}{2} \int_{[K_\rho \cap \Omega]\times\{t\}} (\theta_\epsilon - k)_+^2 \zeta^p - \frac{1}{2} \int_{[K_\rho \cap \Omega]\times\{-\tau\}} (\theta_\epsilon - k)_+^2 \zeta^p \\
- \frac{p}{2} \int_{-\tau}^t \int_{K_\rho \cap \Omega} (\theta_\epsilon - k)_+^2 \zeta^{p-1} \partial_t \zeta\ .
\end{aligned} \tag{10}$$

Concerning the other term, we have

$$\int_{-\tau}^{t}\int_{K_\rho\cap\Omega}|\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla[(\theta_\epsilon-k)_+\zeta^p]=\int_{-\tau}^{t}\int_{K_\rho\cap\Omega}\left|\nabla(\theta_\epsilon-k)_+\,\zeta\right|^p$$

$$+p\int_{-\tau}^{t}\int_{K_\rho\cap\Omega}|\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla\zeta\left[\zeta^{p-1}(\theta_\epsilon-k)_+\right]$$

$$\geq\frac{1}{2}\int_{-\tau}^{t}\int_{K_\rho\cap\Omega}\left|\nabla(\theta_\epsilon-k)_+\,\zeta\right|^p-C(p)\int_{-\tau}^{t}\int_{K_\rho\cap\Omega}(\theta_\epsilon-k)_+^p\,|\nabla\zeta|^p\ , \tag{11}$$

using the inequality of Young

$$ab\leq\frac{\varepsilon^p}{p}a^p+\frac{1}{p'\varepsilon^{p'}}b^{p'}\ , \tag{12}$$

valid for $a,b\geq 0$ and $\varepsilon>0$, with the choices

$$a=(\theta_\epsilon-k)_+|\nabla\zeta|\ ,\qquad b=\left|\nabla(\theta_\epsilon-k)_+\,\zeta\right|^{p-1}$$

and ε appropriate. Since $t\in(-\tau,0)$ is arbitrary, we can combine estimates (10) and (11) to obtain (9).

■

Remark 1 An entirely analogous estimate holds for $(\theta_\epsilon-k)_-$, with $k<-\epsilon$.

4. The intrinsic rescaling

The proof of the equicontinuity will follow from these estimates, by adapting the technique introduced by DiDenedetto (cf. [4]). It consists essentially in showing that for every point $(x_0,t_0)\in\Sigma$ we can find a sequence of nested and shrinking cylinders $(x_0,t_0)+Q(\tau_n,\rho_n)$, such that the essential oscillation of each function θ_ϵ in the intersection of these cylinders with Q goes to zero as $n\to\infty$ in a way that is quantitatively independent of ϵ. This can be achieved, roughly speaking, by considering the equation in a geometry dictated by its own structure. This means that, instead of the usual cylinders, we have to work in cylinders whose dimensions take the degeneracy of the equation into account. Let's make this idea precise. From now on we will drop the ϵ in θ_ϵ.

Consider a point $(x_0,t_0)\in\Sigma$ and a cylinder $(x_0,t_0)+Q(R^{p-1},2R)$, where $R>0$ is so small that $t_0-R^{p-1}>0$. To simplify, change variables and assume $(x_0,t_0)=(0,0)$. Set

$$\mu_-:=\operatorname*{ess\,inf}_{Q(R^{p-1},2R)\cap Q}\theta\ ;\qquad\mu_+:=\operatorname*{ess\,sup}_{Q(R^{p-1},2R)\cap Q}\theta\ ;\qquad\omega:=\operatorname*{ess\,osc}_{Q(R^{p-1},2R)\cap Q}\theta=\mu_+-\mu_-$$

and construct the cylinder

$$Q(dR^p,R)\ ,\qquad\text{with}\quad d=\left(\frac{\omega}{2^{s^*}}\right)^{2-p}\ ,$$

where s^* is to be chosen later. Note that for $p = 2$, i.e. in the non degenerate case, $d = 1$ and these are the standard parabolic cylinders. We will assume, without loss of generality, that $\omega < 1$ and also that

$$\frac{1}{d} = \left(\frac{\omega}{2^{s^*}}\right)^{p-2} > R \ . \tag{13}$$

If this doesn't hold, we have

$$\left(\frac{\omega}{2^{s^*}}\right)^{p-2} \leq R \qquad \text{so} \qquad \omega \leq 2^{s^*} R^{\frac{1}{p-2}}$$

