THE UNDECIDABILITY OF THE TRACE CODING PROBLEM AND SOME DECIDABLE CASES*

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Abstract. We introduce the notion of a weak morphism of trace semigroups and use it to deal with the problem of deciding the existence of a coding between trace semigroups. We prove that this problem is in general undecidable thereby answering the question raised by Ochmański in 1988. We also show its decidability when the first dependence alphabet is a forest and some facts about the existence of weak codings.

1. Introduction and Notations

In [8] Mazurkiewicz proposed trace semigroups, introduced earlier by Cartier and Foata [4] in combinatorics, for modeling concurrent processes. In this context, the notion of a coding of trace semigroups becomes particularly important since it corresponds to a simulation of concurrent systems. In [9] Ochmański raised a problem to find an algorithm deciding for each pair of trace semigroups whether there is a coding between them. In the above view, this is equivalent to deciding for two concurrent systems if one of them can be simulated by the other. Up to now, only partial decidability results were obtained in [1], [2] and [6]. The problem was completely solved just for the case of strong morphisms (defined in [3]) by Diekert et al. [6].

We use another notion of a weak morphism to deal with this problem. We prove that the corresponding problem for weak morphisms is even more complex than the original one and show its decidability for some classes of semigroups. On the other hand, we construct weak codings showing that both problems are in general undecidable.

The paper is organized as follows. In Section 2 we recall basic notions and known results which will be used throughout the paper. For a more comprehensive introduction see for instance [5]. Notice that some definitions and formulations of results are modified to fit our purposes. Further we introduce the notion of a weak morphism with some simple observations about it presented in Section 3. The aim of Section 4 is to construct a reduction of the trace coding problem to the weak trace coding problem. Some special cases for which the weak trace coding problem is decidable are considered in Section 5. The undecidability result is proved in Sections 6 and 7. First, we encode the Post's correspondence problem (PCP) into the problem of deciding whether there exists a weak morphism with prescribed

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contents of images of letters. Then we show that these content requirements can be replaced by an enrichment of the independence alphabets and finally we use the result from Section 4 in the reverse direction to get a reduction of the PCP to the trace coding problem. Section 8 summarizes main results on the subject.

Let Σ be a finite set. We denote by Σ^+ the semigroup of non-empty words over Σ. In this context elements of Σ are often called letters. Let $\mathsf{c} : \Sigma^+ \to 2^\Sigma$ be the content mapping and $\preceq$ the prefix relation on Σ^+. Let $u \in \Sigma^+$. We denote by $\mathsf{h}\,(u)$ the first letter of u, by $\mathsf{t}\,(u)$ the last one, by $|u|$ the length of u and by $\overleftarrow{u}$ its left-right mirror image. For $X \subseteq \Sigma$, let $\pi_X\,(u)$ be the projection of u to X^+ and $|u|_X = |\pi_X\,(u)|$ (often we write instead of X a list of its elements).

The cardinality of a set A is denoted $|A|$. For an $m \times n$-matrix K, $M \subseteq \{1, \ldots, m\}$ and $N \subseteq \{1, \ldots, n\}$, we mean by $K\,(M, N)$ the submatrix of K consisting of the lines in M and columns in N. For a binary relation ρ, we denote by $\operatorname{sym} \rho$ its symmetric closure.

2. Basic Notions

Definition 1. Let Σ be a finite set and I be a symmetric reflexive binary relation on Σ. We call I an *independence relation* on Σ and the pair (Σ, I) an *independence alphabet*. The complement $D = (\Sigma \times \Sigma) \setminus I$ is called a *dependence relation* and the pair (Σ, D) a *dependence alphabet*.

For the rest of this section, let (Σ, I), (Σ', I') be two independence alphabets. The corresponding dependence relations will be denoted D and D' and we will derive their names in this way also further on. Dependence and independence alphabets will be often considered as graphs. For $x \in \Sigma$, we denote $D\,(x) = \{y \in \Sigma \mid (x, y) \in D\}$ and for $X \subseteq \Sigma$, $D\,(X) = \bigcup_{x \in X} D\,(x)$.

Definition 2. Let $\sim_I$ be the congruence on Σ^+ generated by the relation $\{(xy, yx) \mid (x, y) \in I\}$. The quotient semigroup $\Sigma^+/\sim_I$ will be denoted $\mathbb{T}\,(\Sigma, I)$ and called a *free partially commutative* (or *trace*) semigroup. Elements of this semigroup are called *traces*.

A solution of the word problem for trace semigroups is given by the following so called *Projection Lemma*.

Lemma 1. *Let $u, v \in \Sigma^+$. Then $u \sim_I v$ if and only if $\forall x \in \Sigma : |u|_x = |v|_x$ and $\forall\,(x, y) \in D : \pi_{x,y}\,(u) = \pi_{x,y}\,(v)$. In particular, trace semigroups are cancellative.*

In the sequel we will not distinguish between letters $x \in \Sigma$ and their one-element congruence classes $(x \sim_I)$. As the content and the length of a word and the number of occurrences of a letter in a word are preserved by the congruence $\sim_I$, it makes sense to consider these notions also for traces.

Definition 3. For $s \in \mathbb{T}\,(\Sigma, I)$ we define its *initial* and *final alphabet* by

$$\operatorname{init}\,(s) = \{\mathsf{h}\,(u) \mid u \in s\}\,, \quad \operatorname{fin}\,(s) = \{\mathsf{t}\,(u) \mid u \in s\}\,.$$

Definition 4. A trace $t \in \mathbb{T}(\Sigma, I)$ is called a *subtrace* of $s \in \mathbb{T}(\Sigma, I)$ if there exist $r, r' \in \mathbb{T}(\Sigma, I)$ such that $s = rtr'$. In such case, the triple (r, t, r') is called an *occurrence* of t in s.

Definition 5. Let $x \in \Sigma$, $s \in \mathbb{T}(\Sigma, I)$. An *$x$-block* in s is an occurrence of a subtrace of the form $(t, (x^n)\sim_I, t')$ for some $n \in \mathbb{N}$ and $t, t' \in \mathbb{T}(\Sigma, I)$ satisfying $x \notin \mathrm{fin}(t) \cup \mathrm{init}(t')$.

For $u \in \Sigma^+$, $n \in \mathbb{N}$, let us denote by $u[n]$ the word consisting of the first n blocks of u.

We extend the relation I to $\mathbb{T}(\Sigma, I)$ in the following way.

Definition 6. If $s, t \in \mathbb{T}(\Sigma, I)$ satisfy $\mathsf{c}(s) \times \mathsf{c}(t) \subseteq I$, we call s and t *independent* and write $(s, t) \in I$.

Notice that, for $s, t \in \mathbb{T}(\Sigma, I)$, $(s, t) \in I$ implies $st = ts$.

Lemma 2. *Let $u, v \in \Sigma^+$ satisfy*

$$\forall (x, y) \in D : \underset{x,y}{\pi}(u) \preceq \underset{x,y}{\pi}(v) \ \text{ or } \ \underset{x,y}{\pi}(v) \preceq \underset{x,y}{\pi}(u)\,. \tag{1}$$

Let u' and v' be the words resulting from u and v if we take just the first $\min\{|u|_x, |v|_x\}$ *occurrences of the letter x for all $x \in \Sigma$. Then $u' \sim_I v'$ and there exist (up to $\sim_I$ unique) words $v \backslash u, u \backslash v \in \Sigma^+$ such that*

$$u \sim_I u' \cdot (v \backslash u)\,, \qquad v \sim_I v' \cdot (u \backslash v)\,, \qquad \mathsf{c}(v \backslash u) \times \mathsf{c}(u \backslash v) \subseteq (I \setminus id_\Sigma)$$

and consequently also $u \cdot (u \backslash v) \sim_I v \cdot (v \backslash u)$. ☐

From the previous lemma we immediately obtain

Lemma 3. *Let $u, v \in \Sigma^+$. Then there exist $s, t \in \Sigma^+$ such that $us \sim_I vt$ if and only if the condition* (1) *is satisfied.* ☐

Definition 7. A trace $s \in \mathbb{T}(\Sigma, I)$ is called *connected* if the corresponding graph $\left(\mathsf{c}(s), D \cap \mathsf{c}(s)^2\right)$ is connected. It is called *primitive* if it is connected and $s = t^n$, $t \in \mathbb{T}(\Sigma, I)$, implies $n = 1$.

Proposition 1 ([7]). *Every connected trace is a power of a unique primitive trace.*

If a connected trace $s \in \mathbb{T}(\Sigma, I)$ is a power of a primitive trace t, then t is called a *primitive root* of s. It is clear that every trace $s \in \mathbb{T}(\Sigma, I)$ can be uniquely decomposed into a product of independent connected traces called *connected components* of s. Let us denote by $\mathcal{PR}(s)$ the set of all primitive roots of connected components of s.

Proposition 2 ([7]). *Let $s_1, \dots, s_n \in \mathbb{T}(\Sigma, I)$ satisfy $s_i s_j = s_j s_i$ for all $i, j = 1, \dots, n$. Then the traces in $P = \bigcup_{i=1}^{n} \mathcal{PR}(s_i)$ are pairwise independent, i.e. for all $t, t' \in P$ either $(t, t') \in I$ or $t = t'$.*

Remark 1. Since trace semigroups are defined by presentations, it is clear that every morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is uniquely determined by a mapping $\varphi' : \Sigma \to (\Sigma')^+$, which satisfies

$$\forall (x, y) \in I : \varphi'(x)\, \varphi'(y) \sim_{I'} \varphi'(y)\, \varphi'(x) . \tag{2}$$

Conversely, every mapping $\varphi' : \Sigma \to (\Sigma')^+$ satisfying (2) extends to a morphism. Further on, we will write, for $u \in \Sigma^+$ and $v \in (\Sigma')^+$, simply $\varphi(u) = v$ instead of $\varphi(u \sim_I) = v \sim_{I'}$; in particular, the mapping φ' will be also denoted by φ.

Definition 8. A morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is called a *coding* if it is injective.

Remark 2. We use trace semigroups $\mathbb{T}(\Sigma, I)$ rather than more common monoids $\mathbb{M}(\Sigma, I)$ with the neutral element 1 adjoint to avoid technical difficulties. It is clear that the main results on codings hold for monoids as well as for semigroups.

