A nonstandard linear finite element method for a planar elasticity problem

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Abstract

The aim of this work is to present a nonstandard linear finite element method for a planar elasticity problem. The error for the solution computed with this method is estimated with respect to H^1-norm and second convergence order is shown.

1 Introduction

In most physical applications, quantities of interested are governed by systems of partial differential equations, not just a single equation. For example in solid mechanics the deformations and stresses of elastic and inelastic bodies subject to load are governed by systems of partial differential equations.

On the computation of a numerical approximation for the solution of a system of partial differential equations the numerical methods usually used are the finite element methods or the finite difference methods being the properties of the numerical approximation studied using the knowledge of the properties of these methods for scalar case.

In this paper, our goal is to study a new linear finite element method for a planar elasticity problem presented for the scalar case by one of the author in [6]. This method enable us to compute the numerical approximation to the displacement with a improved accuracy when compared with standard linear finite element methods described in the literature as for instance in [1], [3],[11]. This new method has two main features: it is based on a family of triangulations of the domain which not needs to be quasi-uniform and regular and the finite element solution computed presents second convergence order with respect to H^1 norm. This last property of the linear finite element method means that the gradient of each component of the displacement is superconvergent.

Already about to decades ago Zlámal ([21]) has found the superconvergence of the gradient for certain quadrature finite element solutions on nearly rectangular grids. Also Brandts ([2]) has found superconvergence of the gradient considering a triangular grid but regular and quasi-ufimorm.

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Attending that the nonstandard finite element method that we study in this work is equivalent to a finite difference method carefully defined we conclude that this last method is supraconvergent. Supraconvergent finite difference schemes has been largely studied in the literature and without be exhaustive we mention [5]-[10],[12],[13], [15],[16], [20].

The papers is organized as follows. In section 2 we present the problem that we intent to solve. The nonstandard linear finite element method is described in section 3. In section 4 we present a finite difference method equivalent to the linear finite element method describe in section 3. The study of the H^1 norm of the error is considered in section 5.

2 The boundary value problem

We begin with some notation. Let $v = (v_1, v_2)$ be a function of two variables. We define

$$div(v) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \; grad(v) = \begin{bmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} \\ \dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_2}{\partial y} \end{bmatrix}$$

Let $A = [a_{ij}]_{i,j=1,2}$ be a matrix with a_{ij}, $i, j = 1, 2$, functions of two variables. We define

$$div(A) = \begin{bmatrix} \dfrac{\partial a_{11}}{\partial x} + \dfrac{\partial a_{12}}{\partial y} \\ \dfrac{\partial a_{21}}{\partial x} + \dfrac{\partial a_{22}}{\partial y} \end{bmatrix} \tag{1}$$

and in the vectorial space of matrices of order two we consider the following inner product

$$A : B = \sum_{i,j=1}^{2} a_{ij} b_{ij} \tag{2}$$

where $A = [a_{ij}], B = [b_{ij}]$ are matrices of second order. By $tr(A)$ we represent the trace of the matrix A.

Let us define now the boundary value problem that we consider in this work. By $\Omega \subset \mathbb{R}^2$ we denote a union of rectangles and by $\partial\Omega$ we represent its boundary. We consider an isotropic material in the configuration space Ω and a body force f. By the static theory of linear elasticity, the the displacement u is solution of the partial differential equation

$$-\,div(\sigma(u)) = f \text{ in } \Omega \tag{3}$$

with the displacement boundary condition

$$u = g \text{ on } \partial\Omega. \tag{4}$$

In (3), $\sigma(u)$ denotes the stress tensor defined by

$$\sigma(u) = 2\mu\epsilon(u) + \lambda tr(\epsilon(u)) I_2$$

where

$$\epsilon(u) = \frac{1}{2}(grad(u) + grad(u)^t)$$

and μ, λ are the Lamé constants.

In order two define the variational formulation of the pure displacement problem (3) and (4) we consider the space $\mathbf{H} = H^1(\Omega) \times H^1(\Omega)$ with the inner product

$$(u, v)_{\mathbf{H}\times\mathbf{H}} = (u_1, v_1)_{H^1(\Omega)} + (u_2, v_2)_{H^1(\Omega)}$$

for $u, v \in \mathbf{H}$. In $\mathbf{L}^2 = L^2(\Omega) \times L^2(\Omega)$ we consider the inner product

$$(u, v)_{\mathbf{L}^2} = (u_1, v_1)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\Omega)}$$

for $u, v \in \mathbf{L}^2$.