and there is nothing to prove because the oscillation goes to zero with the radius. Now, (13) implies that $Q(dR^p, R) \subset Q(R^{p-1}, 2R)$ and the relation

$$\operatorname*{ess\ osc}_{Q(dR^p,R)} \ \theta \leq \omega \tag{14}$$

which will be the starting point of an iteration process that leads to our main results. Note that we had to consider the cylinder $Q(R^{p-1}, 2R)$ and assume (13), so that (14) would hold for the rescalled cylinder $Q(dR^p, R)$. This is in general not true for a given cylinder since its dimensions would have to be intrinsically defined in terms of the essential oscillation of the function within it. The proof of our main result follows from showing that starting from $Q(dR^p, R)$ and going down to a smaller cylinder the oscillation decreases by a small factor that we can exhibit.

We conclude this section recalling a well known lemma due to DeGiorgi that will be essential in the sequel. Given a continuous function $v : \Omega \to \mathbb{R}$ and a pair of real numbers $k < l$, we define

$$\begin{aligned} [v > l] \ &\equiv \ \{x \in \Omega \ : \ v(x) > l\} \ , \\ [v < k] \ &\equiv \ \{x \in \Omega \ : \ v(x) < k\} \ , \\ [k < v < l] \ &\equiv \ \{x \in \Omega \ : \ k < v(x) < l\} \ . \end{aligned} \tag{15}$$

Lemma 1 (DeGiorgi, [2]) *Let $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, with $\rho > 0$ and $x_0 \in \mathbb{R}^N$ and k and l real numbers such that $k < l$. There exists a constant C, depending only on N and p and independent of ρ, x_0, v, k and l, such that*

$$(l - k) \left|[v > l]\right| \leq C \, \frac{\rho^{N+1}}{|[v < k]|} \int_{[k<v<l]} |\nabla v| \ .$$

The conclusion of the lemma remains valid for functions $v \in W^{1,1}(\Omega) \cap C(\Omega)$, provided Ω is convex. We will use it in the case of a cube. We also remark that the continuity is not essential. For a function merely in $W^{1,1}(\Omega)$ we define (15) through any representative in the equivalence class. It can be shown that the conclusion is independent of that choice.

5. The continuity up to the boundary

From now on, we'll assume, without loss of generality that

$$\mu_+ - \frac{\omega}{4} > 0 \ . \tag{16}$$

If both inequalities
$$\mu_+ - \frac{\omega}{4} \le 0 \qquad \text{and} \qquad \mu_- + \frac{\omega}{4} \ge 0 \tag{17}$$
were true, then by subtraction we would get
$$\frac{\omega}{2} = \mu_+ - \frac{\omega}{4} - \mu_- - \frac{\omega}{4} \le 0$$
and there would be nothing to prove. If it is the second of (17) that is violated, the reasoning is similar. We start the proof of the boundary regularity with the following proposition, which has a double scope. It will determine the height of our rescaled cylinder $Q(dR^p, R)$, by fixing the number s^*, and at the same time provide a level such that the measure of the set where θ is above such a level in a comparable cylinder can be controlled.

Proposition 2 *Given $\nu_0 \in (0,1)$, there exists $s^* > 2$, depending only on the data, such that*
$$\left| (x,t) \in Q\Big(\frac{d}{2}R^p, R\Big) \ : \ \theta(x,t) > \mu_+ - \frac{\omega}{2^{s^*}} \right| \le \nu_0 \left| Q\Big(\frac{d}{2}R^p, R\Big) \right| .$$