Definition 9. A morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is called *strong* if

$$\forall (x, y) \in I \setminus id_\Sigma : \mathsf{c}(\varphi(x)) \times \mathsf{c}(\varphi(y)) \subseteq I' \setminus id_{\Sigma'} . \tag{3}$$

It is called *weak* if $\forall x \in \Sigma : (\varphi(x), \varphi(x)) \in I'$.

Remark 3. The correspondence from Remark 1 works also for strong and weak morphisms if we replace the condition (2) with respectively (3) and

$$\forall (x, y) \in I : (\varphi(x), \varphi(y)) \in I'. \tag{4}$$

Lemma 4. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism and $u \in \Sigma^+$. Then $\varphi(\overleftarrow{u}) \sim_{I'} \overleftarrow{\varphi(u)}$.* ☐

Definition 10. The (*strong, weak*) *trace coding problem* asks to decide for given two independence alphabets (Σ, I) and (Σ', I') whether there exists a (strong, weak) coding $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$. The (*strong, weak*) *trace code problem* asks to decide for a given (strong, weak) morphism $\varphi : (\Sigma, I) \to (\Sigma', I')$ whether it is a coding.

Remark 4. The (strong) trace code problem is well known to be undecidable even in the very special case when $\mathbb{T}(\Sigma, I)$ is free.

A solution for the strong trace coding problem is provided by the following result.

Theorem 1 ([6]). *There is a strong coding $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ if and only if there is a relational morphism of graphs*

$$H : (\Sigma', D' \cup id_{\Sigma'}) \to (\Sigma, D \cup id_\Sigma)$$

surjective on vertices and edges.

Definition 11. Let $\mu, \nu : \Sigma \to 2^{\Sigma'}$ be arbitrary mappings. A weak morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is called (μ, ν)*-weak* whenever it satisfies

$$\forall x \in \Sigma : \mu(x) \subseteq \mathsf{c}(\varphi(x)) \subseteq \nu(x).$$

It is called ν*-weak* if $\forall x \in \Sigma : \mathsf{c}(\varphi(x)) \subseteq \nu(x)$.

3. Weak Codings

In this section we investigate properties of equalities showing that a weak morphism is not a coding.

Lemma 5. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism such that $\forall (x, y) \in D : \varphi(xy) \not\sim_{I'} \varphi(yx)$. If $u, v \in \Sigma^+$ satisfy $\varphi(u) \sim_{I'} \varphi(v)$, then* $\operatorname{init}(u) \times \operatorname{init}(v) \subseteq I$. ☐

As an immediate consequence we get

Lemma 6. *Every weak morphism $\varphi : \Sigma^+ \to \mathbb{T}(\Sigma', I')$ satisfying $\forall (x, y) \in D : \varphi(xy) \not\sim_{I'} \varphi(yx)$ is a coding.* ☐

Lemma 7. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism which is not a coding. Then one of the following cases arises:*

1. *There is $X \subseteq \Sigma$ such that $X \times X \subseteq I$ and the system $(\varphi(x))_{x \in X}$ of elements of the free commutative semigroup generated by $\bigcup_{x \in X} \mathsf{c}(\varphi(x))$ is linearly dependent.*
2. *There exist $u, v \in \Sigma^+$ such that $u \not\sim_I v$, $\varphi(u) \sim_{I'} \varphi(v)$, $u \sim_I$ is connected and $\forall x \in \Sigma : |u|_x = |v|_x$.*

Proof. We can assume $\forall (x, y) \in D : \varphi(xy) \not\sim_{I'} \varphi(yx)$, otherwise the case 2 arises. Take $u, v \in \Sigma^+$ shortest words satisfying $u \not\sim_I v$ and $\varphi(u) \sim_{I'} \varphi(v)$. If $uv \not\sim_I vu$, we prove that the condition 2 is true by taking any connected components of uv and vu with the same content which are not equivalent. So, suppose $uv \sim_I vu$. From Proposition 2 and the minimality of u and v we get $\mathcal{PR}(u \sim_I) \times \mathcal{PR}(v \sim_I) \subseteq I$. If there are $x, y \in \mathsf{c}(uv)$, xDy, there are also $a, b \in \mathsf{c}(\varphi(u)) = \mathsf{c}(\varphi(v))$, $aD'b$. Thus for some $s \in \mathcal{PR}(u \sim_I)$ and $t \in \mathcal{PR}(v \sim_I)$ we have $a \in \mathsf{c}(\varphi(s))$, $b \in \mathsf{c}(\varphi(t))$ which contradicts $(s, t) \in I$. Hence $\mathsf{c}(uv)$ is a clique in (Σ, I) and the case 1 arises. ☐

Lemma 8. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism and $x \in \Sigma$ such that the restriction of φ to $\mathbb{T}\left(\Sigma \setminus \{x\}, I \cap (\Sigma \setminus \{x\})^2\right)$ is a coding. Let $u, v \in \Sigma^+$ satisfy $\varphi(u) \sim_{I'} \varphi(v)$, $|u|_x = |v|_x$ and $\pi_{x,y}(u) = \pi_{x,y}(v)$ for all $y \in \Sigma$ such that $(x, y) \in D$. Then $u \sim_I v$.*

Proof. As letters dependent on $\varphi(x)$ occur only in the images of elements dependent on x, from the equality of projections we can deduce that letters from $\varphi(x)$ for the corresponding occurrences of x in u and v reside on the same positions (with respect to dependent letters) in $\varphi(u)$ and $\varphi(v)$. Hence if we remove these letters, we get $\varphi\left(\pi_{\Sigma \setminus \{x\}}(u)\right) \sim_{I'} \varphi\left(\pi_{\Sigma \setminus \{x\}}(v)\right)$. Since the restriction of φ is a coding, we obtain $\pi_{\Sigma\setminus\{x\}}(u) \sim_I \pi_{\Sigma\setminus\{x\}}(v)$, which is enough to verify $u \sim_I v$. □

Problem 1. Does the previous lemma still hold when the morphism φ is not weak?

Lemma 9. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a coding, $N : \Sigma \to \mathbb{N}$ a mapping. Then the rule $\varphi^N(x) = \varphi(x)^{N(x)}$ for $x \in \Sigma$ defines a coding $\varphi^N : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$.*

Proof. It is clear that $id^N_{\mathbb{T}(\Sigma,I)}$ is a coding and $\varphi^N = \varphi \circ id^N_{\mathbb{T}(\Sigma,I)}$. □

Lemma 10. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism. If φ^N is a coding for some $N : \Sigma \to \mathbb{N}$, then φ is a coding as well.*

Proof. Let $n = \operatorname{lcm}\{N(x) \mid x \in \Sigma\}$ and define $N' : \Sigma \to \mathbb{N}$ by $N'(x) = n/N(x)$ for all $x \in \Sigma$. Then we get a coding $\varphi^N \circ id^{N'}_{\mathbb{T}(\Sigma,I)} = \varphi \circ id^{N_n}_{\mathbb{T}(\Sigma,I)} = id^{N_n}_{\mathbb{T}(\Sigma',I')} \circ \varphi$, where N_n maps all elements of Σ (Σ' respectively) to n. Thus φ is injective. □

Notice that if φ is not weak, the previous lemma does not have to hold:

Example 1. Take $\Sigma = \{x, y, z\}$, $\Sigma' = \{a, b\}$. Let $\varphi : \Sigma^+ \to (\Sigma')^+$ be the morphism given by $\varphi(x) = a$, $\varphi(y) = b$, $\varphi(z) = ab$ and $N(x) = 2$, $N(y) = N(z) = 1$. Then φ^N is a coding, but φ is not.

Lemma 11. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak coding and $A \subseteq \Sigma'$ such that $\forall x \in \Sigma : |\mathsf{c}(\varphi(x)) \cap A| \leq 1$. Then there is a weak coding $\varphi^A : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ satisfying $\forall x \in \Sigma : \mathsf{c}\left(\varphi^A(x)\right) = \mathsf{c}(\varphi(x))$ and $|\varphi^A(x)|_A \leq 1$.*

Proof. Let $n = \operatorname{lcm}\{|\varphi(x)|_A \mid x \in \Sigma,\ \mathsf{c}(\varphi(x)) \cap A \neq \emptyset\}$ and define $N : \Sigma \to \mathbb{N}$ and $N' : \Sigma' \to \mathbb{N}$ by the rules

$$N(x) = \begin{cases} n/|\varphi(x)|_A & \text{if } \mathsf{c}(\varphi(x)) \cap A \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

$$N'(a) = \begin{cases} n & \text{for } a \in A \\ 1 & \text{for } a \notin A. \end{cases}$$

Let φ^A be the morphism defined by

$$\varphi^A(x) = \begin{cases} a \cdot \pi_{\Sigma'\setminus A}\left(\varphi^N(x)\right) & \text{if } a \in \mathsf{c}(\varphi(x)) \cap A \\ \varphi^N(x) & \text{if } \mathsf{c}(\varphi(x)) \cap A = \emptyset. \end{cases}$$

Then by Lemma 9 $\varphi^N = id^{N'}_{\mathbb{T}(\Sigma',I')} \circ \varphi^A$ is a coding and so φ^A must be injective. □

Definition 12. Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism. Let $u, v \in \Sigma^+$. We say that (u, v) is a *semi-equality* (for φ) if $\operatorname{init}(u) \cap \operatorname{init}(v) = \emptyset$

and there exist $s, t \in (\Sigma')^+$ such that $\varphi(u)\, s \sim_{I'} \varphi(v)\, t$. It is called *non-trivial* if there are no words $w, r \in \Sigma^+$ such that $uw \sim_I vr$. The pair $(\varphi(v) \backslash \varphi(u), \varphi(u) \backslash \varphi(v))$ is called the *state* of (u, v).

Lemma 12. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak morphism and let $u, v \in \Sigma^+$ be shortest words satisfying $u \not\sim_I v$ and $\varphi(u) \sim_{I'} \varphi(v)$. If $w \preceq u$ and $r \preceq v$, then (w, r) is a semi-equality.* □

Lemma 13. *Let (u, v) be a non-trivial semi-equality for a weak morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ satisfying*

$$(\mathsf{c}(\varphi(v) \backslash \varphi(u) \cdot \varphi(u) \backslash \varphi(v)))^2 \subseteq I'.$$

Then φ is not a coding.