Considering the sesquilinear form

$$a(u, v) = \int_\Omega (2\mu\epsilon(u) : \epsilon(v) + \lambda div(u) div(v)) dxdy$$

defined in $\mathbf{H} \times \mathbf{H}$, we define the variational problem:

Find $u \in \mathbf{H}$ such that $u = g$ on $\partial\Omega$ and

$$a(u, v) = \int_\Omega f v dxdy, \ \forall v \in \mathbf{H}_0. \tag{5}$$

The solution u is obtained considering $w \in \mathbf{H}$ and $u^* \in \mathbf{H}_0$ such that

$$a(u, v) = \int_\Omega f v dxdy - a(w, v), \ \forall v \in \mathbf{H}_0.$$

Attending to that, in what follows we consider Dirichlet homogeneous boundary conditions.

The first Korn's inequality enable us to conclude the elipticity of $a(.,.)$ in $\mathbf{H}_0 \times \mathbf{H}_0$ and so the next result:

Theorem 1 *If $f \in \mathbf{H}^{-1}$ then exits a unique $u \in \mathbf{H}_0$ satisfying (5).* ■

3 The finite element method

Preparing the definition of the nonstandard linear finite element method presented in [6] for the scalar case, we consider $h = (h_j)_{\mathbb{Z}}$ and $k = (k_\ell)_{\mathbb{Z}}$ two sequences of positive numbers. We define the grid

$$\mathbb{R}_1 = \{x_j \in \mathbb{R} : x_{j+1} = x_j + h_j, \, j \in \mathbb{Z}\}$$

with $x_0 \in \mathbb{R}$ given and a corresponding grid $\mathbb{R}_2$ with the meshsize vector k in place of h. Let

$$\mathbb{R}_H = \mathbb{R}_1 \times \mathbb{R}_2 \subset \mathbb{R}^2.$$

Define also

$$\Omega_H := \Omega \cap \mathbb{R}_H, \qquad \partial\Omega_H := \partial\Omega \cap \mathbb{R}_H, \qquad \overline{\Omega}_H := \overline{\Omega} \cap \mathbb{R}_H.$$

The grid $\overline{\Omega}_H$ is assumed to satisfy the following regularity condition with respect to the region Ω.

(Reg) Let $\Box$ denote any subrectangle $(x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ formed by the grid $\mathbb{R}_H$. Then $\Box \cap \partial\Omega$ is empty.

By W_H we denote the space of grid functions defined on $\overline{\Omega}_H$ and by $\mathring{\mathbf{W}}_H$ we represent the set of grid functions in W_H null on the boundary points $\partial\Omega_H$.

We consider two special triangulations related to the set $\overline{\Omega}_H$, which we call $\mathcal{T}_H^{(1)}$ and $\mathcal{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \,\dot{\cup}\, \mathbb{R}_H^{(2)},$$

where the sum $j + \ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even or odd, respectively. To simplify the following definition we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. With each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the triangles $\Delta_{j,\ell}^{(i)}$, $i = 1, 2, 3, 4$, which have a right angle at (x_j, y_ℓ) and two of the four closest neighbor grid points of (x_j, y_ℓ) as further vertices. We then define the triangulations

$$\mathcal{T}_H^{(s)} := \{\Delta_{j,\ell}^{(i)} \subset \bar{\Omega} \quad , \ (x_j, y_\ell) \in \mathbb{R}_H^{(s)} \quad , \ i \in \{1, 2, 3, 4\}\} \tag{6}$$

of $\overline{\Omega}$ ($\mathring{\Delta}$ denotes the interior of Δ). Figure 1 shows an example of one of these triangulations.

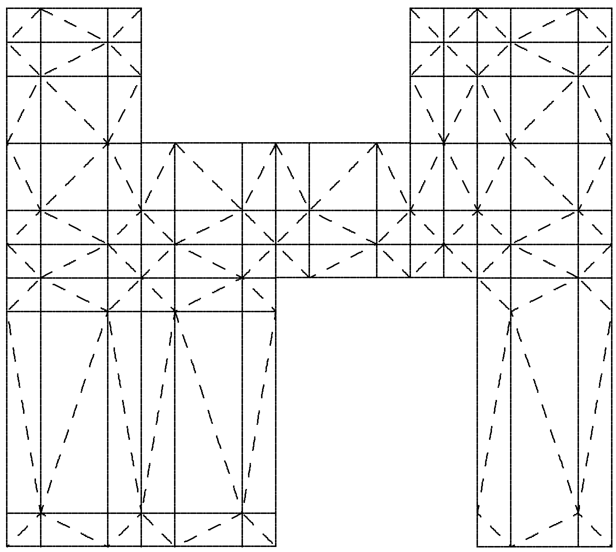

Figure 1: Triangulation $\mathcal{T}_H^{(s)}$.

