TRIQUOTIENT MAPS VIA ULTRAFILTER CONVERGENCE

MARIA MANUEL CLEMENTINO AND DIRK HOFMANN

Abstract. In this paper we characterize triquotient maps as those that are surjective on chains of convergent ultrafilters, extending the result known for triquotient maps between finite topological spaces.

1. Introduction

Triquotient maps, introduced by Michael [11], fit very nicely among classes of special quotient maps:

– proper maps and open maps are triquotient maps;
– triquotient maps are effective descent maps, which in turn are biquotient maps.

A recent study of Janelidze and Sobral on the behaviour of the mentioned classes of morphisms, when defined between finite topological spaces, led to very interesting characterizations based on point convergence (see Theorem 2.2). Among these characterizations, it was established the following

Theorem I. *If X and Y are finite topological spaces, a continuous map $f : X \to Y$ is a triquotient map if and only if it is surjective on chains of convergent points.*

To pass from the finite to the infinite case one must replace points by (ultra)filters – or (ultra)nets – and, besides the case of effective descent and triquotient maps (and partially local homeomorphisms), the characterizations are straightforward. To establish a general characterization of triquotient maps that includes the former Theorem, some new notions and techniques are needed.

This is the central part of this paper: we introduce and study a new category, defined via ultrafilter convergence, endowed with a special endofunctor that is used to define *chains of convergent ultrafilters*, and that finally leads to the

Theorem II. *A continuous map $f : X \to Y$ is a triquotient map if and only if it is surjective on chains of convergent ultrafilters.*

Moreover, these characterizations turn out to be very effective on proving stability properties for special kinds of limits, since initial structures – in particular limit structures – are easily described by convergent ultrafilters. This gives rise to unified proofs of results obtained separately, and will be the subject of a forthcoming note.

Acknowledgment. We thank George Janelidze, Manuela Sobral and Walter Tholen for valuable discussions on the subject of this paper.

1991 *Mathematics Subject Classification.* 54C10, 54A20, 54B30, 18A20, 18B30.

Key words and phrases. Biquotient map, effective descent map and triquotient map, convergent structure.

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra. The first author also thanks Project PRAXIS XXI 2/2.1/MAT/46/94.

2. Basic definitions and results

For a topological space X we denote its topology by $\mathcal{O}X$. For $x \in X$, $\mathcal{O}(x)$ denotes the set of open subsets of X containing x. If x and y are points of X, by $y \to x$ we mean that $x \in \overline{\{y\}}$.

Definitions 2.1. A topological continuous map $f : X \to Y$ is:

1. a *biquotient map* if, whenever $y \in Y$ and $\mathcal{A}$ is an open covering of $f^{-1}(y)$, then finitely many $f(A)$, with $A \in \mathcal{A}$, cover some neighbourhood of y in Y;
2. *effective descent* (*descent*) if the pullback functor $f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$, that assigns to each $g : Z \to Y$ its pullback along f, is (pre)monadic: see [8];
3. a *triquotient map* if there exists a map $(\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y$ such that:
   (T1) $(\forall U \in \mathcal{O}X)\ U^\sharp \subseteq f(U)$,
   (T2) $X^\sharp = Y$,
   (T3) $(\forall U, V \in \mathcal{O}X)\ U \subseteq V \Rightarrow U^\sharp \subseteq V^\sharp$,
   (T4) $(\forall U \in \mathcal{O}X)\ (\forall y \in U^\sharp)\ (\forall \Sigma \subseteq \mathcal{O}X \text{ directed})\ f^{-1}(y) \cap U \subseteq \bigcup \Sigma \Rightarrow \exists S \in \Sigma : y \in S^\sharp$;
4. *proper* (*perfect*) if it is closed and has compact fibres (and Hausdorff, i.e. if $f(x) = f(x')$ and $x \neq x'$ there exist $U \in \mathcal{O}(x)$ and $V \in \mathcal{O}(x')$ with $U \cap V = \emptyset$).

Concerning the notion of triquotient map, we note that (T3) is implied by (T4).

We also remark that every proper map $f : X \to Y$ is triquotient: take $U^\sharp := Y - f(X - U)$ for $U \in \mathcal{O}X$, as well as every open map: take $U^\sharp := f(U)$. Plewe showed that – both on topological spaces and locales – triquotient maps are effective descent (see [12]). These latter are descent maps, which are exactly the biquotient maps (see [8]) introduced independently by Michael [10], Hájek [5], by the name of *limit lifting maps*, and Day and Kelly [4], as *universal quotient maps*.

The following results may be found in [7] and [3]:

Theorem 2.2. *If X and Y are finite topological spaces and $f : X \to Y$ is a continuous map, then:*

1. *f is a biquotient map if and only if, for each $y_1 \to y_0$ in Y, there exists $x_1 \to x_0$ in X with $f(x_i) = y_i$ for $i = 0, 1$;*
2. *f is effective descent if and only if, for each $y_2 \to y_1 \to y_0$ in Y, there exists $x_2 \to x_1 \to x_0$ in X with $f(x_i) = y_i$ for $i = 0, 1, 2$;*
3. *f is a triquotient map if and only if, for each chain $y_n \to \ldots \to y_0$ ($n \in \mathbb{N}$) in Y, there exists a chain $x_n \to \ldots \to x_0$ in X with $f(x_i) = y_i$ for each $i = 0, 1, \ldots, n$;*
4. *f is proper (perfect) if and only if, for each $x_1 \in X$ and $f(x_1) \to y_0$ in Y, there exists (a unique) x_0 in X with $x_1 \to x_0$ and $f(x_0) = y_0$;*
5. *f is open (local homeomorphism) if and only if, for each $x_0 \in X$ and $y_1 \to f(x_0)$ in Y, there exists (a unique) x_1 in X with $x_1 \to x_0$ and $f(x_1) = y_1$.*

