Some remarks on complete congruence lattices of join-infinite distributive lattices

Jorge Picado *

Abstract

In [5] G. Grätzer and E. T. Schmidt raised the problem of characterizing the complete congruence lattices of complete lattices satisfying the Join-Infinite Distributive Identity (JID) and the Meet-Infinite Distributive Identity (MID) and proved the theorem: *Any complete lattice with more than two elements and with a meet-irreducible zero cannot be represented as the lattice of complete congruence relations of a complete lattice satisfying (JID) and (MID).* In this note we generalize this result by showing that the complete congruence lattice of every complete lattice satisfying (JID) and (MID) is a zero-dimensional complete lattice satisfying (JID). Some consequences are discussed.

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1. Introduction

A *complete congruence* on a complete lattice L is an equivalence relation Θ on L which satisfies the Infinite Substitution Property, that is, if $(a_i, b_i) \in \Theta$ for all $i \in I$, then

$$\Big(\bigvee\{a_i \mid i \in I\}, \bigvee\{b_i \mid i \in I\}\Big) \in \Theta$$

and

$$\Big(\bigwedge\{a_i \mid i \in I\}, \bigwedge\{b_i \mid i \in I\}\Big) \in \Theta.$$

The partially ordered set (by inclusion) of all complete congruences on L, which we denote by $\mathcal{CC}(L)$, is closed under intersections hence it is a complete lattice.

In 1983, R. Wille [11] raised the problem on the possibility of representing every complete lattice as the lattice of complete congruences of a suitable complete lattice L. In 1988, this was answered in the affirmative by G. Grätzer [3]. In [4], G. Grätzer and E. T. Schmidt proved that L can always be chosen as a complete distributive lattice and in [5, Problem 1] they raised the same representation problem for the two following special classes of complete distributive lattices: the complete distributive lattices satisfying the Join-Infinite Distributive Identity

$$\text{(JID)}\ a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$$

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and the complete distributive lattices satisfying the Meet-Infinite Distributive Identity

$$\text{(MID)}\ a \vee \bigwedge S = \bigwedge \{a \vee s \mid s \in S\}.$$

As a motivation for this problem they proved the following theorem:

Theorem 1. (Grätzer and Schmidt [5]) *Let K be a complete lattice with more than two elements and with a meet-irreducible zero. Then K cannot be represented as the lattice of complete congruence relations of a complete distributive lattice L satisfying (JID) and (MID).*

The complete distributive lattices satisfying (JID) constitute a very special class of lattices. They are usually called *frames* and constitute the object of study of pointfree topology [7]. Moreover, frames are precisely the so-called complete Heyting algebras, i.e. complete lattices endowed with a *relative pseudocomplementation*, that is, with a binary operation "$\rightarrow$" satisfying the equivalence

$$c \leq a \rightarrow b \Leftrightarrow c \wedge a \leq b \quad [6].$$

In particular, $a \rightarrow 0 = \bigvee\{c \in L \mid c \wedge a = 0\}$ is the pseudocomplement of a, usually denoted by a^*. If $a \vee a^* = 1$ then a^* is the complement of a.

We say that a complete distributive lattice L is a *double frame* if it satisfies (JID) and (MID). So a double frame L is a frame which is also a co-frame, that is, its dual lattice L^∂ is also a frame. They are called *completely distributive frames* in [9] and *join-continuous frames* in [8].

Further, we recall that a frame is said to be *zero-dimensional* whenever its elements are joins of complemented elements [6].

The main purpose of this note is to improve Theorem 1, by proving that:

Theorem 2. *The lattice of complete congruence relations of every double frame is a zero-dimensional frame.*

This generalizes Theorem 1. Indeed, any zero-dimensional complete lattice with a meet-irreducible zero has, at most, two elements, because the existence of a meet-irreducible zero implies that 0 and 1 are the only complemented elements.

2. Proof of Theorem 2

We begin by recalling a lemma (whose proof is straightforward) which allows us, when working with a pair of elements a and b, to assume without loss of generality that $a \leq b$.