With respect to the defined triangulation the continuous piecewise linear interpolations $P_H^{(s)}v_H$ of $v_H \in W_H$, $s = 1, 2$, are well-defined.

In the following we consider the set of grid functions $v_H = (v_{H,1}, v_{H,2})$ defined on $\overline{\Omega}_H \times \overline{\Omega}_H$ and vanishing on the boundary $\partial\Omega_H$ which we denote by $\overset{\circ}{\mathbf{W}}_H$. For $v_H \in \overset{\circ}{\mathbf{W}}_H$ by $P_H^{(s)}v_H$ we represent $(P_H^{(s)}v_{H,1}, P_H^{(s)}v_{H,2})$.

Now we define the sesquilinear form $a_H(.\,,.)$ in $\overset{\circ}{\mathbf{W}}_H \times \overset{\circ}{\mathbf{W}}_H$ as the arithmetical mean

$$a_H = \frac{1}{2}(a_H^{(1)} + a_H^{(2)}) \tag{7}$$

of two sesquilinear forms, each of which is defined by

$$a_H^{(s)}(u_H, v_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} \int_\Delta \Big(2\mu\epsilon(P_H^{(s)}u_H) : \epsilon(P_H^{(s)}v_H) + \lambda div(P_H^{(s)}u_H) div(P_H^{(s)}v_H)\Big)\, dxdy, \tag{8}$$

for $s = 1, 2$.

For $u_H, v_H \in \overset{\circ}{W}_H$ we consider the following sesquilinear forms

$$a_{xx}^{(s)}(u_H, v_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} \int_\Delta (P_H^{(s)}u_H)_x (P_H^{(s)}v_H)_x dxdy\,, \tag{9}$$

$$a_{yy}^{(s)}(u_H, v_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} \int_\Delta (P_H^{(s)}u_H)_y (P_H^{(s)}v_H)_y dxdy\,, \tag{10}$$

and

$$a_{xy}^{(s)}(u_H, v_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} \int_\Delta (P_H^{(s)}u_H)_x (P_H^{(s)}v_H)_y dxdy\,. \tag{11}$$

We define $a_{yx}^{(s)}(.,.)$ as $a_{xy}^{(s)}(.,.)$ changing the position of the variables x and y.

Using sesquilinear forms $a_{xx}^{(s)}, a_{yy}^{(s)}, a_{xy}^{(s)}$ and $a_{yx}^{(s)}$, is easy to show that for $u_H, v_H \in \overset{\circ}{\mathbf{W}}_H$ we have

$$\begin{aligned}
a_H^{(s)}(u_H, v_H) &= (2\mu + \lambda)\left(a_{xx}^{(s)}(u_{H,1}, v_{H,1}) + a_{yy}^{(s)}(u_{H,2}, v_{H,2})\right) \\
&+ \mu\left(a_{xx}^{(s)}(u_{H,2}, v_{H,2}) + a_{yy}^{(s)}(u_{H,1}, v_{H,1})\right) \\
&+ \mu\left(a_{xy}^{(s)}(u_{H,2}, v_{H,1}) + a_{yx}^{(s)}(u_{H,1}, v_{H,2})\right) \\
&+ \lambda\left(a_{yx}^{(s)}(u_{H,2}, v_{H,1}) + a_{xy}^{(s)}(u_{H,1}, v_{H,2})\right).
\end{aligned}$$

Then our variational formulation is:

Find $u_H \in \overset{\circ}{\mathbf{W}}_H$ such that

$$a_H(u_H, v_H) = (f, v_H)_H \tag{12}$$

for all $v_H \in \overset{\circ}{\mathbf{W}}_H$.

In (12), $(.,.)_H$ denotes the inner product

$$(w_H, v_H)_H = \sum_{(x_i,y_j)\in\Omega_H} \omega_{ij}(w_{1,ij}v_{1,ij} + w_{2,ij}v_{2,ij})$$

with $\omega_{ij} = \dfrac{h_i + h_{i+1}}{2}\dfrac{k_j + k_{j+1}}{2}$.