The statements 1, 4, and partially 5 can be easily generalized using ultrafilters, while a possible generalization of 2 is the well-known Reiterman-Tholen characterization of topological effective descent maps [13]:

Theorem 2.3. *If $f : X \to Y$ is a continuous map, then:*

1. *f is a biquotient map if and only if, for each ultrafilter $\mathfrak{b} \to y$ in Y, there exists an ultrafilter $\mathfrak{a} \to x$ in X with $f(\mathfrak{a}) = \mathfrak{b}$ and $f(x) = y$:*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\ \mathfrak{b} \arrow[r] & y \end{tikzcd}$$

2. *f is an effective descent map if and only if, for each family of ultrafilters $(\mathfrak{b}_i)_{i\in I}$ and each ultrafilter $\mathfrak{u}$ on I, whenever $\mathfrak{b}_i \to y_i$, for $i \in I$, and $y_i \overset{\mathfrak{u}}{\to} y$ in Y, there exists an ultrafilter $\mathfrak{a}$ on X such that $\mathfrak{a} \to x \in f^{-1}(y)$ and, for each $U \in \mathfrak{u}$, $\bigcup_{i\in U}(f^{-1}(y_i) \cap \mathsf{adh}(f^{-1}(\mathfrak{b}_i))) \in \mathfrak{a}$.*
3. *f is a proper (perfect) map if and only if, for each ultrafilter $\mathfrak{a}$ on X with $f(\mathfrak{a}) \to y$ in Y, there exists (a unique) $x \in f^{-1}(y)$ such that $\mathfrak{a} \to x$:*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\ f(\mathfrak{a}) \arrow[r] & y \end{tikzcd}$$

4. *f is an open map if and only if, for each $x \in X$ and each ultrafilter $\mathfrak{b}$ on Y with $\mathfrak{b} \to f(x)$ in Y, there exists an ultrafilter $\mathfrak{a}$ such that $\mathfrak{a} \to x$ in X and $f(\mathfrak{a}) = \mathfrak{b}$:*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\ \mathfrak{b} \arrow[r] & f(x) \end{tikzcd}$$

By similarity with the finite case, and also with the characterizations of proper and perfect maps, it seems natural that one may obtain a characterization of local homeomorphisms imposing in the above one a uniqueness condition. We do not know if this really holds. In fact, what we know at the moment is:

Theorem 2.4. *For a continuous map $f : X \to Y$, if f is a local homeomorphism, then, for each $x \in X$ and each ultrafilter $\mathfrak{b}$ on Y with $\mathfrak{b} \to f(x)$ in Y, there exists a unique ultrafilter $\mathfrak{a}$ such that $\mathfrak{a} \to x$ in X and $f(\mathfrak{a}) = \mathfrak{b}$.*

It is also possible to characterize these classes of maps by their behaviour on lifting convergent (ultra)nets:

Theorem 2.5. *Let $f : X \to Y$ be a topological continuous map.*

1. *f is a biquotient map if and only if, for each net $y_\lambda \to y$ in Y, there exists a net $x_\gamma \to x$ in X such that $(f(x_\gamma))$ is a subnet of (y_λ) and $f(x) = y$.*
2. *f is a proper (perfect) map if and only if, for each ultranet $(x_\lambda)_{\lambda\in\Lambda}$ in X with $f(x_\lambda) \to y$ in Y, there exists (a unique) $x \in f^{-1}(y)$ such that $x_\lambda \to x$.*
3. *f is an open map if and only if, for each $x \in X$ and each ultranet $(y_\lambda)_{\lambda\in\Lambda}$ converging to $f(x)$ in Y, there exists $(x_\gamma)_{\gamma\in\Gamma}$ in X and $\phi : \Gamma \to \Lambda$ such that $x_\gamma \to x$ and $(f(x_\gamma))$ is a subnet of (y_λ), via ϕ, such that $\phi(\uparrow \gamma) = \uparrow \phi(\gamma)$.*
4. *f is a local homeomorphism if and only if, for each $x \in X$ and each net $y_\lambda \to f(x)$ in Y, there exists an* essentially unique *net $(x_\lambda)_{\lambda\in\Lambda}$ in X such that $x_\lambda \to x$ and $f(x_\lambda) = y_\lambda$ for every $\lambda \in \Lambda$.*

(By *essentially unique* we mean that, if (x_λ) and (x'_λ) satisfy the conditions above, then there exists $\lambda_0 \in \Lambda$ such that, for $\lambda \geq \lambda_0$, $x_\lambda = x'_\lambda$.)

The approach with ultrafilter convergence gives a more elegant and unified way of describing biquotient, proper, perfect and open maps. This is the reason why we preferred them to nets, and investigated similar characterizations for effective descent and triquotient maps. This is the aim of the next sections.

First we recall some known facts on ultrafilters. If X is a set, the set $\mathcal{U}(X)$ of ultrafilters on X may be endowed with the Zariski closure, becoming a compact Hausdorff space (see [9]). For a map $f : X \to Y$ and $\mathfrak{a} \in \mathcal{U}(X)$, $f(\mathfrak{a})$ denotes the filter generated by $\{f(A) \,|\, A \in \mathfrak{a}\}$, which is automatically an ultrafilter since $\mathfrak{a}$ is. The map $\mathcal{U}f : \mathcal{U}(X) \to \mathcal{U}(Y)$, $\mathfrak{a} \mapsto f(\mathfrak{a})$ is continuous.