Proof. From Lemma 4 one can see that $\varphi(u \overleftarrow{v}) \sim_{I'} \varphi(v \overleftarrow{u})$. □

4. Reduction into the Weak Coding Equivalent

In this section we enrich (Σ', I') in order to replace the primitive roots of the images of elements of Σ under a morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ with letters.

For a graph (Σ, D), we denote

$$\mathcal{C}(\Sigma, D) = \left\{ X \subseteq \Sigma \mid |X| \geq 2 \ \&\ \left(X, D \cap X^2\right) \text{ is connected} \right\}.$$

Definition 13. Let (Σ, I) and (Σ', I') be two independence alphabets. We define new independence alphabet $({\Sigma'}_\Sigma, {I'}_\Sigma)$ as follows. Let

$${\Sigma'}_\Sigma = \Sigma' \cup \left(\mathcal{C}(\Sigma', D') \times \Sigma\right)$$

and, for a word $u \in ({\Sigma'}_\Sigma)^+$, write

$$\overline{\mathsf{c}}(u) = \left(\mathsf{c}(u) \cap \Sigma'\right) \cup \bigcup \left\{ A \mid (A, x) \in \mathsf{c}(u) \setminus \Sigma' \right\}$$

and call it the *extended content* of u. Finally, for $\alpha, \beta \in {\Sigma'}_\Sigma$, set

$$(\alpha, \beta) \in {I'}_\Sigma \iff \overline{\mathsf{c}}(\alpha) \times \overline{\mathsf{c}}(\beta) \subseteq I' \text{ or } \alpha = \beta.$$

Remark 5. Notice that if for some word $u \in ({\Sigma'}_\Sigma)^+$ we have $\overline{\mathsf{c}}(u) \times \overline{\mathsf{c}}(u) \subseteq I'$, then in fact $u \in (\Sigma')^+$. As $\sim_{{I'}_\Sigma}$ preserves extended content, the notion of extended content can be used also for traces.

Lemma 14. *Let (Σ, I) and (Σ', I') be two independence alphabets. Let $\alpha, \beta \in {\Sigma'}_\Sigma$, $\alpha {I'}_\Sigma \beta$, $\alpha \neq \beta$, $a \in \overline{\mathsf{c}}(\alpha)$ and $b \in \overline{\mathsf{c}}(\beta)$. Then $aI'b$ and $a \neq b$.* □

Proposition 3. *Let (Σ, I) and (Σ', I') be two independence alphabets. Then there is a coding $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ if and only if there is a weak coding $\psi : \mathbb{T}(\Sigma, I) \to \mathbb{T}({\Sigma'}_\Sigma, {I'}_\Sigma)$.*

Proof. "$\Longrightarrow$" Assume that $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is a coding. For every $A \in \mathcal{C}(\Sigma', D')$ choose any mapping $e_A : \Sigma \to \Sigma$ such that for all $x, y \in \Sigma$

$$e_A(x) = e_A(y) \iff x = y \text{ or } \exists\, s \in \mathcal{PR}(\varphi(x)) \cap \mathcal{PR}(\varphi(y)) : \mathsf{c}(s) = A.$$

Such mappings certainly exist since the condition on the right defines an equivalence relation on Σ. Take an arbitrary letter $x \in \Sigma$. Consider the decomposition of $\varphi(x)$ into its primitive roots

$$\varphi(x) = \left(\prod_{s \in \mathcal{PR}(\varphi(x)) \setminus \Sigma'} s^{i_s} \right) \cdot \left(\pi_{\mathcal{PR}(\varphi(x)) \cap \Sigma'} (\varphi(x)) \right)$$

and define

$$\psi(x) = \left(\prod_{s \in \mathcal{PR}(\varphi(x)) \setminus \Sigma'} \left(\mathsf{c}(s), e_{\mathsf{c}(s)}(x)\right)^{i_s} \right) \cdot \left(\pi_{\mathcal{PR}(\varphi(x)) \cap \Sigma'} (\varphi(x)) \right).$$

We will prove that ψ extends to a desired coding.

First, let us show that it defines a weak morphism. Let $(x, y) \in I$. Then $\varphi(x)\,\varphi(y) \sim_{I'} \varphi(y)\,\varphi(x)$ and by Proposition 2

$$\forall s, t \in \mathcal{PR}(\varphi(x)) \cup \mathcal{PR}(\varphi(y)),\, s \neq t : (s, t) \in I'. \tag{5}$$

Taking arbitrary $a \in \mathsf{c}(\psi(x)) \cap \Sigma'$, $b \in \mathsf{c}(\psi(y)) \cap \Sigma'$, $s \in \mathcal{PR}(\varphi(x)) \setminus \Sigma'$ and $t \in \mathcal{PR}(\varphi(y)) \setminus \Sigma'$, from (5) we immediately deduce that $(a, b) \in {I'}_\Sigma$ and $\left(a, \left(\mathsf{c}(t), e_{\mathsf{c}(t)}(y)\right)\right) \in {I'}_\Sigma$. If $s \neq t$ then $\mathsf{c}(s) \times \mathsf{c}(t) \subseteq I'$ by (5) and if $s = t$ then $e_{\mathsf{c}(s)}(x) = e_{\mathsf{c}(t)}(y)$ by the definition of $e_{\mathsf{c}(s)}$. In both cases we obtain $\left(\left(\mathsf{c}(s), e_{\mathsf{c}(s)}(x)\right), \left(\mathsf{c}(t), e_{\mathsf{c}(t)}(y)\right)\right) \in {I'}_\Sigma$. This verifies that $\mathsf{c}(\psi(x)) \times \mathsf{c}(\psi(y)) \subseteq {I'}_\Sigma$.

Now it remains to show that ψ is injective. We define a strong morphism $\sigma : \mathbb{T}({\Sigma'}_\Sigma, {I'}_\Sigma) \to \mathbb{M}(\Sigma', I')$ as follows. For $a \in \Sigma'$ and $(A, x) \in {\Sigma'}_\Sigma \setminus \Sigma'$

$$\sigma(a) = a$$

$$\sigma((A, x)) = \begin{cases} s & \text{if } \exists\, y \in \Sigma \text{ such that } s \in \mathcal{PR}(\varphi(y)), \mathsf{c}(s) = A, e_A(y) = x \\ 1 & \text{otherwise.} \end{cases}$$

Notice that the second rule is unambiguous due to the definition of e_A. From Definition 13 one can immediately see that σ really extends to a morphism. It is clear that $\varphi = \sigma \circ \psi$ and the injectivity of ψ follows from that of φ.

"$\Longleftarrow$" Let $\psi : \mathbb{T}(\Sigma, I) \to \mathbb{T}({\Sigma'}_\Sigma, {I'}_\Sigma)$ be a weak coding. As the condition

$$(a, \alpha) \in H \iff a \in \bar{\mathsf{c}}(\alpha)$$

defines a surjective relational morphism

$$H : \left(\Sigma', D' \cup id_{\Sigma'}\right) \to \left({\Sigma'}_\Sigma, {D'}_\Sigma \cup id_{{\Sigma'}_\Sigma}\right),$$

by Theorem 1 there is a strong coding $\sigma : \mathbb{T}({\Sigma'}_\Sigma, {I'}_\Sigma) \to \mathbb{T}(\Sigma', I')$. Therefore we obtain a desired coding by composition $\sigma \circ \psi$. ☐

Corollary 1. *There is an effective reduction of the trace coding problem to the weak trace coding problem.* ☐

5. Decidable Cases

Lemma 15. *Let $m, n \in \mathbb{N}$, $Z \subseteq \{1, \dots, m\} \times \{1, \dots, n\}$. Then there exists an $m \times n$-matrix $A = (a_{ij})$ over $\mathbb{N}_0$ such that*

1. *$a_{ij} = 0$ if and only if $(i,j) \in Z$,*
2. *for every $M \subseteq \{1, \dots, m\}$ and $N \subseteq \{1, \dots, n\}$ with $|M| = |N|$, the submatrix $A(M,N)$ of A is regular if and only if there is a bijection $\tau : M \to N$ satisfying $(i, \tau(i)) \notin Z$ for all $i \in M$.*

Proof. Without loss of generality assume $m \leq n$. Let us consider the increasing sequence S of natural numbers $s_k = m^{m^{2k}}$, $k \in \mathbb{N}_0$, and complete the matrix A on non-Z coordinates with elements of this sequence, taking each of them only once. The "only if" part of the condition 2 is clear since if there is no such bijection, then the determinant of $A(M,N)$ is zero by the condition 1. For the converse, let $A(M,N)$ be a square submatrix of A and consider its determinant as the usual sum of products of its elements. Then some of these products are non-zero thanks to the existence of a bijection τ. Take any two of them p and q. Let s_k be the greatest element of S which occurs in one of them (say in p) but not in the other. Then $p/q \geq s_k / s_{k-1}^m \geq m!$. As there is at most $m!$ products in the sum, the value of the greatest one is bigger then the sum of the others and thus the determinant cannot be zero. ☐

Definition 14. Let (Σ, I) and (Σ', I') be two independence alphabets. Let $f : \Sigma \to 2^{\Sigma'}$ and $x \in X \subseteq \Sigma$. The set of *central letters* for x in X with respect to f is defined as

$$X_f(x) = \big\{ a \in f(x) \mid \big(\exists y \in X, b \in f(y) : (a,b) \in D'\big) \& \\ \&\big(a \in f(y), y \in X \implies x = y\big)\big\}.$$

Definition 15. Let (Σ, I) and (Σ', I') be two independence alphabets and $f : \Sigma \to 2^{\Sigma'}$ a mapping. We call f a *wlt-mapping* if

1. $\forall x, y \in \Sigma : (x,y) \in I \iff f(x) \times f(y) \subseteq I'$,
2. for all $X \subseteq \Sigma$ such that $X \times X \subseteq I$, there exists an injective mapping $\rho_X : X \to \Sigma'$ satisfying $\forall x \in X : \rho_X(x) \in f(x)$,
3. for all $X \subseteq \Sigma$ such that the graph $(X, D \cap (X \times X))$ is a tree, there exist $x \in X$ and an injective mapping $\sigma_{X,x} : D(x) \cap X \to D'(X_f(x))$ satisfying $\forall y \in D(x) \cap X : \sigma_{X,x}(y) \in f(y)$. Such letter x will be called *X-deciphering.*