The following stability result is consequence of the definition of $a_H(.,.)$:

Theorem 2 *Exists a positive constant C such that*

$$\|P_H v_H\|_1 \leq C \sup_{0\neq w_H \in \overset{\circ}{\mathbf{W}}_H} \frac{|a_H(v_H, w_H)|}{\|P_H w_H\|_1} \tag{13}$$

for all $v_H \in \overset{\circ}{\mathbf{W}}_H$.

■

4 An equivalent finite difference method

We are now preparing the definition of a finite difference approximation "equivalent" to the last variational problem which is useful to implement the defined finite element method. For each grid point $(x_j, y_\ell) \in \mathbb{R}_H$ we define the central finite difference quotients

$$\delta_x^{(1/2)} w_{j,\ell} = \frac{w_{j+1/2,\ell} - w_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}}, \quad \delta_x^{(1/2)} w_{j+1/2,\ell} = \frac{w_{j+1,\ell} - w_{j,\ell}}{x_{j+1} - x_j},$$

$$\delta_x w_{j,\ell} = \frac{w_{j+1,\ell} - w_{j-1,\ell}}{x_{j+1} - x_{j-1}},$$

where $x_{j+1/2} := x_j + h_j/2$, $x_{j-1/2} := x_j - h_{j-1}/2$. Correspondingly, the central finite difference quotients with respect to the variable y are defined.

We introduce now the following finite difference problem:

Find $u_H \in \overset{\circ}{\mathbf{W}}_H$ such that

$$A_H u_H = f \quad \text{in } \Omega_H, \tag{14}$$

with

$$A_H u_H := \begin{bmatrix} (2\mu+\lambda)\delta_x^{(1/2)}\delta_x^{(1/2)} u_{H,1} + \mu\delta_x\delta_y u_{H,2} + \lambda\delta_y\delta_x u_{H,2} + \mu\delta_y^{(1/2)}\delta_y^{(1/2)} u_{H,1} \\ (2\mu+\lambda)\delta_y^{(1/2)}\delta_y^{(1/2)} u_{H,2} + \lambda\delta_y\delta_x u_{H,1} + \mu\delta_x\delta_y u_{H,1} + \mu\delta_x^{(1/2)}\delta_x^{(1/2)} u_{H,2} \\ \end{bmatrix}$$

Attending to the definition of $a_H(.,.)$ and to A_H is easy to show the next result:

Proposition 1 *Let the sesquilinear form $a_H(\cdot\,,\cdot)$ be defined by (8). With A_H defined by (14) , holds the equality*

$$a_H(v_H, w_H) = (A_H v_H, w_H)_H$$

for $w_H, v_H \in \overset{\circ}{\mathbf{W}}_H$.

$\blacksquare$

We consider in the following a sequence of grids $\mathbb{R}_H$ defined using a sequence Λ of $H = (h, k)$ such that the maximal mesh-size H_{max} tends to zero.

Attending to the last proposition and to the stability inequality (13) we conclude that A_H is invertible and stable in the following sense: exists a positive constant C such that

$$\|P_H v_H\|_1 \leq C \sup_{0 \neq w_H \in \overset{\circ}{\mathbf{W}}_H} \frac{|(A_H v_H, w_H)_H|}{\|P_H w_H\|_1} \tag{15}$$

for $v_H \in \overset{\circ}{\mathbf{W}}_H$ and for $H \in \Lambda$. The constant C is independent of $H \in \Lambda$.

In the following example we show the performance of the method defined by (12) or equivalently the performance of the finite difference scheme (14).

Example 1 *Let us consider the boundary value problem (3) defined on the rectangle $\Omega = (0, 1) \times (0, 1)$, with $\lambda = 1, \mu = 0.5$,*

$$f_1(x, y) = f_2(x, y) = \pi^2[-0.4cos(\pi(x + y)) + 0.1cos(\pi(x - y))]$$

and $g = 0$. This planar elasticity problem as the following solution $u = (0.2sin(\pi x)sin(\pi y)$.

In the following table are presented the number of point considered in the x and y nonuniform grids which are denoted respectively by N and M, the maximum step-size in both direction x and y, and the norm $\|.\|_1$ of the error.