3. The category **URS**

In order to characterize triquotient maps using convergence, we will need a combination of ultrafilters convergence, as it is already the case of effective descent maps, but in a higher order: 2-chains for effective descent maps between finite spaces give rise to the combination of 2-sorts of convergent ultrafilters while n-chains, for triquotient maps, will give rise to infinite chains of convergent ultrafilters.

To make the description as simple as possible, we introduce the category of ultrarelational spaces, whose particularity – that distinguishes them from pseudotopological spaces – is the fact that principal ultrafilters do not need to converge.

Definition 3.1. An *ultrarelation* on a set X is a subset $r \subseteq \mathcal{U}(X) \times X$. An *ultrarelational space* is a set X equipped with an ultrarelation r on X. Given ultrarelational spaces (X, r) and (Y, s), a map $f : X \to Y$ is *continuous* if, for each $(\mathfrak{a}, x) \in r$, $(f(\mathfrak{a}), f(x)) \in s$.
We denote by **URS** the category of ultrarelational spaces and continuous maps.

We will often use the more suggestive notation $\mathfrak{a} \to x$ instead of $(\mathfrak{a}, x) \in r$.

The category **URS** is equipped with a canonical faithful functor $|\_| : \mathbf{URS} \to \mathbf{Set}$ sending (X, r) to X. The construct $(\mathbf{URS}, |\_|)$ is topological (in the sense of [1]) and therefore concretely complete and cocomplete. It contains **Top** as a full and concrete subcategory: each topology τ on X defines an ultrarelation $r_{(X,\tau)}$ by

$$r_{(X,\tau)} = \{(\mathfrak{a}, x) \mid \mathfrak{a} \to x \text{ w.r.t. the topology } \tau\}.$$

We remark that the notation introduced for points, $y \to x$ if $x \in \overline{\{y\}}$, is consistent with the notation for ultrafilters: $y \to x$ if and only if $\eta(y) \to x$, where $\eta(y)$ denotes the principal ultrafilter defined by y.

For an ultrarelational space (X, r), we consider the projection map $p_{(X,r)} : r \to X$, with $(\mathfrak{a}, x) \mapsto x$, and define the following ultrarelation $R_{(X,r)}$ on r:

$$R_{(X,r)} = \{(\mathfrak{A}, (\mathfrak{a}, x)) \in \mathcal{U}(r) \times r \,|\, p_{(X,r)}(\mathfrak{A}) = \mathfrak{a}\}.$$

We denote the ultrarelational space $(r, R_{(X,r)})$ by $\mathrm{Ult}(X, r)$. Obviously, the map $p_{(X,r)} : \mathrm{Ult}(X, r) \to (X, r)$ is continuous, by definition of the ultrarelational structure on r.

Some extra conditions on these spaces will give us back well-known structures:

Definition 3.2. An ultrarelational space (X, r) is called

1. *weak reflexive* if, for each $x \in X$, there exists an $\mathfrak{a} \in \mathcal{U}(X)$ such that $(\mathfrak{a}, x) \in r$;
2. *reflexive* if, for each $x \in X$, $(\dot{x}, x) \in r$;
3. *fibre-closed* if, for each $x \in X$, $\{\mathfrak{a} \in \mathcal{U}(X) \,|\, (\mathfrak{a}, x) \in r\}$ is closed in $\mathcal{U}(X)$ with respect to the Zariski topology;
4. *transitive* if the map

$$\mu_{(X,r)} : R_{(X,r)} \to \mathcal{U}(X) \times X,\ (\mathfrak{A}, (\mathfrak{a}, x)) \mapsto (\bigcup_{\mathcal{A} \in \mathfrak{A}} \bigcap_{(\mathfrak{a}', x') \in \mathcal{A}} \mathfrak{a}', x)$$

   factors via the inclusion $r \hookrightarrow \mathcal{U}(X) \times X$.

An ultrarelational space (X,r) is weak reflexive if and only if $p_{(X,r)}:\mathrm{Ult}(X,r)\to(X,r)$ is surjective. Hence $\mathrm{Ult}(X,r)$ is weak reflexive provided that (X,r) is: for an $(\mathfrak{a},x)$, $p^{-1}_{(X,r)}(\mathfrak{a})$ is a filter base and any ultrafilter $\mathfrak{A}$ in $\mathrm{Ult}(X,r)$ containing it converges to $(\mathfrak{a},x)$. Moreover, $\mathrm{Ult}(X,r)$ is always fibre-closed.

The choice of the name transitive needs some justification. Assume that a chain $x_2\to x_1\to x_0$ in X is given. Hence $(\mathfrak{A},(\mathfrak{a},x_0))\in R_{(X,r)}$ with $\mathfrak{a}=\eta(x_1)$ and $\mathfrak{A}=\eta(\eta(x_2),x_1)$. Then

$$\eta(x_2)\subseteq\bigcup_{\mathcal{A}\in\mathfrak{A}}\ \bigcap_{(\mathfrak{a}',x')\in\mathcal{A}}\mathfrak{a}'$$

and therefore $\mu_{(X,r)}(\eta(\eta(x_2),x_1),(\eta(x_1),x_0))=(\eta(x_2),x_0)$.

These properties define full subcategories of **URS**: the category **PsTop** (**PrTop**; **Top**) of pseudotopological (pretopological; topological) spaces is concretely isomorphic to the full subcategory of **URS** consisting of all reflexive (reflexive and fibre-closed; reflexive and transitive) ultrarelational spaces.