Lemma 16. *Let (Σ, I) and (Σ', I') be two independence alphabets and $f : \Sigma \to 2^{\Sigma'}$ a mapping satisfying the condition 1 of the previous definition. Let $x \in X \subseteq \Sigma$ and let $Y \subseteq X$ be the subset consisting of all elements of X whose distance from x in the graph $\left(X, D \cap X^2\right)$ is at most 2. Then x is X-deciphering if and only if it is Y-deciphering.* ☐

Lemma 17. *Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a weak coding. Then $f = \mathsf{c} \circ \varphi|_\Sigma$ is a wlt-mapping.*

Proof. The first condition is clear and the second one is an immediate consequence of Proposition 2. Suppose that the condition 3 does not hold. We will show that φ is not a coding. Let $X \subseteq \Sigma$ be a subset which falsifies 3. Then for all $x \in X$ the system

$$\left(\underset{D'\left(X_{f(x)}\right)}{\pi} \left(\varphi\left(y\right)\right) \right)_{y \in D(x) \cap X} \tag{6}$$

of elements of the subsemigroup of $\mathbb{T}\left(\Sigma', I'\right)$ generated by $\bigcup f\left(D\left(x\right) \cap X\right)$, which is free commutative, is linearly dependent. Therefore we can split $D\left(x\right) \cap X$ into disjoint sets L_x and R_x such that there exist $n_x^y \in \mathbb{N}_0$, for $y \in D\left(x\right) \cap X$, not all of them equal to 0, satisfying

$$\prod_{y \in L_x} \left(\underset{D'\left(X_{f(x)}\right)}{\pi} \left(\varphi\left(y\right)\right) \right)^{n_x^y} = \prod_{y \in R_x} \left(\underset{D'\left(X_{f(x)}\right)}{\pi} \left(\varphi\left(y\right)\right) \right)^{n_x^y} \tag{7}$$

Without loss of generality we can assume that $\forall x, y \in X : y \in L_x \iff x \in R_y$. As the dependence relation restricted on X forms a tree, we can choose a subset $Y \subseteq X$, $|Y| \geq 2$, satisfying the condition $\forall x \in Y, y \in D\left(x\right) \cap X :$ $(n_x^y \neq 0 \iff y \in Y)$. Further take numbers $m_x \in \mathbb{N}$ for $x \in Y$ such that $x, y, z \in Y,\ y, z \in D\left(x\right) \implies m_y / m_z = n_x^y / n_x^z$. Such numbers can be easily constructed inductively while adding vertices of the tree one by one.

Now let $u \in Y^+$ be any word consisting of exactly one x-block of length m_x for every letter $x \in Y$ such that for $x, y \in Y$, $(x, y) \in D$, y precedes x in u if and only if $y \in L_x$. Then clearly $u \not\sim_I \overleftarrow{u}$. We will verify that $\varphi\left(u\right) \sim_{I'} \varphi\left(\overleftarrow{u}\right)$. Let $a, b \in \bigcup f\left(Y\right)$, $(a, b) \in D'$. Then one of these letters (say a) is central for some x in X since occurrences of both a and b in the contents of images of at least two letters from X would produce a cycle of length 4 in X. Hence, $\varphi\left(u\right)$ contains only one a-block and the number of occurrences of b preceding (succeeding) this block in $\varphi\left(u\right)$ is $\sum\limits_{y \in L_x} \left(m_y \cdot |\varphi\left(y\right)|_b\right)$, $\sum\limits_{y \in R_x} \left(m_y \cdot |\varphi\left(y\right)|_b\right)$ respectively. By (7) these numbers are equal and so $\pi_{a,b}\left(\varphi\left(u\right)\right) = \pi_{a,b}\left(\varphi\left(\overleftarrow{u}\right)\right)$. $\square$

Proposition 4. *Let* (Σ, I) *and* (Σ', I') *be two independence alphabets satisfying one of the following conditions.*

1. $\mathbb{T}\left(\Sigma, I\right)$ *is a product of free semigroups.*
2. (Σ, D) *is a forest.*
3. (Σ, D) *does not contain cycles of length* 3 *or* 4 *and* $\mathbb{T}\left(\Sigma', I'\right)$ *is a product of free semigroups on at most* 2 *generators.*

Then there is a weak coding $\varphi : \mathbb{T}\left(\Sigma, I\right) \to \mathbb{T}\left(\Sigma', I'\right)$ *if and only if there is a wlt-mapping* $f : \Sigma \to 2^{\Sigma'}$. *In particular, the weak trace coding problem is decidable in these cases.*

Remark 6. Notice that in the case 1 the condition 3 in Definition 15 is redundant.

Proof. The "only if" part is a consequence of Lemma 17. We prove the converse in the case 1 first. Let $\mathbb{T}(\Sigma, I) = {\Sigma_1}^+ \times \cdots \times {\Sigma_n}^+$. By the condition 2 of Definition 15 and Lemma 15, we can choose a weak morphism φ satisfying $f = \mathsf{c} \circ \varphi|_\Sigma$ such that for all $X \subseteq \Sigma$ with $X \times X \subseteq I$, the system $(\varphi(x))_{x \in X}$ of elements of the free commutative semigroup generated by $\bigcup f(X)$ is linearly independent. This shows that the case 1 of Lemma 7 cannot happen. As the condition 1 of Definition 15 makes the case 2 also impossible by Lemma 6, φ is a coding.

In the case 2, we start by using the conditions 2 and 3 of Definition 15 and Lemma 15 to construct a weak morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ such that the same conditions as in the previous case are satisfied and in addition for all $X \subseteq \Sigma$ such that $(X, D \cap (X \times X))$ is a tree, there is $x \in X$ for which the system (6) is linearly independent. As these conditions are valid also for all subgraphs of (Σ, D), to prove that φ is a coding, we can use induction with respect to the number of vertices of the graph. So, let us assume that φ is a coding on $\mathbb{T}(Y, I \cap Y^2)$ for all proper subsets Y of Σ. If (Σ, D) has no edges, then φ is a coding due to the condition for independence cliques. If it is not the case, let X be one of its non-trivial connected components and consider the vertex $x \in X$ guaranteed by the condition for trees. Let $u, v \in \Sigma^+$ satisfy $\varphi(u) \sim_{I'} \varphi(v)$. Since X is a connected component of (Σ, D), the letters from $X_f(x)$ occur just in the image of x and the letters from $D'(X_f(x))$ just in the images of elements of $D(x)$. As $X_f(x) \neq \emptyset$, $|u|_x = |v|_x$. Further we deduce that

$$\underset{X_f(x) \cup D'(X_f(x))}{\pi}\left(\varphi\left(\underset{\{x\} \cup D(x)}{\pi}(u)\right)\right) \sim_{I'} \underset{X_f(x) \cup D'(X_f(x))}{\pi}\left(\varphi\left(\underset{\{x\} \cup D(x)}{\pi}(v)\right)\right)$$

and using the independence of the set (6) we obtain $\pi_{x,y}(u) = \pi_{x,y}(v)$ for every $y \in D(x)$. Because φ is a coding on $\mathbb{T}\left(\Sigma \setminus \{x\}, I \cap (\Sigma \setminus \{x\})^2\right)$, we can use Lemma 8 to conclude $u \sim_I v$.

To deal with the case 3, we will modify the wlt-mapping to make it suitable for construction of a coding. First notice that for every pair of dependent letters in Σ' employed by f, at least one of them occurs in the image of only one element of Σ, thus being central for it in Σ. Let us choose some letters in Σ', which will never be considered central, even if they in fact are, in such way that for every pair of dependent letters exactly one is central in Σ with respect to f for some element of Σ. We have to show that f is a wlt-mapping also in the new sense since the condition 3 of Definition 15 becomes more strict. Let X be a subtree of (Σ, D) and consider $x \in X$ and $a \in X_f(x)$ which should be set non-central. Then the letter b dependent on a satisfies $b \in \Sigma_f(y)$ for some $y \in D(x)$. That is why our change affects the validity of the condition only if $y \in X$. Let us consider the maximal subtree Y of X satisfying $y \in Y$ but $x \notin Y$. We will show that Y contains an X-deciphering vertex, thus proving the validity of the condition 3. Suppose that there is not an X-deciphering vertex in $Y \setminus \{y\}$. We will inductively construct a mapping $\sigma_{X,y}$. Let $\mathcal{B}$ be the set of all maximal subtrees of Y which do not contain y. In every step let us have a subset $\mathcal{C}$ of $\mathcal{B}$ and an injective mapping

$$\sigma_{X,y}^{\mathcal{C}} : D(y) \cap \bigcup \mathcal{C} \to D'(X_f(y)) \setminus \bigcup f\left(\bigcup (\mathcal{B} \setminus \mathcal{C})\right)$$

such that $\sigma_{X,y}^{\mathcal{C}}(v) \in f(v)$ for all $v \in D(y) \cap \bigcup \mathcal{C}$. Denote $Z = \bigcup (\mathcal{B} \setminus \mathcal{C}) \cup \{y\}$. There exists a Z-deciphering vertex z. The distance between y and z is at most 1, otherwise z would be X-deciphering by Lemma 16. If $z = y$, we can define

$$\sigma_{X,y}(v) = \begin{cases} \sigma_{Z,y}(v) & \text{for } v \in D(y) \cap \bigcup (\mathcal{B} \setminus \mathcal{C}) \\ \sigma_{X,y}^{\mathcal{C}}(v) & \text{for } v \in D(y) \cap \bigcup \mathcal{C} \\ a & \text{for } v = x \end{cases}$$

since $Z_f(y) \subseteq X_f(y)$. If $z \in D(y)$, then there is $c \in Z_f(z) \setminus X_f(z)$. It follows that $D'(c) \cap \Sigma_f(y) \neq \emptyset$ and we can extend $\sigma_{X,y}^{\mathcal{C}}$ to $\sigma_{X,y}^{\mathcal{C} \cup \{Z\}}$, where $z \in Z \in \mathcal{B}$, by the rule $\sigma_{X,y}^{\mathcal{C} \cup \{Z\}}(z) = c$.