Lemma 1. *Let Θ be a complete congruence on L. Then:*

(a) *If $(a, b) \in \Theta$ then $(a \wedge b, a \vee b) \in \Theta$.*

(b) *If $(a, b) \in \Theta$ and $a \leq c \leq b$ then $(a, c) \in \Theta$ and $(c, b) \in \Theta$.* □

For $a, b \in L$, let $\Theta_{a,b} = \bigwedge\{\Theta \in \mathcal{CC}(L) \mid (a,b) \in \Theta\}$, that is, the complete congruence generated by (a,b). By Lemma 1, $\Theta_{a,b} = \Theta_{a\wedge b, a\vee b}$. Then, obviously,

$$\Theta = \bigvee\{\Theta_{a,b} \mid (a,b) \in \Theta, a \leq b\}$$

for each $\Theta \in \mathcal{CC}(L)$.

From now on let L denote a double frame. Particular complete congruences on L associated with each $a \in L$ are

$$\nabla_a = \{(x,y) \mid x \vee a = y \vee a\}$$

and

$$\Delta_a = \{(x,y) \mid x \wedge a = y \wedge a\},$$

also characterized as the congruences generated by $(0,a)$ and $(a,1)$, respectively.

The following results give us some distributivity properties.

Lemma 2. *Let Θ, Θ_i $(i \in I)$ be elements of $\mathcal{CC}(L)$ and let $a \in L$. Then:*

(a) $\nabla_a \vee \Theta = \{(x,y) \in L \times L \mid (x \vee a, y \vee a) \in \Theta\}$;

(b) $\nabla_a \vee \bigwedge\{\Theta_i \mid i \in I\} = \bigwedge\{\nabla_a \vee \Theta_i \mid i \in I\}$.

Proof. (a) The inclusion $\nabla_a \vee \Theta \subseteq \{(x,y) \in L \times L \mid (x \vee a, y \vee a) \in \Theta\}$ is clear because $\{(x,y) \in L \times L \mid (x \vee a, y \vee a) \in \Theta\}$ is indeed a complete congruence that contains ∇_a and Θ.

Conversely let (x,y) be such that $(x \vee a, y \vee a) \in \Theta$. Since $(x, x \vee a) \in \nabla_a$ and $(y \vee a, y) \in \nabla_a$ we may conclude that $(x,y) \in \nabla_a \vee \Theta$.

(b) Follows immediately from (a). □

Similarly, one gets:

Lemma 3. *Let Θ, Θ_i $(i \in I)$ be elements of $\mathcal{CC}(L)$ and let $a \in L$. Then:*

(a) $\Delta_a \vee \Theta = \{(x,y) \in L \times L \mid (x \wedge a, y \wedge a) \in \Theta\}$;

(b) $\Delta_a \vee \bigwedge\{\Theta_i \mid i \in I\} = \bigwedge\{\Delta_a \vee \Theta_i \mid i \in I\}$. □

Based on these elementary lemmata, we can now prove Theorem 2:

Proof of Theorem 2. Let L be a double frame. In order to prove that $\mathcal{CC}(L)$ is a frame we need to show that there exists, for each $\Theta_1, \Theta_2 \in \mathcal{CC}(L)$, the relative pseudocomplement $\Theta_1 \to \Theta_2$. We do this by proving that

$$\bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\}$$

is indeed the element $\Theta_1 \to \Theta_2$.

As the first step, we show that

$$\Theta_1 \wedge \bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\} \leq \Theta_2.$$

Consider $(a,b) \in \Theta_1 \wedge \bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\}$ with $a \leq b$. Then $(a,b) \in \nabla_a \vee \Delta_b \vee \Theta_2$, which implies, by Lemma 2(a), $(a, a \vee b) \in \Delta_b \vee \Theta_2$. Thus, by Lemma 3(a), $(a \wedge b, (a \vee b) \wedge b) = (a,b) \in \Theta_2$.