4. The functor Ult

Each ultrarelational continuous map $f:(X,r)\to(Y,s)$ induces a map

$$\mathrm{Ult}(f):r\to s,\ (\mathfrak{a},x)\mapsto(f(\mathfrak{a}),f(x)),$$

and the diagram

$$\begin{tikzcd} r \arrow[r, "\mathrm{Ult}(f)"] \arrow[d, "p_{(X,r)}"'] & s \arrow[d, "p_{(Y,s)}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. It is then clear that $\mathrm{Ult}(f):\mathrm{Ult}(X,r)\to\mathrm{Ult}(Y,s)$ is continuous. Moreover, the equalities $\mathrm{Ult}(1)=1$ and $\mathrm{Ult}(f\circ g)=\mathrm{Ult}(f)\circ\mathrm{Ult}(g)$ hold, hence $\mathrm{Ult}:\mathbf{URS}\to\mathbf{URS}$ is a functor and $(p_{(X,r)})_{(X,r)\in\mathsf{Ob}\,\mathbf{URS}}:\mathrm{Ult}\to\mathrm{Id}_{\mathbf{URS}}$ is a natural transformation.

Note that we have $\mathrm{Ult}(p_{(X,r)})=p_{\mathrm{Ult}(X,r)}$ for each ultrarelational space (X,r), that is, (Ult,p) is a *well copointed endofunctor* (see [6]).

Hence, we may define endofunctors Ult^α and natural transformations p^α_β for ordinal numbers α,β with $\beta\le\alpha$, by:

- $\mathrm{Ult}^0=\mathrm{Id}_{\mathbf{URS}}$, $p^0_0=1_{\mathrm{Ult}^0}$;
- $\mathrm{Ult}^{\alpha+1}=\mathrm{Ult}(\mathrm{Ult}^\alpha)$, $p^{\alpha+1}_\beta=p^\alpha_\beta\cdot p\mathrm{Ult}^\alpha$ and $p^{\alpha+1}_{\alpha+1}=1_{\mathrm{Ult}^{\alpha+1}}$, for $\beta\le\alpha$;
- $\mathrm{Ult}^\lambda=\lim\limits_{\beta\le\alpha<\lambda}p^\alpha_\beta$, $p^\lambda_\beta=$ the limit projection and $p^\lambda_\lambda=1_{\mathrm{Ult}^\lambda}$, for every limit ordinal λ and every $\beta<\lambda$.

From now on, since we usually work with only one ultrarelation on a set X, for an ultrarelational space we relax our notation and write X instead of (X,r). Also, we will denote $\mathrm{Ult}^\alpha(X)$ by X_α and $\mathrm{Ult}^\alpha(f)$ by f_α, for every continuous map $f:X\to Y$ between ultrarelational spaces.

This transfinite construction can be easily described: for each ultrarelational space X and each ordinal α,

$$X_\alpha=\{((\mathfrak{a}_\beta)_{\beta\in\alpha},x)\in\prod_{\beta\in\alpha}\mathfrak{U}(X_\beta)\times X\mid\mathfrak{a}_0\to x\text{ and }(\forall\gamma\le\beta<\alpha)\ (p^\beta_\gamma)_X(\mathfrak{a}_\beta)=\mathfrak{a}_\gamma\},$$

for each $\beta\le\alpha$, the projection $(p^\alpha_\beta)_X:X_\alpha\to X_\beta$ is defined by

$$(p^\alpha_\beta)_X((\mathfrak{a}_\gamma)_{\gamma\in\alpha},x)=((\mathfrak{a}_\gamma)_{\gamma\in\beta},x),$$

and the ultrarelational structure in X_α is defined by

$$\mathfrak{a}_\alpha\to((\mathfrak{a}_\beta)_{\beta\in\alpha},x)\iff(\forall\beta\in\alpha)\ (p^\alpha_\beta)_X(\mathfrak{a}_\alpha)=\mathfrak{a}_\beta.$$

Finally, if $f : X \to Y$ is a continuous map, then, for each ordinal α and each $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$, $f_\alpha((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) = ((f_\beta(\mathfrak{a}_\beta))_{\beta\in\alpha}, f(x))$.

We remark that, for each ordinal α and each ultrarelational space X, an element of $X_{\alpha+1}$ is given by an ultrafilter $\mathfrak{a}_\alpha \in \mathcal{U}(X_\alpha)$ and an element $x \in X$ such that $(p_0^\alpha)_X(\mathfrak{a}_\alpha) \to x$. The map $f_{\alpha+1} : X_{\alpha+1} \to Y_{\alpha+1}$ is surjective if and only if, for each ultrafilter $\mathfrak{b}_\alpha$ on Y_α and each $y \in Y$ such that $(p_0^\alpha)_Y(\mathfrak{b}_\alpha) \to y$, there exist an ultrafilter $\mathfrak{a}_\alpha$ on X_α and an $x \in f^{-1}(y)$ such that $(p_0^\alpha)_X(\mathfrak{a}_\alpha) \to x$ and $f_\alpha(\mathfrak{a}_\alpha) = \mathfrak{b}_\alpha$.

Hence, by Theorem 2.3, if X and Y are topological spaces, a continuous map $f : X \to Y$ is a biquotient map if and only if f_1, and then also f_0, is surjective. Next we will show that this kind of surjectivity condition characterizes also effective descent and triquotient maps in **Top**. Therefore, we introduce the following

Definition 4.1. If α is an ordinal number, an ultrarelational continuous map $f : X \to Y$ is said to be *α-surjective* if $f_\beta : X_\beta \to Y_\beta$ is surjective for every $\beta \in \alpha$. The map f is called *Ω-surjective* if f_α is surjective for every ordinal α.