Proof. We write the energy inequality (9) for the function $(\theta - k)_+$ in the cylinder $Q(dR^p, 2R)$, with
$$k = \mu_+ - \frac{\omega}{2^s} ,$$
where $2 \le s \le s^*$ and s^* is to be chosen later. Observe that, due to (16), we have
$$k = \mu_+ - \frac{\omega}{2^s} \ge \mu_+ - \frac{\omega}{4} > 0$$
so $k > \epsilon$, for ϵ sufficiently small. We take a piecewise smooth cutoff function $0 < \zeta \le 1$, defined in the cylinder and satisfying the assumptions
$$\begin{array}{ll} \zeta = 1 \ \text{ in } \ Q(\frac{d}{2}R^p, R) & \qquad \zeta = 0 \ \text{ on } \ \partial_p Q(dR^p, 2R) \\[2ex] |\nabla \zeta| \le \frac{1}{R} & \qquad 0 \le \partial_t \zeta \le \frac{2}{dR^p} \end{array} .$$
Observe that, since $(\theta - k)_+$ vanishes on $Q(dR^p, 2R) \cap \Sigma$, because the same happens with θ and $k > 0$, we can extend it by zero to the whole cylinder $Q(dR^p, 2R)$. With this in mind, the inequalities read, for the indicated choices,
$$\int_{-\frac{d}{2}R^p}^{0} \int_{K_R} |\nabla (\theta - k)_+|^p$$
$$\le \frac{C}{R^p} \int_{-dR^p}^{0} \int_{K_{2R}} (\theta - k)_+^p + \frac{C}{dR^p} \int_{-dR^p}^{0} \int_{K_{2R}} (\theta - k)_+^2 . \tag{18}$$
neglecting the first term on the left hand side of (9) and using the assumptions on ζ. We estimate the two terms on the right hand side with
$$\frac{C}{R^p} \int_{-dR^p}^{0} \int_{K_{2R}} (\theta - k)_+^p \le \frac{C}{R^p} \Big(\frac{\omega}{2^s}\Big)^p \Big| Q\Big(\frac{d}{2}R^p, R\Big) \Big|$$

and, due to the definition of d and the fact that $s \le s^*$,

$$\frac{C}{dR^p}\int_{-dR^p}^{0}\int_{K_{2R}}(\theta-k)_+^2 \le \frac{C}{R^p}\Big(\frac{\omega}{2^{s^*}}\Big)^{p-2}\Big(\frac{\omega}{2^s}\Big)^2\Big|Q(dR^p,2R)\Big|$$

$$\le \frac{C}{R^p}\Big(\frac{\omega}{2^s}\Big)^p\,\Big|Q\Big(\frac{d}{2}R^p,R\Big)\Big|\ ,$$

and obtain from (18)

$$\int\int_{A_s}|\nabla\theta|^p \le \frac{C}{R^p}\Big(\frac{\omega}{2^s}\Big)^p\,\Big|Q\Big(\frac{d}{2}R^p,R\Big)\Big|\ , \tag{19}$$

where

$$A_s=\Big\{(x,t)\in Q\Big(\frac{d}{2}R^p,R\Big)\ :\ \theta(x,t)>\mu_+-\frac{\omega}{2^s}\Big\}\ .$$

We next apply lemma 1 to the function $\theta(\cdot,t)$, for $-\frac{d}{2}R^p\le t\le 0$, and with

$$k=\mu_+-\frac{\omega}{2^s}\ ,\qquad l=\mu_+-\frac{\omega}{2^{s+1}}\ ,\qquad l-k=\frac{\omega}{2^{s+1}}\ .$$

Putting

$$A_s(t)=\Big\{x\in K_R\ :\ \theta(x,t)>\mu_+-\frac{\omega}{2^s}\Big\}$$

we automatically have

$$\Big|A_s(t)\Big|\le(1-\nu^*)\,\Big|K_R\Big|\ ,\ \ \forall t\in Q\Big(\frac{d}{2}R^p,R\Big)$$

since $\Big(\theta-\mu_++\frac{\omega}{2^s}\Big)_+$ vanishes outside of $K_R\cap\Omega$ and $\partial\Omega$ satisfies the property of positive geometric density (6). As a consequence

$$\Big|\Big\{x\in K_R\ :\ \theta(x,t)\le\mu_+-\frac{\omega}{2^s}\Big\}\Big|\equiv|K_R|-\Big|A_s(t)\Big|\ge\nu^*\,|K_R|\ ,$$

and we get from the lemma

$$\frac{\omega}{2^{s+1}}|A_{s+1}(t)|\le\frac{C\,R^{N+1}}{\nu^*\,|K_R|}\int_{A_s(t)\setminus A_{s+1}(t)}|\nabla\theta|\ ,$$

for $t\in(-\frac{d}{2}R^p,0)$. Integrating over this interval, we conclude that

$$\frac{\omega}{2^{s+1}}|A_{s+1}|\le\frac{C}{\nu^*}R\int\int_{A_s\setminus A_{s+1}}|\nabla\theta|\le\frac{C}{\nu^*}R\Big(\int\int_{A_s}|\nabla\theta|^p\Big)^{\frac{1}{p}}\,|A_s\setminus A_{s+1}|^{\frac{p-1}{p}}$$

$$\le\frac{C}{\nu^*}\Big(\frac{\omega}{2^s}\Big)\Big|Q\Big(\frac{d}{2}R^p,R\Big)\Big|^{\frac{1}{p}}\,|A_s\setminus A_{s+1}|^{\frac{p-1}{p}}\ ,$$

using also (19). Raising both members to the power $\frac{p}{p-1}$ and dividing by $(\frac{\omega}{2^{s+1}})^{\frac{p}{p-1}}$, we get

$$|A_{s+1}|^{\frac{p}{p-1}} \leq C\,(\nu^*)^{-\frac{p}{p-1}} \left|Q\Big(\frac{d}{2}R^p, R\Big)\right|^{\frac{1}{p-1}} |A_s \setminus A_{s+1}| \ .$$