Finally, we prove that for any $\Theta \in \mathcal{CC}(L)$, $\Theta \leq \bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\}$ whenever $\Theta_1 \wedge \Theta \leq \Theta_2$. Let $(a,b) \in \Theta_1$ with $a \leq b$, and assume $\Theta_1 \wedge \Theta \leq \Theta_2$. Then

$$\begin{aligned}
\nabla_a \vee \Delta_b \vee \Theta_2 &= \Theta_2 \vee \nabla_a \vee \Delta_b \vee (\Theta \wedge \Theta_1) \\
&= \Theta_2 \vee \nabla_a \vee \Big((\Delta_b \vee \Theta) \wedge (\Delta_b \wedge \Theta_1)\Big) \\
&= \Theta_2 \vee \Big((\nabla_a \vee \Delta_b \vee \Theta) \wedge (\nabla_a \vee \Delta_b \vee \Theta_1)\Big).
\end{aligned}$$

Since $(0,a) \in \nabla_a$, $(a,b) \in \Theta_1$ and $(b,1) \in \Delta_b$, the pair $(0,1)$ belongs to $\nabla_a \vee \Delta_b \vee \Theta_1$, that is, $\nabla_a \vee \Delta_b \vee \Theta_1 = \nabla$. Therefore $\nabla_a \vee \Delta_b \vee \Theta_2 = \Theta_2 \vee \nabla_a \vee \Delta_b \vee \Theta$, that is, $\Theta \leq \nabla_a \vee \Delta_b \vee \Theta_2$. Hence $\Theta \leq \bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\}$, and we may conclude that $\mathcal{CC}(L)$ is indeed a frame.

Now we prove that $\mathcal{CC}(L)$ is zero-dimensional. Since $\Theta_{a,b} = \Delta_a \wedge \nabla_b$ for $a \leq b$ [5, Lemma 1], we have $\Theta = \bigvee\{\Delta_a \wedge \nabla_b \mid (a,b) \in \Theta, a \leq b\}$ for every $\Theta \in \mathcal{CC}(L)$. Further, ∇_a and Δ_a are complementary to each other, that is, $\nabla_a \wedge \Delta_a = \{(x,x) \mid x \in L\}$, the zero of $\mathcal{CC}(L)$, and $\nabla_a \vee \Delta_a = L \times L$, the unit of $\mathcal{CC}(L)$. Hence $\mathcal{CC}(L)$ is zero-dimensional. $\square$

This proof also reveals the importance of the complete congruences of the form ∇_a and Δ_a; they (suprema) generate the complete congruence lattice.

3. Remarks

3.1. In case L is finite, the complete lattice $\mathcal{CC}(L)$, which coincides with the congruence lattice $Cong(L)$ of L, is the Boolean closure of L.

3.2. Clearly $\nabla : L \to \mathcal{CC}(L)$, defined by $\nabla(a) = \nabla_a$, and $\Delta : L^\partial \to \mathcal{CC}(L)$, given by $\Delta(a) = \Delta_a$, are injective complete homomorphisms. Furthermore, for any complemented element $a \in L$, $\Delta_a = \nabla_{a^*}$. It follows immediately that, in case L is zero-dimensional, $\mathcal{CC}(L)$ is Boolean and isomorphic to L. Indeed, for any $\Theta \in \mathcal{CC}(L)$, Θ equals

$$\bigvee_{i \in I} (\Delta_{a_i} \wedge \nabla_{b_i}) = \bigvee_{i \in I} \Big(\Delta_{(\bigvee_{j \in J_i} c^i_j)} \wedge \nabla_{b_i}\Big) = \bigvee_{i \in I} \Big((\bigwedge_{j \in J_i} \nabla_{(c^i_j)^*}) \wedge \nabla_{b_i}\Big) = \nabla_{\bigvee_{i \in I} \bigwedge_{j \in J_i} (c^i_j)^*},$$

where each c^i_j is complemented, and therefore $\mathcal{CC}(L) = \{\nabla_a \mid a \in L\}$ and the map ∇ is an isomorphism. Hence L and $\mathcal{CC}(L)$ are isomorphic if and only if L is a complete Boolean algebra. This also implies that a zero-dimensional frame is a double frame if and only if it is a complete Boolean algebra.