Hence, 1-surjective are just surjective maps, while our observation above means that a biquotient map in **Top** is a 2-surjective map.

5. 3-SURJECTIVE MAPS

The continuous maps $f : X \to Y$ between topological spaces such that f_2 (and then also f_0 and f_1) is surjective are very well-known: they are exactly the effective descent maps in **Top**, as we show below. For that we will make use of Reiterman-Tholen characterization (Theorem 2.3). We first start showing that the data they used may be easily interpreted using the functor Ult.

Lemma 5.1. *If Y is a topological space and $\mathcal{F}_Y = \{(I, \mathfrak{u}, (\mathfrak{f}_i), (y_i), y) \,|\, \mathfrak{u}$ ultrafilter on I, $\mathfrak{f}_i \to y_i$ and $y_i \xrightarrow{\mathfrak{u}} y\}$, there are maps $\Phi : Y_2 \to \mathcal{F}_Y$ and $\Psi : \mathcal{F}_Y \to Y_2$ such that $\Psi \cdot \Phi = 1_{Y_2}$.*

Proof. For any $(\mathfrak{B}, (\mathfrak{b}, y))$ in Y_2, $\mathfrak{B}$ is an ultrafilter on Y_1 and, for $\phi = p : Y_1 \to Y$,

$$\Phi(\mathfrak{B}, (\mathfrak{b}, y)) := (Y_1, \mathfrak{B}, (\mathfrak{f})_{(\mathfrak{f},y')\in Y_1}, (y')_{(\mathfrak{f},y')\in Y_1}, y)$$

belongs to $\mathcal{F}_Y$ since $(\mathfrak{f}, y') \in Y_1$, that is $\mathfrak{f} \to y'$, and $y' \xrightarrow{\mathfrak{B}} y$ by the definition of the ultrarelational structure on Y_1.

On the other hand, if $(I, \mathfrak{u}, (\mathfrak{f}_i), (y_i), y) \in \mathcal{F}_Y$, with $\phi : I \to Y$ inducing $y_i \xrightarrow{\mathfrak{u}} y$, for $\psi : I \to Y_1$ defined by $\psi(i) = (\mathfrak{f}_i, y_i)$, we may define

$$\Psi(I, \mathfrak{u}, (\mathfrak{f}_i), (y_i), y) := (\psi(\mathfrak{u}), (\phi(\mathfrak{u}), y)),$$

and it is easy to check that $\Psi \cdot \Phi = 1_{Y_2}$. $\square$

Theorem 5.2. *A topological continuous map $f : X \to Y$ is effective descent if and only if it is 3-surjective, that is:*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A} \arrow[r, dashed] \arrow[d, dash] & \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\ \mathfrak{B} \arrow[r] & \mathfrak{b} \arrow[r] & y \end{tikzcd}$$

Proof. Assume first that f_2 is surjective and let I be an index set, $\mathfrak{b}_i$ $(i \in I)$ be a family of ultrafilters on Y converging to y_i and $y_i \xrightarrow{\mathfrak{u}} y$ with $\mathfrak{u}$ ultrafilter on I. Considering its corresponding element $(\mathfrak{B}, (\mathfrak{b}, y))$ in Y_2, since f_2 is surjective there exist an element $x \in f^{-1}(y)$ and ultrafilters $\mathfrak{a}$ on X and $\mathfrak{A}$ on X_1 such that

$$\mathfrak{a} \to x,\ \mathfrak{A} \to (\mathfrak{a}, x),\ f_1(\mathfrak{A}) = \mathfrak{B}.$$

Hence we have, for each $U \in \mathfrak{u}$,

$$\bigcup_{i\in U}(f^{-1}(y_i)\cap \mathsf{adh}(f^{-1}(\mathfrak{b}_i))) = p_X(\mathrm{Ult}(f)^{-1}(\psi(U))) \in \mathfrak{a}.$$

Assume now that f is effective descent. Let $(\mathfrak{B},(\mathfrak{b},y)) \in Y_2$. For its corresponding data $(I,\mathfrak{u},(\mathfrak{f}_i),y_i,y)$, since f is effective descent, there exist an ultrafilter $\mathfrak{a}$ on X and an element $x \in f^{-1}(y)$ such that $\mathfrak{a} \to x$ and, for each $\mathcal{B} \in \mathfrak{B}$,

$$p_X(\mathrm{Ult}(f)^{-1}(\mathcal{B})) = \bigcup_{(\mathfrak{b}',y')\in\mathcal{B}}(f^{-1}(y')\cap \mathsf{adh}(f^{-1}(\mathfrak{b}'))) \in \mathfrak{a}.$$

Hence $\mathrm{Ult}(f)^{-1}(\mathfrak{B}) \cup p_X^{-1}(\mathfrak{a})$ induces a filter on X_1 which can be refined to an ultrafilter $\mathfrak{A}$, that clearly satisfies the conditions $\mathfrak{A} \to (\mathfrak{a},x)$ and $f_1(\mathfrak{A}) = \mathfrak{B}$. □

From an argument like the one used in [9] for the ultrafilter monad, one can prove the following:

Lemma 5.3. *For each ultrarelational continuous map $f : (X,r) \to (Y,s)$, the diagram*

$$\begin{tikzcd}
R_{(X,r)} \arrow[r, "\mathrm{Ult}^2(f)"] \arrow[d, "\mu_{(X,r)}"'] & R_{(Y,s)} \arrow[d, "\mu_{(Y,s)}"] \\
\mathfrak{U}(X)\times X \arrow[r, "\mathfrak{U}f\times f"] & \mathfrak{U}(Y)\times Y
\end{tikzcd}$$

commutes. □

Now the "Key Lemma 4.1" of [13] is an obvious consequence of the result above:

Corollary 5.4. *Let $f : (X,r) \to (Y,s)$ be a continuous map such that f_2 is surjective. If (X,r) is transitive, then so is (Y,s).* □

To prove the "Key Lemma 4.2" of [13] using our techniques, one uses the following

Proposition 5.5. *For every pullback diagram in* **URS**

$$\begin{tikzcd}
X\times_Z Y \arrow[r, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd} \tag{1}$$

the canonical map $k : (X\times_Z Y)_1 \to X_1 \times_{Z_1} Y_1$ is a perfect surjection.