Now we can add all non-central letters a to $f(x)$ for all $x \in \Sigma$ satisfying $(x, y) \in I \implies f(y) \cap D'(a) = \emptyset$ without violating the conditions of Definition 15. From now on, the mapping f will be completely determined (up to isomorphisms of $\mathbb{T}(\Sigma', I')$) by the values $|\Sigma_f(x)|$ for $x \in \Sigma$. If there is at most one of these numbers equal to zero in every connected component of (Σ, D), then we can use Lemma 15 and the condition 2 of Definition 15 to construct a coding φ such that $\mathsf{c} \circ \varphi|_\Sigma = f$. To see that it is really a coding, we can use Lemma 7 since in the case 2 for every element $x \in \operatorname{init}(u) \setminus \operatorname{init}(v)$ there cannot be central letters in $f(x)$ as for each $y \in D(x)$, every central letter in $f(x)$ is dependent on some letter in $f(y)$.

Notice that the condition of X-decipherability of x can now be formulated as $|\Sigma_f(x)| \geq |D(x) \cap X|$. We will prove that if there are at least two vertices x and y in the same connected component of (Σ, D) satisfying $|\Sigma_f(x)| = |\Sigma_f(y)| = 0$, then we can construct a wlt-mapping g such that the number of vertices without central letters with respect to g is less than with respect to f. If we apply this construction repeatedly, we eventually reach the case of the previous paragraph, thus completing the proof. Let P be the set of vertices of a simple path p between x and y. There must be $z \in P$ with $|\Sigma_f(z)| \geq 2$. Suppose that z is the nearest one to x on p satisfying this condition and move one of the letters from $\Sigma_f(z)$ to $f(x)$ to obtain a mapping g such that $|\Sigma_g(z)| = |\Sigma_f(z)| - 1$, $|\Sigma_g(x)| = 1$ and $|\Sigma_g(v)| = |\Sigma_f(v)|$ for $v \notin \{x, z\}$.

It remains to prove that g is a wlt-mapping. The validity of the conditions 1 and 2 is clear. In order to prove the condition 3, let X be a subtree of (Σ, D). The only non-trivial case is when $z \in X$. If $x \in X$ too, then any Y-deciphering vertex with respect to f for the subtree Y of X consisting of all vertices whose simple path to z in X contains x is also X-deciphering with respect to g. If $x \notin X$, consider the set Q of all letters on the initial part of p starting in x and finishing in the first vertex s dependent on some element of X. Take some $t \in X \cap D(s)$ with maximal distance from z in X and consider the subtree Y of X consisting of all vertices whose simple path in X to z contains t. Then $Z = Y \cup Q$ is a subtree of (Σ, D) and any Z-deciphering vertex with respect to f is also X-deciphering with respect to g (notice that if $z \in Y$ then $|D(z) \cap Z| = |D(z) \cap X| + 1$ and for all $v \in Y \setminus \{z\}$ we have $|D(v) \cap Z| = |D(v) \cap X|$). □

Problem 2. In the case 3 of the previous proposition, is it true that for every wlt-mapping f there exists a coding φ such that $\mathsf{c} \circ \varphi|_\Sigma = f$?

The most interesting special case of the (weak) trace coding problem which is still open is that when the first semigroup is a free product of free commutative semigroups. The condition analogous to the existence of a wlt-mapping was proved accurate for trace codings if all of these free commutative semigroups have at most 2 generators in [2]. But even in this case the existence of a wlt-mapping does not guarantee the existence of a weak coding as shown by the following example.

Example 2. Let

$$\Sigma = \{x, y, z, r, s, t\}, \ I = id_\Sigma \cup \{(x, y), (y, x)\},$$
$$\Sigma' = \{a, b, c, d, e, f\} \text{ and } D' = \operatorname{sym}\{(a, d), (b, e), (c, f)\}.$$

It can be shown that wlt-mappings $f : \Sigma \to 2^{\Sigma'}$ are (up to symmetry) exactly the following: $f(x) \in \{\{a, b\}, \{a, b, c\}\}$, $f(y) \in \{\{a, c\}, \{a, b, c\}\}$, $f(z) = \{d, e, f\}$, $f(r) = \{a, e, f\}$, $f(s) = \{d, b, f\}$, $f(t) = \{d, e, c\}$. Consider any weak morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ such that $\mathsf{c} \circ \varphi|_\Sigma = f$. We will prove that it is not injective. By Lemma 11 we can assume that $|\varphi(x)|_a = |\varphi(y)|_a = |\varphi(z)|_d = |\varphi(s)|_d = |\varphi(t)|_d = 1$ and without loss of generality also $|\varphi(x)|_b \geq |\varphi(y)|_b$ and $|\varphi(x)|_c \leq |\varphi(y)|_c$. If some of these inequalities is satisfied as an equality, then $\varphi(xty) \sim_{I'} \varphi(ytx)$ ($\varphi(xsy) \sim_{I'} \varphi(ysx)$ respectively). If it is not the case, we have $\varphi(x) = ab^i c^j$, $\varphi(y) = ab^k c^l$, $\varphi(s) = db^m f^n$, $\varphi(t) = de^o c^p$ for some $j, k \in \mathbb{N}_0$ and $i, l, m, n, o, p \in \mathbb{N}$ and

$$\varphi\left(x^{mp} t^{m(l-j)} s^{p(i-k)} y^{mp}\right) \sim_{I'} \varphi\left(y^{mp} s^{p(i-k)} t^{m(l-j)} x^{mp}\right).$$

6. Encoding of the PCP

We will denote by $\mathcal{P}$ the following instance of the Post's correspondence problem (PCP). We are given n pairs $(u_1, v_1), \ldots, (u_n, v_n)$ of non-empty words over a finite alphabet Ξ such that every product of them contains no subword of the form x^k with $x \in \Xi$ and $k \geq 2$. We have to decide whether there is some finite sequence $i_1, \ldots, i_m$ of natural numbers from $\{1, \ldots, n\}$ satisfying $u_1 u_{i_1} u_{i_2} \cdots u_{i_m} = v_1 v_{i_1} v_{i_2} \cdots v_{i_m}$. Notice that the general case of the PCP can be easily reduced to this one. In fact, we obtain an equivalent instance of this form by introducing one new letter # into Ξ and performing the substitution $x \mapsto x\#$ for all $x \in \Xi$. This restriction allows us not to care about powers of letters when looking for an encoding of the PCP into another problem.

For $i = 1, \dots, n$, let us denote by i_u (i_v) the length of u_i (v_i respectively) and by x_{ij} ($\overline{x}_{ik}$) the j-th letter of the word u_i (k-th letter of v_i respectively), for $j = 1, \dots, i_u$, $k = 1, \dots, i_v$. For the rest of this section, when writing indices i, ij or ik, we implicitly assume that they run through all values as in the previous sentence.

Now we define two independence alphabets (Σ, I) and (Σ', I'). Let

$$\begin{aligned}\Sigma = \{&\alpha_h, \beta_{1h}, \dots, \beta_{8h}, \gamma_{1h}, \gamma_{2h}, \gamma_{3h}, \delta_{i1h}, \delta_{i2h}, \varepsilon_{ij1h}, \varepsilon_{ij2h},\\ &\overline{\varepsilon}_{ik1h}, \overline{\varepsilon}_{ik2h}, \zeta_{ij1h}, \zeta_{ij2h}, \overline{\zeta}_{ik1h}, \overline{\zeta}_{ik2h}, \eta_{ij1h}, \eta_{ij2h}, \eta_{ij3h},\\ &\overline{\eta}_{ik1h}, \overline{\eta}_{ik2h}, \overline{\eta}_{ik3h}, \vartheta_{ijh}, \overline{\vartheta}_{ikh}, \iota_{ih}, \kappa_{1h}, \kappa_{2h}, \xi_{ij1h}, \xi_{ij2h}, \xi_{ij3h},\\ &\overline{\xi}_{ik1h}, \overline{\xi}_{ik2h}, \overline{\xi}_{ik3h}, \lambda_{i1h}, \lambda_{i2h}, \lambda_{i3h} \mid h = 1, 2\},\end{aligned}$$

$$\begin{gathered}S = \left\{s, s_f, s_i, s_{ij}, \overline{s}_{ik}, t_{ij}, \overline{t}_{ik}\right\},\ A = \{a, a_{ij}, \overline{a}_{ik}\},\\ E = \{e, e_{ij}, \overline{e}_{ik}\},\ F = \left\{f, f_{ij}, \overline{f}_{ik}\right\},\ G = \{g_i, g_{ij}, \overline{g}_{ik}\},\\ P = \{p_{ij}, \overline{p}_{ik}\},\ Q = \{q_i, q_{ij}, \overline{q}_{ik}\},\end{gathered}$$

$$\begin{aligned}\Omega = \{&\alpha, \alpha_0, \beta_1, \dots, \beta_8, \gamma_1, \gamma_2, \gamma_3, \delta_{i1}, \delta_{i2}, \varepsilon_{ij1}, \varepsilon_{ij2}, \overline{\varepsilon}_{ik1}, \overline{\varepsilon}_{ik2},\\ &\zeta_{ij1}, \zeta_{ij2}, \overline{\zeta}_{ik1}, \overline{\zeta}_{ik2}, \eta_{ij1}, \eta_{ij2}, \eta_{ij3}, \overline{\eta}_{ik1}, \overline{\eta}_{ik2}, \overline{\eta}_{ik3}, \vartheta_{ij}, \overline{\vartheta}_{ik},\\ &\iota_i, \kappa_1, \kappa_2, \xi_{ij1}, \xi_{ij2}, \xi_{ij3}, \overline{\xi}_{ik1}, \overline{\xi}_{ik2}, \overline{\xi}_{ik3}, \lambda_{i1}, \lambda_{i2}, \lambda_{i3}\},\end{aligned}$$

$$\Sigma'_0 = \{l_1, l_2, r_1, r_2, b, c, d\} \cup S \cup A \cup E \cup F \cup G \cup P \cup Q \cup \Xi \cup \{\star\} \cup \Omega,$$

let Γ be an arbitrary finite set of new letters and define $\Sigma' = \Sigma'_0 \cup \Gamma$. Let $I = id_\Sigma \cup \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$. In the following we set the dependences between elements of Σ', all pairs not mentioned below are considered independent.

$$\begin{gathered}l_1 D' l_2,\ r_1 D' r_2,\ bD'c,\ bD'd,\\ p_{ij} D' e_{ij},\ \overline{p}_{ik} D' \overline{e}_{ik},\ q_i D' f,\ q_{ij} D' f_{ij},\ \overline{q}_{ik} D' \overline{f}_{ik},\end{gathered}$$

$$\begin{gathered}S^2 \cap I' = id_S,\ (\Xi \cup \{\star\})^2 \cap I' = id_{\Xi \cup \{\star\}},\\ \Omega^2 \cap I' = id_\Omega \cup \operatorname{sym}(\{\alpha, \alpha_0\} \times \{\beta_1, \dots, \beta_4\}),\end{gathered}$$

$$\begin{aligned}&(A \cup E \cup F \cup G)^2 \cap I' = id_{A \cup E \cup F \cup G} \cup \operatorname{sym}\{(e, a), (e_{ij}, a_{ij}), (\overline{e}_{ik}, \overline{a}_{ik}),\\ &(f, a), (f_{ij}, a_{ij}), \left(\overline{f}_{ik}, \overline{a}_{ik}\right), (g_i, a), (g_i, a_{i1}), (g_{ij}, a_{ij}), (g_{ij}, a_{ij+1}),\\ &(\overline{g}_{ik}, \overline{a}_{ik}), (\overline{g}_{ik}, \overline{a}_{ik+1}), (g_i, f_{i1}), (g_{ij}, f_{ij+1}), \left(\overline{g}_{ik}, \overline{f}_{ik+1}\right)\},\end{aligned}$$

where $a_{ii_u+1} = \overline{a}_{i1}$, $\overline{a}_{ii_v+1} = a$, $f_{ii_u+1} = \overline{f}_{i1}$ and $\overline{f}_{ii_v+1} = f$.