Number of points	h^2_{max}	k^2_{max}	$\|P_H R_H u - P_H u_H\|_1$
$N = 8$, $M = 8$	0.01	0.01	0.0427604
$N = 16$, $M = 16$	0.0025	0.0025	0.0111163
$N = 32$, $M = 32$	0.000625	0.000625	0.00269675

5 Estimating the error

The discretization defined by A_H is, even for smooth function u, on nonuniform grids of first order consistent. We show that the last discretization is second order convergence, that is, $\|P_H R_H u - P_H u_H\|_1 = O(H^2_{max})$ where u and u_H are respectively the finite difference solution (also finite element solution) and the solution of the elasticity problem. By R_H we represent the pontwise restriction operator.

Let us estimate now $\|P_H R_H u - P_H u_H\|_1$. Attending to (13), we have

$$\|P_H R_H u - P_H u_H\|_1 \leq C \sup_{v_H \in \overset{\circ}{\mathbf{W}}_H} \frac{|a_H(R_H u, v_H) - (f, v_H)_H}{\|P_H v_H\|_1} \tag{16}$$

and an estimate to the error $\|P_H R_H u - P_H u_H\|_1$ is obtained estimating

$$a_H(R_H u, v_H) - (f, v_H)_H \tag{17}$$

for $v_H \in \overset{\circ}{\mathbf{W}}_H$.

Let us remark that, from [6], we have

$$a_{xx}^{(s)}(R_H u_i, v_{H,i}) = -(R_H \frac{\partial^2 u_i}{\partial x^2}, v_{H,i})_H + R_{xxi}, \tag{18}$$

for $i = 1, 2$,

$$a_{yy}^{(s)}(R_H u_i, v_{H,i}) = -(R_H \frac{\partial^2 u_i}{\partial y^2}, v_{H,i})_H + R_{yyi}, \tag{19}$$

for $i = 1, 2$, and

$$a_{xy}^{(s)}(R_H u_i, v_{H,j}) = -(R_H \frac{\partial^2 u_i}{\partial y \partial x}, v_{H,i})_H + R_{xyi}, \tag{20}$$

for $i = 1, j = 2$ and $i = 2, j = 1$, where

$$|R_{xxi}| \leq C \left(\sum_{\Delta \in \mathcal{T}_H^{(s)}} |\Delta| h_\Delta^4 \| \frac{\partial^3 u_i}{\partial x^3} \|_{1,\infty,\Delta}^2 \right)^{1/2} \|P_H v_{H,i}\|_1, \tag{21}$$

and by $|\Delta|$ we represent the area of the triangle Δ. The bound for remainder term R_{yyi} is obtained by taking k_Δ and $\partial/\partial y$ in place of h_Δ and $\partial/\partial x$ respectively in (21), and

$$|R_{xyi}| \leq C \left(\sum_{\Delta \in \mathcal{T}_H^{(s)}} |\Delta| (h_\Delta^4 + k_\Delta^4)(\| \frac{\partial^3 u_i}{\partial^2 y \partial x} \|_{1,\infty,\Delta}^2 + \| \frac{\partial^3 u_i}{\partial y \partial^2 x} \|_{1,\infty,\Delta}^2 \right)^{1/2} \|P_H v_{H,j}\|_1. \tag{22}$$

The bound for the remainder term R_{yxi} is obtained by taking $\partial/\partial y$ in place of $\partial/\partial x$ in (22).

Altogether we have proved the following result:

Proposition 2 *Let u be in $C^4(\overline{\Omega}) \times C^4(\overline{\Omega})$. Then*

$$a_H(R_H u, v_H) = -(R_H[div(\sigma(u))], v_H)_H + \tau(u, v_H)$$

with

$$\tau(u, v_H) \leq C \left(\sum_{\Delta \in \mathcal{T}_H} |\Delta| (diam \Delta)^4 \|u\|_{4,\infty,\Delta}^2 \right)^{1/2} \|P_H v_H\|_1$$

where C is independent of the triangulation $\mathcal{T}_H$ and of u.

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Finally, from the last proposition and attending to inequality (16) we conclude the main result of this work:

Theorem 3 *If the solution of (3) and (4) is in* $C^4(\overline{\Omega}) \times C^4(\overline{\Omega})$ *with* $g = 0$, *then the variational problem (12) and the finite difference problem (14) have a unique solution* u_H *in* $\overset{\circ}{\mathbf{W}}_H$ *satisfying the error estimate*

$$\|P_H R_H u - P_H u_H\|_1 \leq C \left(\sum_{\Delta \in \mathcal{T}_H} |\Delta| (diam \Delta)^4 \|u\|^2_{4,\infty,\Delta} \right)^{1/2} \tag{23}$$

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