Proof. Consider the diagram

$$\begin{tikzcd}
(X\times_Z Y)_1 \arrow[rr, "\rho_1"] \arrow[dr, "k"] \arrow[dd, "\pi_1"'] & & Y_1 \arrow[dd, "g_1"] \\
& X_1\times_{Z_1} Y_1 \arrow[ur, "\rho'"'] \arrow[dl, "\pi'"] & \\
X_1 \arrow[rr, "f_1"] & & Z_1
\end{tikzcd}$$

and let $\mathfrak{A}$ be a filter on $(X\times_Z Y)_1$ such that $k(\mathfrak{A}) \to ((\mathfrak{a},x),(\mathfrak{b},y))$ in $X_1\times_{Z_1} Y_1$. Hence, $\pi_1(\mathfrak{A}) \to (\mathfrak{a},x)$ and $\rho_1(\mathfrak{A}) \to (\mathfrak{b},y)$, therefore, for $\mathfrak{c} := p_{X\times_Z Y}(\mathfrak{A})$, $\mathfrak{c} \to (x,y)$ in $X\times_Z Y$, since $\pi(\mathfrak{c}) = p_X(\pi_1(\mathfrak{A})) = \mathfrak{a} \to x$ and $\rho(\mathfrak{c}) = p_Y(\rho_1(\mathfrak{A})) = \mathfrak{b} \to y$. This means that $\mathfrak{A} \to (\mathfrak{c},(x,y))$. Since $k(\mathfrak{c},(x,y)) = ((\mathfrak{a},x),(\mathfrak{b},y))$, and $(\mathfrak{c},(x,y))$ is the only possible choice, k is perfect as claimed. The surjectivity of k follows from the fact that each pullback diagram satisfies Beck-Chevalley condition (see [12]). □

Now, observing that, given a pullback diagram (1), in the diagrams below

$$\begin{tikzcd}
(X\times_Z Y)_1 \arrow[rr, "\rho_1"] \arrow[dr, "k"] \arrow[dd, "\pi_1"'] & & Y_1 \arrow[dd, "g_1"] \\
& X_1\times_{Z_1} Y_1 \arrow[ur, "\rho'"'] \arrow[dl, "\pi'"] & \\
X_1 \arrow[rr, "f_1"] & & Z_1
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\times_Z Y)_2 \arrow[drrr, "\rho_2"] \arrow[dr, "k_1"] \arrow[dddr, "\pi_2"'] & & & \\
& (X_1\times_{Z_1} Y_1)_1 \arrow[rr, "\rho_1'"] \arrow[dr, "k'"] \arrow[dd, "\pi_1'"'] & & Y_2 \arrow[dd, "g_2"] \\
& & X_2\times_{Z_2} Y_2 \arrow[ur, "\rho''"'] \arrow[dl, "\pi''"] & \\
& X_2 \arrow[rr, "f_2"] & & Z_2
\end{tikzcd}$$

$\rho_1 = \rho' \cdot k$, $\rho_2 = \rho'' \cdot k' \cdot k_1$, where k, k' are perfect surjections, $k_1 = \mathrm{Ult}(k)$ is surjective and ρ' (ρ'') is the pullback of f_1 (f_2) along g_1 (g_2), we conclude that:

Corollary 5.6. *3-surjective maps are pullback-stable.* $\square$

6. Ω-surjective maps

We are now going to characterize topological triquotient maps inside **URS** as the Ω-surjective maps. First we state an auxiliary result.

Lemma 6.1. *Let X be a weak reflexive ultrarelational space. Then, for each ordinal α, X_α is weak reflexive and $(p^\alpha_\beta)_X$ is a surjection for each $\beta \leq \alpha$.*

Proof. It follows immediately from the preservation of weak reflexivity by Ult and from the construction of X_λ for every limit ordinal λ. $\square$

Proposition 6.2. *Let $f : X \to Y$ be a topological continuous map together with a map $(\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y$ satisfying (T1) and (T4)*[1]*. Then, for each ordinal α and each $U \in \mathcal{O}X$, $(p^\alpha_0)^{-1}(U^\sharp) \subseteq f_\alpha((p^\alpha_0)^{-1}(U))$.*

Proof. For $\alpha = 0$, the assertion follows from the fact that $U^\sharp \subseteq f(U)$ for each $U \in \mathcal{O}X$. For $\alpha > 0$ assume that the condition above holds for each $\beta \in \alpha$. Let $U \in \mathcal{O}X$, $y \in U^\sharp$ and $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in Y_\alpha$. We define

$$\Sigma = \{S \in \mathcal{O}X \mid \exists\beta \in \alpha : f_\beta((p^\beta_0)^{-1}_X(S)) \notin \mathfrak{b}_\beta\}.$$