Next we define a mapping $\nu_0 : \Sigma \to 2^{\Sigma'}$.

$$\nu_0(\alpha_1) = \{l_1, r_1, b, s_1, a_{11}, \alpha\}, \qquad \nu_0(\alpha_2) = \{l_1, r_1, b, s_1, a_{11}, \alpha\},$$
$$\nu_0(\beta_{11}) = \{l_2, r_1, \alpha_0, \beta_1\}, \qquad \nu_0(\beta_{12}) = \{l_2, r_2, \beta_1\},$$
$$\nu_0(\beta_{21}) = \{l_2, \alpha_0, \beta_2\}, \qquad \nu_0(\beta_{22}) = \{l_2, r_2, \alpha, \beta_2\},$$
$$\nu_0(\beta_{31}) = \{l_1, r_2, \alpha_0, \beta_3\}, \qquad \nu_0(\beta_{32}) = \{l_2, r_2, \beta_3\},$$
$$\nu_0(\beta_{41}) = \{r_2, \alpha_0, \beta_4\}, \qquad \nu_0(\beta_{42}) = \{l_2, r_2, \alpha, \beta_4\},$$
$$\nu_0(\beta_{51}) = \{l_2, s_f, \beta_5\}, \qquad \nu_0(\beta_{52}) = \{r_2, s_1, \beta_5\},$$
$$\nu_0(\beta_{61}) = \{l_2, s_1, \beta_6\}, \qquad \nu_0(\beta_{62}) = \{r_2, s_f, a, \beta_6\},$$
$$\nu_0(\beta_{71}) = \{l_2, c, \beta_7\}, \qquad \nu_0(\beta_{72}) = \{r_2, b, \beta_7\},$$
$$\nu_0(\beta_{81}) = \{l_2, b, \beta_8\}, \qquad \nu_0(\beta_{82}) = \{r_2, d, \beta_8\},$$
$$\nu_0(\gamma_{11}) = \{l_2, s_f, \gamma_1\}, \qquad \nu_0(\gamma_{12}) = \{l_1, s_f, \gamma_1\},$$
$$\nu_0(\gamma_{21}) = \{r_1, s_f, \gamma_2\}, \qquad \nu_0(\gamma_{22}) = \{r_2, s_f, \gamma_2\},$$
$$\nu_0(\gamma_{31}) = \{a_{11}, s_f, \gamma_3\}, \qquad \nu_0(\gamma_{32}) = \{a, s_f, \gamma_3\},$$

$$\nu_0(\delta_{i11}) = \{l_2, c, s_i, \delta_{i1}\}, \qquad \nu_0(\delta_{i12}) = \{r_2, d, s_{i+1}, q_i, \delta_{i1}\},$$
$$\nu_0(\delta_{i21}) = \{l_2, c, s_i, q_i, \delta_{i2}\}, \qquad \nu_0(\delta_{i22}) = \{r_2, d, s_{i+1}, \delta_{i2}\},$$
$$\nu_0(\varepsilon_{ij11}) = \{l_2, s_{ij}, \varepsilon_{ij1}\}, \qquad \nu_0(\varepsilon_{ij12}) = \{r_2, s_{ij+1}, p_{ij}, \varepsilon_{ij1}\},$$
$$\nu_0(\varepsilon_{ij21}) = \{l_2, s_{ij}, p_{ij}, \varepsilon_{ij2}\}, \qquad \nu_0(\varepsilon_{ij22}) = \{r_2, s_{ij+1}, \varepsilon_{ij2}\},$$
$$\nu_0(\overline{\varepsilon}_{ik11}) = \{l_2, \overline{s}_{ik}, \overline{\varepsilon}_{ik1}\}, \qquad \nu_0(\overline{\varepsilon}_{ik12}) = \{r_2, \overline{s}_{ik+1}, \overline{p}_{ik}, \overline{\varepsilon}_{ik1}\},$$
$$\nu_0(\overline{\varepsilon}_{ik21}) = \{l_2, \overline{s}_{ik}, \overline{p}_{ik}, \overline{\varepsilon}_{ik2}\}, \qquad \nu_0(\overline{\varepsilon}_{ik22}) = \{r_2, \overline{s}_{ik+1}, \overline{\varepsilon}_{ik2}\},$$
$$\nu_0(\zeta_{ij11}) = \{l_2, t_{ij}, \zeta_{ij1}\}, \qquad \nu_0(\zeta_{ij12}) = \{r_2, t_{ij+1}, q_{ij}, \zeta_{ij1}\},$$
$$\nu_0(\zeta_{ij21}) = \{l_2, t_{ij}, q_{ij}, \zeta_{ij2}\}, \qquad \nu_0(\zeta_{ij22}) = \{r_2, t_{ij+1}, \zeta_{ij2}\},$$
$$\nu_0\left(\overline{\zeta}_{ik11}\right) = \left\{l_2, \overline{t}_{ik}, \overline{\zeta}_{ik1}\right\}, \qquad \nu_0\left(\overline{\zeta}_{ik12}\right) = \left\{r_2, \overline{t}_{ik+1}, \overline{q}_{ik}, \overline{\zeta}_{ik1}\right\},$$
$$\nu_0\left(\overline{\zeta}_{ik21}\right) = \left\{l_2, \overline{t}_{ik}, \overline{q}_{ik}, \overline{\zeta}_{ik2}\right\}, \qquad \nu_0\left(\overline{\zeta}_{ik22}\right) = \left\{r_2, \overline{t}_{ik+1}, \overline{\zeta}_{ik2}\right\},$$

$$\nu_0(\eta_{ij11}) = \{l_2, s, p_{ij}, x_{ij}, \eta_{ij1}\}, \qquad \nu_0(\eta_{ij12}) = \{r_2, s, e_{ij}, \eta_{ij1}\},$$
$$\nu_0(\eta_{ij21}) = \{l_2, s, e_{ij}, \eta_{ij2}\}, \qquad \nu_0(\eta_{ij22}) = \{r_2, s, p_{ij}, \eta_{ij2}\},$$
$$\nu_0(\eta_{ij31}) = \{l_2, s, p_{ij}, \eta_{ij3}\}, \qquad \nu_0(\eta_{ij32}) = \{r_2, s, e_{ij}, \eta_{ij3}\},$$
$$\nu_0(\overline{\eta}_{ik11}) = \{l_2, s, \overline{p}_{ik}, \overline{\eta}_{ik1}\}, \qquad \nu_0(\overline{\eta}_{ik12}) = \{r_2, s, \overline{e}_{ik}, \overline{x}_{ik}, \overline{\eta}_{ik1}\},$$
$$\nu_0(\overline{\eta}_{ik21}) = \{l_2, s, \overline{e}_{ik}, \overline{\eta}_{ik2}\}, \qquad \nu_0(\overline{\eta}_{ik22}) = \{r_2, s, \overline{p}_{ik}, \overline{\eta}_{ik2}\},$$
$$\nu_0(\overline{\eta}_{ik31}) = \{l_2, s, \overline{p}_{ik}, \overline{\eta}_{ik3}\}, \qquad \nu_0(\overline{\eta}_{ik32}) = \{r_2, s, \overline{e}_{ik}, \overline{\eta}_{ik3}\},$$
$$\nu_0(\vartheta_{ij1}) = \{l_2, s, a_{ij+1}, x_{ij}, \vartheta_{ij}\}, \qquad \nu_0(\vartheta_{ij2}) = \{r_2, s, a_{ij}, g_{ij}, \vartheta_{ij}\},$$
$$\nu_0\left(\overline{\vartheta}_{ik1}\right) = \left\{l_2, s, \overline{a}_{ik+1}, \overline{\vartheta}_{ik}\right\}, \qquad \nu_0\left(\overline{\vartheta}_{ik2}\right) = \left\{r_2, s, \overline{a}_{ik}, \overline{g}_{ik}, \overline{x}_{ik}, \overline{\vartheta}_{ik}\right\},$$
$$\nu_0(\iota_{i1}) = \{l_2, s, a_{i1}, \iota_i\}, \qquad \nu_0(\iota_{i2}) = \{r_2, s, a, g_i, \iota_i\},$$
$$\nu_0(\kappa_{11}) = \{l_2, s, \star, \kappa_1\}, \qquad \nu_0(\kappa_{12}) = \{r_2, s_f, e, \star, \kappa_1\},$$
$$\nu_0(\kappa_{21}) = \{l_2, s_f, \star, \kappa_2\}, \qquad \nu_0(\kappa_{22}) = \{r_2, s, e, \star, \kappa_2\},$$