Let $\mathcal{B}(\mathcal{CC}(L))$ be the complete Boolean algebra $\{\Theta^* \mid \Theta \in \mathcal{CC}(L)\}$ (the so-called *booleanization* [1] of $\mathcal{CC}(L)$). Then:

Proposition. $\mathcal{B}(\mathcal{CC}(L)) = \{\Theta \in \mathcal{CC}(L) \mid \Theta = \bigwedge_i (\nabla_{a_i} \vee \Delta_{b_i})$ *for some* $a_i, b_i \in L\}$.

Proof. In the proof of Theorem 2 we observed that

$$\Theta_1 \to \Theta_2 = \bigwedge\{\nabla_a \vee \Delta_b \vee \Theta_2 \mid a \leq b, (a,b) \in \Theta_1\}$$

for every $\Theta_1, \Theta_2 \in \mathcal{CC}(L)$; in particular, $\Theta^* = \bigwedge\{\nabla_a \vee \Delta_b \mid a \leq b, (a,b) \in \Theta\}$ for every $\Theta \in \mathcal{CC}(L)$.

The conclusion of the converse inclusion follows from the fact that $\nabla_a \vee \Delta_b$ is complemented with complement $\Delta_a \wedge \nabla_b$:

$$\Theta = \bigwedge_i (\nabla_{a_i} \vee \Delta_{b_i}) = \bigwedge_i (\Delta_{a_i} \wedge \nabla_{b_i})^* = \Big(\bigvee_i (\Delta_{a_i} \wedge \nabla_{b_i})\Big)^* \in \mathcal{B}(\mathcal{CC}(L)). \qquad \square$$

3.3. An element c of a frame L is called *compact* (*finite* in Johnstone [6]) whenever, for every subset $S \subseteq L$ with $c \leq \bigvee S$, there exists a finite $F \subseteq S$ with $c \leq \bigvee F$. Equivalently, c is compact whenever, for every up-directed subset $S \subseteq L$ with $c \leq \bigvee S$, there exists $s \in S$ with $c \leq s$. L is said to be *compact* if its unit 1 is compact. Under the Axiom of Choice, a frame L satisfies the Ascending Chain Condition if and only if each of its elements is compact. By this reason, a frame is called *noetherian* whenever each of its elements is compact. A *coherent frame* [6] is a compact frame L in which every element is expressible as a join of compact elements and finite meets of compact elements are compact. This is equivalent to saying that L is isomorphic to the frame of ideals of a distributive lattice [6].

Evidently, in any compact frame, each complemented element is compact; the converse is true if, in addition, the frame is zero-dimensional. Therefore $\mathcal{CC}(L)$ is compact if and only if the complemented elements of $\mathcal{CC}(L)$ are exactly the compact ones. Moreover:

Proposition. *Let L be a double frame. The following assertions are equivalent:*

(i) $\mathcal{CC}(L)$ *is compact.*

(ii) L *and* L^∂ *are noetherian.*

(iii) $\mathcal{CC}(L) = Cong(L)$.

(iv) $\mathcal{CC}(L)$ *is a coherent frame.*

Proof. (i)⇒(ii) Since $\nabla : L \to \mathcal{CC}(L)$ and $\Delta : L^\partial \to \mathcal{CC}(L)$ are frame embeddings and, for every $a \in L$, $\nabla(a)$ and $\Delta(a)$ are complemented thus compact, the conclusion follows.

(ii)⇒(iii) L being noetherian implies that arbitrary joins in L are actually finite joins; dually, L being co-noetherian, arbitrary meets in L are actually finite meets. Hence $\mathcal{CC}(L) = Cong(L)$.