Σ is directed since all $\mathfrak{b}_\beta$ ($\beta \in \alpha$) are ultrafilters and all $(p^\beta_\gamma)_X$ ($\gamma \leq \beta < \alpha$) are surjective. We are now going to show that $y \notin S^\sharp$ for each $S \in \Sigma$. Assume that $y \in S^\sharp$ for some $S \in \Sigma$. Then we have $S^\sharp \in \mathfrak{b}_0$ and therefore, for all $\beta \in \alpha$, $(p^\beta_0)^{-1}_Y(S^\sharp) \in \mathfrak{b}_\beta$. But this is impossible since, by induction hypothesis, we have

$$(p^\beta_0)^{-1}_Y(S^\sharp) \subseteq f_\beta((p^\beta_0)^{-1}_X(S)) \notin \mathfrak{b}_\beta.$$

By (T4), there exists $x \in f^{-1}(y) \cap U$ such that, for all $S \in \Sigma$, $x \notin S$. Hence for each $V \in \mathcal{O}(x)$ and each $\beta \in \alpha$ we have $f_\beta((p^\beta_0)^{-1}_X(V)) \in \mathfrak{b}_\beta$ and therefore $(p^\beta_0)^{-1}_X(\mathcal{O}(x)) \cup f^{-1}_\beta(\mathfrak{b}_\beta)$ induces a filter $\mathfrak{f}_\beta$ on X_β. For each $\beta \in \alpha$ we put

$$\mathcal{M}_\beta = \{\mathfrak{a} \in \mathcal{U}(X_\beta) \mid \mathfrak{a} \supseteq \mathfrak{f}_\beta\}.$$

Each $\mathcal{M}_\beta$ ($\beta \in \alpha$) is non-empty and Zariski-closed, hence, since a codirected limit of non-empty compact Hausdorff spaces is non-empty (cf. [2]), there exists $(\mathfrak{a}_\beta)_{\beta\in\alpha}$ with $\mathfrak{a}_\beta$ ultrafilter on X_β such that $(p^\beta_{\beta'})_X(\mathfrak{a}_\beta) = \mathfrak{a}_{\beta'}$ for all $\beta' \leq \beta \in \alpha$. We have by definition $\mathfrak{a}_0 \supseteq \mathcal{O}(x)$ and $f_\beta(\mathfrak{a}_\beta) = \mathfrak{b}_\beta$ for all $\beta \in \alpha$, hence $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$ and $f_\alpha((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) = ((\mathfrak{b}_\beta)_{\beta\in\alpha}, y)$. $\square$

Since this shows in particular that every triquotient map (between topological spaces) is Ω-surjective, we conclude immediately that triquotient maps are effective descent.

For a set Y, let λ_Y be the least regular cardinal larger than the cardinal of Y.

[1] See Definition 2.1.

Proposition 6.3. *Let $f : X \to Y$ be a topological continuous map. Then, for each $U \in \mathcal{O}X$, the set*

$$U^\sharp = \{y \in Y \mid (\forall \alpha \in \lambda_Y)\, (p_0^\alpha)_Y^{-1}(y) \subseteq f_\alpha((p_0^\alpha)_X^{-1}(U))\tag{2}$$

is open and the map $(\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y$ satisfies (T1) and (T4).

Proof. First we show that $U^\sharp$ is open for every $U \in \mathcal{O}X$. For that, let $y_0 \in \mathsf{cl}(Y - U^\sharp)$. There exists an ultrafilter $\mathfrak{b}_0$ on Y converging to y_0 such that $Y - U^\sharp \in \mathfrak{b}_0$. For each $y \in Y - U^\sharp$ there exist $\alpha_y \in \lambda_Y$ and $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in Y_{\alpha_y}$ such that

$$\nexists((\mathfrak{a}_\beta)_{\beta\in\alpha_y}, x) \in X_{\alpha_y} : (x \in U \,\wedge\, f_\alpha((\mathfrak{a}_\beta)_{\beta\in\alpha_y}, x) = ((\mathfrak{b}_\beta)_{\beta\in\alpha_y}, y),$$

by definition of $U^\sharp$. For $\alpha := \sup_{y\in(Y-U^\sharp)} \alpha_y$, we have $\alpha \in \lambda_Y$. Considering $\mathcal{B} = (p_0^\alpha)_Y^{-1}(Y - U^\sharp) - f_\alpha((p_0^\alpha)_X^{-1}(U))$, one has $(p_0^\alpha)_Y(\mathcal{B}) = Y - U^\sharp \in \mathfrak{b}_0$. Let $\mathfrak{b}_\alpha$ be any ultrafilter containing $\{\mathcal{B}\} \cup (p_0^\alpha)_Y^{-1}(\mathfrak{b}_0)$. Assume that there exist an ultrafilter $\mathfrak{a}_\alpha$ on X_α and an element $x_0 \in f^{-1}(y_0)$ such that $\mathfrak{a}_0 = p_0^\alpha(\mathfrak{a}_\alpha) \to x_0$, $x_0 \in U$ and $f_\alpha(\mathfrak{a}_\alpha) = \mathfrak{b}_\alpha$. Then we have $U \in \mathfrak{a}_0$ and therefore $(p_0^\alpha)_X^{-1}(U) \in \mathfrak{a}_\alpha$. Hence $f_\alpha((p_0^\alpha)_X^{-1}(U)) \cap \mathcal{B} \neq \emptyset$ which contradicts the definition of $\mathcal{B}$. Therefore, we have proved that $y_0 \in Y - U^\sharp$.