$$\begin{aligned}
\nu_0(\xi_{ij11}) &= \{l_2, s, g_{ij-1}, q_{ij}, \xi_{ij1}\}, & \nu_0(\xi_{ij12}) &= \{r_2, s, f_{ij}, \xi_{ij1}\},\\
\nu_0(\xi_{ij21}) &= \{l_2, s, f_{ij}, \xi_{ij2}\}, & \nu_0(\xi_{ij22}) &= \{r_2, s, q_{ij}, \xi_{ij2}\},\\
\nu_0(\xi_{ij31}) &= \{l_2, s, q_{ij}, \xi_{ij3}\}, & \nu_0(\xi_{ij32}) &= \{r_2, s, f_{ij}, \xi_{ij3}\},\\
\nu_0(\overline{\xi}_{ik11}) &= \{l_2, s, \overline{g}_{ik-1}, \overline{q}_{ik}, \overline{\xi}_{ik1}\}, & \nu_0(\overline{\xi}_{ik12}) &= \{r_2, s, \overline{f}_{ik}, \overline{\xi}_{ik1}\},\\
\nu_0(\overline{\xi}_{ik21}) &= \{l_2, s, \overline{f}_{ik}, \overline{\xi}_{ik2}\}, & \nu_0(\overline{\xi}_{ik22}) &= \{r_2, s, \overline{q}_{ik}, \overline{\xi}_{ik2}\},\\
\nu_0(\overline{\xi}_{ik31}) &= \{l_2, s, \overline{q}_{ik}, \overline{\xi}_{ik3}\}, & \nu_0(\overline{\xi}_{ik32}) &= \{r_2, s, \overline{f}_{ik}, \overline{\xi}_{ik3}\},\\
\nu_0(\lambda_{i11}) &= \{l_2, s, \overline{g}_{ii_v}, q_i, \lambda_{i1}\}, & \nu_0(\lambda_{i12}) &= \{r_2, s, f, \lambda_{i1}\},\\
\nu_0(\lambda_{i21}) &= \{l_2, s, f, \lambda_{i2}\}, & \nu_0(\lambda_{i22}) &= \{r_2, s, q_i, \lambda_{i2}\},\\
\nu_0(\lambda_{i31}) &= \{l_2, s, q_i, \lambda_{i3}\}, & \nu_0(\lambda_{i32}) &= \{r_2, s, f, \lambda_{i3}\},
\end{aligned}$$

where $s_{n+1} = s_{11}$, $s_{ii_u+1} = s_{i+11}$, $s_{n+11} = \overline{s}_{11}$, $\overline{s}_{ii_v+1} = \overline{s}_{i+11}$, $\overline{s}_{n+11} = t_{11}$, $t_{ii_u+1} = t_{i+11}$, $t_{n+11} = \overline{t}_{11}$, $\overline{t}_{ii_v+1} = \overline{t}_{i+11}$, $\overline{t}_{n+11} = s$, $a_{ii_u+1} = \overline{a}_{i1}$, $\overline{a}_{ii_v+1} = a$, $g_{i0} = g_i$ and $\overline{g}_{i0} = g_{ii_u}$. A mapping $\mu : \Sigma \to 2^{\Sigma'}$ is defined by the same rules as ν_0 except $\mu(\alpha_1) = \mu(\alpha_2) = \emptyset$ and $\nu : \Sigma \to 2^{\Sigma'}$ is any mapping satisfying $\forall \omega \in \Sigma : \nu(\omega) \setminus \Gamma = \nu_0(\omega)$. Let φ denote the weak morphism from $\mathbb{T}(\Sigma, I)$ to $\mathbb{T}(\Sigma', I')$ satisfying for all $\omega \in \Sigma : \mathsf{c}(\varphi(\omega)) = \nu_0(\omega)$ and all letters in $\varphi(\omega)$ occur only once except $\varphi(\alpha_1) = l_1^2 r_1 b s_1 a_{11}^2 \alpha$ and $\varphi(\alpha_2) = l_1 r_1^2 b s_1^2 a_{11} \alpha$.

Proposition 5. *Let $\mathcal{P}$, (Σ, I), (Σ', I'), μ, ν and φ be as defined above. Then the following statements are equivalent.*

1. *$\mathcal{P}$ has no solution.*
2. *The (μ, ν)-weak morphism $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is a coding.*
3. *There is a (μ, ν)-weak coding $\psi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$.*

Proof. Throughout the proof we will always remember the current semi-equality state we deal with, usually we will write the state without powers of letters as we will see later that they are not essential. Moreover, we will also omit the letters from Γ since it turns out that they have no effect on any of the considerations performed there.

$1 \Longrightarrow 2$. Let $u, v \in \Sigma^+$, $u \not\sim_I v$ and $\varphi(u) \sim_{I'} \varphi(v)$ be shortest such words. We will construct a solution to $\mathcal{P}$. First notice that every element of $\Omega \setminus \{\alpha, \alpha_0\}$ occurs in the images of exactly two letters from Σ and each time only once. It means that the two letters on the same position in u and v with α_1 and α_2 removed are either such two letters or equal and we will consider only those semi-equalities which respect these pairs.

By Lemma 5, u and v start (up to symmetry) with powers of α_1, α_2 respectively, followed by some dependent letters. If there is some α in the state of the semi-equality consisting of the initial αs of u and v, then clearly no letter except β_{11}, β_{12}, $\ldots$, β_{41}, β_{42} can follow. But if some of these βs occurs, then l_1, r_1 respectively, is not present in the state, which implies that r_1, l_1 respectively, resides on the same side as α there. This prevents us also from occurrences of above mentioned βs as their images contain either α_0 or r_2, l_2 respectively.

Thus there is no α and the state after the initial αs of u and v is (up to powers) (l_1a_{11}, r_1s_1). Notice that from now on the letters using r_2 (l_2) cannot occur in u (v respectively). As the next letter in u has to use either s_1 or no letter from S, it is just a matter of verification to see that the next pair of letters in u and v is $(\delta_{111}, \delta_{112})$ or $(\delta_{121}, \delta_{122})$. But after at least one of these pairs, either s_1 occurs in the state no more or only other pairs of δ_1s can follow thanks to c and d. In fact, c and d prevent us from αs occurring in u and v also further on. In the following, elements of $P \cup Q \cup \{c, d\}$ will not be important and when writing states we will omit them too.

So, the state after δ_1s is $(l_1l_2a_{11}, r_1r_2s_2)$. Similarly as before, one can argue that the next pairs in u and v are in sequence δ_2s, … , δ_ns, ε_{11}s, … , $\overline{\varepsilon}_{nn_v}$s, ζ_{11}s, … , $\overline{\zeta}_{nn_v}$s due to the changes of a letter from S in the state.

Now the state is $(l_1l_2a_{11}, r_1r_2s)$. It is easy to see that only pairs of letters using s can occur until some occurrence of κs because the power of s in the state does not change. We will show that every time after some number of pairs from u and v, the letter from A currently in the state changes to another one (with no other used in the meantime). More precisely, after a_{ij} ($\overline{a}_{ik}$) the element a_{ij+1} ($\overline{a}_{ik+1}$ respectively) follows and some occurrences of exactly one letter from Ξ, namely x_{ij} ($\overline{x}_{ik}$) are introduced into $\varphi(u)$ ($\varphi(v)$ respectively). After a there comes some a_{i1} with no letter from Ξ added. This means that between two consecutive appearances of a in the state (or before the first one) exactly some pair (u'_i, v'_i) of letters from Ξ is added to the state, where u'_i (v'_i) can be obtained from u_i (v_i respectively) by adding some powers to letters. Further notice that κs can only be used when of letters from A just a is in the state as it is the only letter independent on e whose first occurrences are introduced by κs and that the words consisting of elements of Ξ accumulated on both sides of the semi-equality before κs must coincide because the first occurrences of $\star$ are added. If we now consider the sequence $a_{i_11}, \ldots, a_{i_m1}$ of the elements of A appearing directly after a, then $i_1, \ldots, i_m$ will be a solution to $\mathcal{P}$ as $u_1u_{i_1}u_{i_2}\cdots u_{i_m}$ and $v_1v_{i_1}v_{i_2}\cdots v_{i_m}$ are obtained from the words accumulated by removing all powers due to the property of our instance of the PCP.

It remains to prove the claim on appearances of letters from A stated in the previous paragraph. Let us assume that the state is $(l_1l_2a_{ij}, r_1r_2s)$, possibly with some occurrences of g_{ij-1} on the right and f_{ij} on the left. Consider the first semi-equality after this such that a_{ij} is no longer an initial letter of the left side of its state. The only pair from u and v which uses elements of A and can occur up to this place is clearly $(\vartheta_{ij1}, \vartheta_{ij2})$ and it is also the last one. As long as a_{ij} is an initial letter of the state, the only elements of $E \cup F \cup G$, which can be used by letters of v, are e_{ij}, f_{ij} and g_{ij}. This immediately implies that of letters from Ξ exactly x_{ij} is introduced as desired. It also means that g_{ij} cannot be introduced from u and thus it has no occurrence on the left side of the state. As the last pair introduces g_{ij} on the right, there can be only letters independent on g_{ij} on the left at the moment, but in $E \cup F \cup G$ only f_{ij+1} satisfies this. And since the last pair introduces also a_{ij+1} on the left, none of e_{ij}, f_{ij} and g_{ij-1} can occur on the right. Altogether, the state is $(l_1 l_2 a_{ij+1}, r_1 r_2 s)$, possibly with some occurrences of g_{ij} on the right and f_{ij+1} on the left.

If the state is $(l_1 l_2 \overline{a}_{ik}, r_1 r_2 s)$, everything is similar. The remaining case is that of the state $(l_1 l_2 a, r_1 r_2 s)$, possibly with $\overline{g}_{ii_v}$ on the right and f on the left. Here the role of $(\vartheta_{ij1}, \vartheta_{ij2})$, e_{ij}, f_{ij+1} and g_{ij} is played respectively by $(\iota_{i'1}, \iota_{i'2})$, e, $f_{i'1}$ and $g_{i'}$ for some i'. To see this we have to use an argument analogous to the one above to prove that no $g_{\hat{i}}$ comes to the left. Thus starting from the first occurrence of the pair $(\iota_{i'1}, \iota_{i'2})$, there is $g_{i'}$ on the right until a is completely replaced by other letters from A. The dependences between letters from A and $g_{i'}$ then disallow occurrences of all ιs except $(\iota_{i'1}, \iota_{i'2})$. Hence we get a situation similar to that of the first case.

$2 \Longrightarrow 3.$ is trivial.