(iii)⇒(iv) Assume that $\mathcal{CC}(L) = Cong(L)$. Let us prove that the unit $\Theta_{0,1} = L \times L$ is compact i.e. $\Theta_{0,1} \leq \bigvee_{i\in I} \Theta_i$ for any up-directed subset of $\mathcal{CC}(L)$ implies $\Theta_{0,1} = \Theta_i$ for some $i \in I$. This follows immediately from the fact that $\mathcal{CC}(L)$ is closed under up-directed unions, which we prove next:

Let $\Gamma = \{\Theta_i \mid i \in I\}$ be an up-directed subset of $\mathcal{CC}(L) = Cong(L)$. Then $\bigcup_{i\in I} \Theta_i \in \mathcal{CC}(L)$; indeed, for any $(a_1, b_1), (a_2, b_2) \in \bigcup_{i\in I} \Theta_i$, there exist $i \in I$ such that $(a_1, b_1), (a_2, b_2) \in \Theta_i$, which implies $(a_1 \vee a_2, b_1 \vee b_2), (a_1 \wedge a_2, b_1 \wedge b_2) \in \Theta_i \subseteq \bigcup_{i\in I} \Theta_i$. Thus $\bigvee_{i\in I} \Theta_i = \bigcup_{i\in I} \Theta_i$.

In conclusion, $\mathcal{CC}(L)$ is compact. Consequently, complemented element means compact element and, therefore, the zero-dimensionality of $\mathcal{CC}(L)$ implies that $\mathcal{CC}(L)$ is coherent.

<u>(iv)⇒(i)</u> Trivial. □

3.4. What happens in the more general case that L only satisfies (JID)? First, the proof of Theorem 2 cannot be extended to this case since, in general, ∇_a is not a complete congruence. However ∇_a is a *frame congruence*, that is,

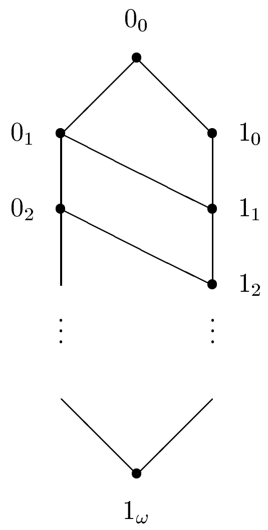
$$\{(a_i, b_i) \mid i \in I\} \subseteq \Theta \Rightarrow \Big(\bigvee\{a_i \mid i \in I\}, \bigvee\{b_i \mid i \in I\}\Big) \in \Theta$$

and

$$(a_1, b_1), (a_2, b_2) \in \Theta \Rightarrow (a_1 \wedge a_2, b_1 \wedge b_2) \in \Theta.$$

Then, as in Section 2, one can easily conclude that the complete lattice $\mathcal{C}(L)$ of frame congruences of L is a zero-dimensional frame. This extends Theorem 2 of Grätzer and Schmidt [5], which asserts that any finite non-Boolean lattice or any complete chain with more than two elements cannot be represented as the lattice of complete congruence relations of a frame K. In fact, in both cases $\mathcal{CC}(K) = \mathcal{C}(K)$, so $\mathcal{CC}(K)$ is zero-dimensional and, clearly, any finite zero-dimensional distributive lattice is Boolean and any zero-dimensional chain has at most two elements.

Secondly, it is even impossible to extend Theorem 2 to frames. Indeed, if L is a frame then $\mathcal{CC}(L)$ is not, in general, pseudocomplemented. To conclude this let K be the dual of the example in [10, p. 16] of a distributive complete lattice which is not pseudocomplemented, that is, the direct product of the 2-chain $\{0, 1\}$ and the dual of the $(\omega + 1)$-chain $\mathbb{N}^\omega = \{0, 1, \ldots, \omega\}$, with the element 0_ω removed:



This is clearly a frame, but not a double frame:

$$1_1 \vee \bigwedge_{n \in \mathbf{N}} 0_n = 1_1$$

and, on the other hand,

$$\bigwedge_{n\in\mathbb{N}} (1_1 \vee 0_n) = 0_1.$$

Consider the complete congruence

$$\Theta = \{(a,a) \mid a \in K\} \cup \{(0_i, 1_i), (1_i, 0_i) \mid 0 \leq i < \omega\}$$

and the family $\{\Theta_n\}_{n\in\mathbb{N}}$ of complete congruences defined by

$$\Theta_n = \{(a,a) \mid a \in K\} \cup \{(0_i, 0_n), (0_n, 0_i) \mid 0 \leq i \leq n\} \cup \{(1_i, 1_n), (1_n, 1_i) \mid 0 \leq i \leq n\}.$$

For any $n \in \mathbb{N}$, $\Theta \wedge \Theta_n = \{(a,a) \mid a \in K\}$ and, on the other hand, $\bigvee_{n\in\mathbb{N}} \Theta_n = K \times K$. Therefore Θ does not admit pseudocomplement.

This example also shows that Corollary 5.4 of [2] is not true, as well as Theorem 5.3 from which it derives, since K is clearly continuous (it satisfies the Ascending Chain Condition). The point is that, contrarily as it is stated in the proof of 5.3, if M is a complete sublattice of a complete lattice L, and for some $x, y \in M$ with $y \leq x$, the weak relative pseudocomplement of x with respect to y exists in L, then the weak relative pseudocomplement of x with respect to y in M does not necessarily exist (this is only true for the plain operation of pseudocomplementation).

3.5. Denoting by (JID_m) the condition that (JID) holds for sets X satisfying $|X| < m$, where m is a regular cardinal with $m > \aleph_o$, and defining (MID_m) dually, then, in case L is an m-complete distributive lattice satisfying (JID_m) and (MID_m), since ∇_a and Δ_a are m-complete congruences of L, it is obvious that all results we presented before can be rephrased in this context. Thus, for instance, the m-complete congruence lattice $\mathcal{CC}_m(L)$ is also a zero-dimensional frame.

References

[1] B. Banaschewski and A. Pultr, *Booleanization*, Cah. Top. Géom. Diff. Cat. 37 (1996) 41-60.

[2] R. Giacobazzi and F. Ranzato, *Some properties of complete congruence lattices*, Algebra Univers. 40 (1998) 189-200.

[3] G. Grätzer, *The complete congruence lattice of a complete lattice*, in Lattices, Semigroups and Universal Algebra (Proc. Intern. Conf. on Lattices, Semigroups and Universal Algebra, Lisbon, 1988) 81-88, Plenum, New York/London, 1990.

[4] G. Grätzer and E. T. Schmidt, *Complete congruence lattices of complete distributive lattices*, J. Algebra 171 (1995) 204-229.

[5] G. Grätzer and E. T. Schmidt, *Complete congruence lattices of join-infinite distributive lattices*, Algebra Univers. 37 (1997) 141-143.

[6] P. T. Johnstone, *Stone Spaces* (Cambridge University Press, Cambridge, 1982).

[7] P. T. Johnstone, *The point of pointless topology*, Bull. Amer. Math. Soc. (N.S.) 8 (1983) 41-53.

[8] J. Picado, *Join-continuous frames, Priestley's duality and biframes*, Appl. Cat. Struct. 2 (1994) 297-313.

[9] A. Pultr and A. Tozzi, *Notes on Kuratowski-Mrówka theorems in point-free context*, Cah. Top. Géom. Diff. Cat. 33 (1992) 3-14.

[10] J. Varlet, *Contribution à l'étude des treillis pseudo-complémentés et des treillis de Stone*, Mémoires de la Société Royale des Sciences de Liège, 5eme série, 8(4), 1963.

[11] R. Wille, *Subdirect decompositions of concept lattices*, Algebra Univers. 17 (1983) 275-287.

Departamento de Matemática
Universidade de Coimbra
3001-454 Coimbra
PORTUGAL
E-mail: picado@mat.uc.pt