This means that (2) defines a map $(\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y$, that satisfies obviously (T1). So it remains to show that it also satisfies (T4). Let $U \in \mathcal{O}X$, $y \in U^\sharp$ and $\Sigma \subseteq \mathcal{O}X$ directed such that, for each $S \in \Sigma$, $y \notin S^\sharp$. There exists an ordinal $\alpha \in \lambda_Y$ such that, for each $S \in \Sigma$, the set $\mathcal{B}_S = (p_0^\alpha)_Y^{-1}(y) - f_\alpha((p_0^\alpha)_X^{-1}(S))$ is non-empty. Since Σ is filtered, $\{\mathcal{B}_S \mid S \in \Sigma\}$ forms a filter base. Let $\mathfrak{b}_\alpha$ be any ultrafilter containing $\{\mathcal{B}_S \mid S \in \Sigma\} \cup (p_0^\alpha)_Y^{-1}(\eta(y))$. Since $y \in U^\sharp$, there exist an ultrafilter $\mathfrak{a}_\alpha$ on X_α and an element $x \in U$ such that $\mathfrak{a}_0 = (p_0^\alpha)_X(\mathfrak{a}_\alpha) \to x$, $f(x) = y$ and $f_\alpha(\mathfrak{a}_\alpha) = \mathfrak{b}_\alpha$. Let $V \in \mathcal{O}(x)$. Then $V \in \mathfrak{a}_0$ and therefore $(p_0^\alpha)_X^{-1}(V) \in \mathfrak{a}_\alpha$. Hence $f_\alpha((p_0^\alpha)_X^{-1}(V)) \cap \mathcal{B}_S \neq \emptyset$ and therefore $V \neq S$, for each $S \in \Sigma$. Hence, Σ does not cover $f^{-1}(y) \cap U$, and (T4) follows. ☐

This proposition gives a general way of defining a map $(\_)^\sharp$ – in fact, the largest possible one –, as required in the definition of triquotient map, but that in general does not satisfy (T2): $X^\sharp = Y$. By the definition of $(\_)^\sharp$ it is clear what this condition means: f is λ_Y-surjective; that is:

$$\begin{tikzcd}
X \arrow[d, "f"'] & \cdots \arrow[r, dashed] & \mathfrak{a}_{\alpha+1} \arrow[r, dashed] \arrow[d, dash] & \mathfrak{a}_\alpha \arrow[r, dashed] \arrow[d, dash] & \cdots \arrow[r, dashed] & \mathfrak{a}_1 \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\
Y & \cdots \arrow[r] \arrow[u, dash] & \mathfrak{b}_{\alpha+1} \arrow[r] & \mathfrak{b}_\alpha \arrow[r] & \cdots \arrow[r] \arrow[u, dash] & \mathfrak{b}_1 \arrow[r] & y
\end{tikzcd}$$

Hence, we may now state the characterization of topological triquotient maps.

Theorem 6.4. *Let $f : X \to Y$ be a continuous map between topological spaces. The following conditions are equivalent:*

(i) *f is a triquotient map;*
(ii) *f is Ω-surjective;*
(iii) *f is λ_Y-surjective.* ☐

In the finite case, since all ultrafilters are fixed, X_n may be described as the set of all $(n+1)$-chains $x_n \to \cdots \to x_0$ of elements of X. The ultrarelational structure is then described by

$$(x_n, \cdots, x_1, x_0) \to (x'_n, \cdots, x'_1, x'_0) \;:\Longleftrightarrow\; (x_{n-1}, \cdots, x_0) = (x'_n, \cdots, x'_1).$$

From Theorem 6.4, we know that, if X and Y are finite, then $f : X \to Y$ is a triquotient map if and only if f_n is surjective for every $n \in \mathbb{N}$, which is exactly Theorem I.

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and concrete categories* (Wiley Interscience, New York 1990).

[2] N. Bourbaki, *Topologie Générale* (Hermann, Paris 1961).

[3] M. M. Clementino, On finite triquotient maps, *J. Pure Appl. Algebra* (to appear).

[4] B. J. Day and G. M. Kelly, On topological quotient maps preserved by pullbacks or products, *Proc. Cambridge Philos. Soc.* **67** (1970) 553-558.

[5] O. Hájek, Notes on quotient maps, *Comment. Math. Univ. Carolinae* **7** (1966) 319-323.

[6] G. M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bull. Austral. Math. Soc.* **22** (1980) 1-83.

[7] G. Janelidze and M. Sobral, Finite preorders and topological descent, Preprint 99-10, Department of Mathematics, University of Coimbra (1999).

[8] G. Janelidze and W. Tholen, How algebraic is the change of base functor?, in: *Proceedings Conference Category Theory*, Como 1990, Lecture Notes in Mathematics **1488** (Springer, Berlin 1991) 174-186.

[9] E.G. Manes, A triple-theoretic construction of compact algebras, in: *Seminar on Triples and Categorical Homology Theory*, Lecture Notes in Mathematics **80** (Springer, Berlin 1969), pp. 91-118.

[10] E. Michael, Bi-quotient maps and cartesian products of quotient maps, *Ann. Inst. Fourier*, Grenoble **18** (1968) 287-302.

[11] E. Michael, Complete spaces and tri-quotient maps, *Illinois J. of Math.* **21** (1977), 716-733.

[12] T. Plewe, Localic triquotient maps are effective descent morphisms, *Math. Proc. Cambridge Philos. Soc.* **122** (1997), 17-43.

[13] J. Reiterman and W. Tholen, Effective descent maps of topological spaces, *Top. Appl.* **57** (1994), 53-69.

Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal
E-mail address: `mmc@mat.uc.pt, dirk@mat.uc.pt`