Being this inequalities valid for $2 \leq s \leq s^*$, we add them for

$$s = 2, 3, \ldots, s^* - 1 \ ,$$

and since the sum on the right hand side can be bounded by $\left|Q\Big(\frac{d}{2}R^p, R\Big)\right|$, we get

$$(s^* - 2)\,|A_{s^*}|^{\frac{p}{p-1}} \leq C\,(\nu^*)^{-\frac{p}{p-1}} \left|Q\Big(\frac{d}{2}R^p, R\Big)\right|^{\frac{p}{p-1}} \ ,$$

i.e.,

$$|A_{s^*}| \leq \frac{C}{\nu^*(s^* - 2)^{\frac{p-1}{p}}} \left|Q\Big(\frac{d}{2}R^p, R\Big)\right| \ .$$

The conclusion of the proposition is reached by choosing s^* sufficiently large such that

$$\frac{C}{\nu^*(s^* - 2)^{\frac{p-1}{p}}} \leq \nu_0 \ .$$

$\blacksquare$

From this proposition we can show that in fact θ is strictly below its supremum μ^+ in a smaller cylinder coaxial with $Q\Big(\frac{d}{2}R^p, R\Big)$. The proof of this fact is similar to lemma 9.1 in [4, Ch.3] and we omit it here. The precise statement is

Proposition 3 *The number ν_0 (and consequently $s^* > 2$) can be chosen, depending only on the data, such that*

$$\theta(x,t) \leq \mu_+ - \frac{\omega}{2^{s^*+1}} \qquad a.e. \quad (x,t) \in Q\Big(\frac{d}{2}\Big(\frac{R}{2}\Big)^p, \frac{R}{2}\Big) \ . \tag{20}$$

Now, as a simple consequence, we have the following

Corollary 1 *There exists a constant $\sigma \in (0,1)$, depending only on the data, such that*

$$\underset{Q\left(\frac{d}{2}(\frac{R}{2})^p, \frac{R}{2}\right)}{ess\ osc}\ \theta \ \leq \ \sigma\,\omega \ .$$

Proof. From Proposition 3 we obtain $s^* > 2$ such that

$$\underset{Q\left(\frac{d}{2}(\frac{R}{2})^p, \frac{R}{2}\right)}{\text{ess sup}}\ \theta \leq \mu^+ - \frac{\omega}{2^{s^*+1}}$$

and consequently

$$\operatorname*{ess\,osc}_{Q\left(\frac{d}{2}(\frac{R}{2})^p,\frac{R}{2}\right)} \theta = \operatorname*{ess\,sup}_{Q\left(\frac{d}{2}(\frac{R}{2})^p,\frac{R}{2}\right)} \theta - \operatorname*{ess\,inf}_{Q\left(\frac{d}{2}(\frac{R}{2})^p,\frac{R}{2}\right)} \theta \le \mu^+ - \mu^- - \frac{\omega}{2^{s^*+1}} = \Big(1 - \frac{1}{2^{s^*+1}}\Big)\omega \ .$$

The corollary follows with $\sigma = (1 - \frac{1}{2^{s^*+1}})$.

■

Using a reasoning similar to the one applied in [3], [4] and [10], we can now construct decreasing sequences $(\omega_n)_n$ and $(R_n)_n$, converging to zero, such that, for all $n = 0, 1, 2, \ldots$,

$$Q_{n+1} \subset Q_n \qquad \text{and} \qquad \operatorname*{ess\,osc}_{Q_n} \theta \le \omega_n \ , \tag{21}$$

where $Q_n = Q(d_n R_n^p, R_n)$. From here the proof of Theorem 1 easily follows (see [3], [4] and [10]) as a consequence of lemma 5.8 of [7, pp. 96-97].

Remark 2 The results can be further extended. In fact, we can obtain a continuous solution at $t = 0$ and at the boundary Σ also for Neumann data.

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