$3 \Longrightarrow 1.$ Assume that $i_1, \ldots, i_m$ is a solution of $\mathcal{P}$ and $\psi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ is a (μ, ν)-weak morphism. We have to show that ψ is not a coding. If it is a coding, then by Lemma 11 there is a (μ, ν)-weak coding $\sigma : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ such that in the image of every letter from $\Sigma \setminus \{\alpha_1, \alpha_2\}$ there is exactly one occurrence of a letter from $\Omega \setminus \{\alpha, \alpha_0\}$. Thus, letters from Σ in equalities for σ must be paired in the same way as in the first part of the proof according to the letters from Ω. To reach a contradiction, we will construct words $u, v \in \Sigma^+$ satisfying $u \not\sim_I v$ and $\sigma(u) \sim_{I'} \sigma(v)$. In every step of the construction we will obtain some non-trivial semi-equality (let us denote it also (u, v)) with no letters from Ω in its state.

To obtain an initial one we have to consider the following ratios:

$$N_l = \frac{|\sigma(\alpha_1)|_{l_1}}{|\sigma(\alpha_2)|_{l_1}} \qquad N_r = \frac{|\sigma(\alpha_1)|_{r_1}}{|\sigma(\alpha_2)|_{r_1}} \qquad N_\alpha = \frac{|\sigma(\alpha_1)|_{\alpha}}{|\sigma(\alpha_2)|_{\alpha}}$$

To have some letter in $\sigma(\alpha_1)$ and $\sigma(\alpha_2)$ dependent on $\sigma(\beta_{12})$, $\sigma(\beta_{21})$ and $\sigma(\beta_{41})$, only one of the ratios can be zero and only one infinity. If $N_l = N_r$ or $N_r = 0/0$, then

$$\sigma\left(\alpha_1^{|\sigma(\alpha_2)|_{l_1}} \beta_{12} \alpha_2^{|\sigma(\alpha_1)|_{l_1}}\right) \sim_{I'} \sigma\left(\alpha_2^{|\sigma(\alpha_1)|_{l_1}} \beta_{12} \alpha_1^{|\sigma(\alpha_2)|_{l_1}}\right).$$

In the same way we get a contradiction for the two symmetric cases using β_{21} and β_{41}. So these ratios are well defined and pairwise different. Suppose that $N_r < N_l < N_\alpha$. Let us consider for (u, v) the semi-equality

$$\left(\alpha_1^{|\sigma(\alpha_2)|_{l_1}}, \alpha_2^{|\sigma(\alpha_1)|_{l_1}}\right)$$

and denote its state (w, w'). The following adjustment of powers will be used in the rest of this proof many times implicitly. That is why we present this construction here in a more general setting than actually needed. Let N be the length of the r_1-block in w'. If we substitute for all letters in (u, v) their $|\sigma(\beta_{11})|_{r_1}$th powers, we get another semi-equality whose state is obtained from (w, w') by the same substitution. Now add β_{11}^N (β_{12}^N) to this altered u (v respectively). It is clear that all occurrences of r_1 in the state are replaced by occurrences of r_2. Now we do the same with (β_{21}, β_{22}) to remove

α and we get a state which satisfies the assumptions of Lemma 13 giving a contradiction. Since the case $N_\alpha < N_l < N_r$ can be handled similarly and the same can be shown also if the medium ratio is N_r using (β_{31}, β_{32}) and (β_{41}, β_{42}), the medium one must be N_α. In particular, $0 < N_\alpha < \infty$ and α is contained in both $\sigma(\alpha_1)$ and $\sigma(\alpha_2)$. Hence we can start with the semi-equality

$$\left(\alpha_1^{|\sigma(\alpha_2)|_\alpha}, \alpha_2^{|\sigma(\alpha_1)|_\alpha}\right).$$

Due to the inequalities between the ratios, l_1 and r_1 occur in its state on different sides. Without loss of generality, let us suppose that l_1 is on the left. If there is not s_1 in the state, then

$$\sigma\left(\alpha_1^{|\sigma(\alpha_2)|_\alpha} \gamma_{12} \alpha_2^{|\sigma(\alpha_1)|_\alpha}\right) \sim_{I'} \sigma\left(\alpha_2^{|\sigma(\alpha_1)|_\alpha} \gamma_{12} \alpha_1^{|\sigma(\alpha_2)|_\alpha}\right).$$

If s_1 occurs on the left, using (β_{51}, β_{52}) we reach a state without s_1, then we remove l_1 and r_1 using $(\gamma_{11}, \gamma_{12})$ and $(\gamma_{21}, \gamma_{22})$ and apply Lemma 13. If there is no a_{11} on the left in the state, we can do the same with (β_{61}, β_{62}) and all pairs of γs. If there is some occurrence of b on the left (right) side of the state, we remove it using (β_{71}, β_{72}) ((β_{81}, β_{82}) respectively). We reach the state $(l_1(l_2)(c)a_{11}, r_1(r_2)(d)s_1)$, where the bracketed letters do not have to occur.

Now we add to (u, v) in sequence δ_1s, … , δ_ns, ε_{11}s, … , $\overline{\varepsilon}_{nn_v}$s, ζ_{11}s, … , $\overline{\zeta}_{nn_v}$s using the standard adjustment of powers to change the letter from S currently in the state to the next one. In every step of this sequence we have to choose one of two pairs of letters which can be used; in fact, each of them introduces a letter from $P \cup Q$ to a different side. In the case of εs, the pair $(\varepsilon_{ij11}, \varepsilon_{ij12})$ should be used if

$$|\sigma(\eta_{ij12})|_{e_{ij}} \cdot |\sigma(\eta_{ij22})|_{p_{ij}} \geq |\sigma(\eta_{ij21})|_{e_{ij}} \cdot |\sigma(\eta_{ij11})|_{p_{ij}}, \tag{8}$$

else $(\varepsilon_{ij21}, \varepsilon_{ij22})$ is the right one. The other choices can be decided in the same way, for δs using the corresponding λs and for ζs using ξs.

The current state is $(l_1 l_2 c a_{11}, r_1 r_2 d s)$ with letters from $P \cup Q$ distributed to both sides in the chosen way (as this property of letters from $P \cup Q$ will remain preserved, we will not mention that any more). We will gradually construct semi-equalities with the following states (without Ξ) and projections of $(\sigma(u), \sigma(v))$ to Ξ :

$$\big(l_1 l_2 c(s) a_{i_l j}, r_1 r_2 d(s)\big), \qquad \big(u'_{i_0} \cdots u'_{i_{l-1}} \cdot u'_{i_l}[j-1], v'_{i_0} \cdots v'_{i_{l-1}}\big), \tag{9}$$

$$\big(l_1 l_2 c(s) \overline{a}_{i_l k}, r_1 r_2 d(s)\big), \qquad \big(u'_{i_0} \cdots u'_{i_l}, v'_{i_0} \cdots v'_{i_{l-1}} \cdot v'_{i_l}[k-1]\big), \tag{10}$$

$$\big(l_1 l_2 c(s) a, r_1 r_2 d(s)\big), \qquad \big(u'_{i_0} \cdots u'_{i_l}, v'_{i_0} \cdots v'_{i_l}\big), \tag{11}$$

for $l = 0, \ldots, m$, where $i_0 = 1$, u'_i and v'_i can be obtained from u_i, v_i respectively, by adding some powers to letters and the corresponding blocks of the projections to Ξ are equal (for all blocks except the last one of the shorter word, this is already forced by the fact of being a semi-equality). Since for $l = m$ the state (11) contains no letters from Ξ, we can use one of the pairs of κs to exchange s for s_f, then γs to remove l_1 and r_1 and finally Lemma 13 to get a contradiction.

It remains to construct the desired semi-equalities. We show how to proceed from the state (9), the cases of (10) and (11) being similar. First, we introduce the letter $x_{i_l j}$. If $u'_{i_0} \cdots u'_{i_{l-1}} \cdot u'_{i_l}[j-1] \succeq v'_{i_0} \cdots v'_{i_{l-1}}$, then we just add the pair $(\vartheta_{i_l j 1}, \vartheta_{i_l j 2})$ to replace $a_{i_l j}$ with $a_{i_l j+1}$. If it is not the case, the block corresponding to the one of $x_{i_l j}$ is already present in $v'_{i_0} \cdots v'_{i_{l-1}}$. Let K be its length and L be the number of occurrences of $a_{i_l j}$ in the state. Let us first suppose that

$$K \cdot |\sigma(\vartheta_{i_l j 2})|_{a_{i_l j}} \geq L \cdot |\sigma(\vartheta_{i_l j 1})|_{x_{i_l j}}. \tag{12}$$

We substitute for every letter in the semi-equality its $|\sigma(\eta_{i_l j 11})|_{x_{i_l j}}$th power and add the pair

$$\left(\vartheta_{i_l j 1}^{|\sigma(\eta_{i_l j 11})|_{x_{i_l j}}}, \vartheta_{i_l j 2}^{|\sigma(\eta_{i_l j 11})|_{x_{i_l j}}}\right)$$

to get $a_{i_l j+1}$ instead of $a_{i_l j}$. Due to (12), we obtain a semi-equality. If (12) is satisfied as an equality, we are done. Else we put new pairs prior to the block of ϑs. Since they leave the state suitable for the addition of these ϑs and none of them affects $a_{i_l j}$, the powers should be always adjusted by the same amount for ϑs and for the letters coming from the original semi-equality. If $p_{i_l j}$ occurs in the state on the right, we add the pairs $(\eta_{i_l j 11}, \eta_{i_l j 12})$ and $(\eta_{i_l j 21}, \eta_{i_l j 22})$ to exchange $p_{i_l j}$ for $e_{i_l j}$ and back (if $p_{i_l j}$ is on the left, we have to use these pairs in the reverse order). If the number of occurrences of $x_{i_l j}$ in the state is insufficient for this construction, $x_{i_l j}$ vanishes from the state during the addition of $(\eta_{i_l j 11}, \eta_{i_l j 12})$ because the whole semi-equality was multiplied by $|\sigma(\eta_{i_l j 11})|_{x_{i_l j}}$ at the beginning and we are done. Suppose that there are still some occurrences of $x_{i_l j}$ remaining. Due to (8), for the resulting semi-equality the number of occurrences of $p_{i_l j}$ in its state is not smaller than in the state of the semi-equality without the new pairs since the number of $e_{i_l j}$s produced by the first pair has to match the number of them removed by the second one. Let M be the length of the first $x_{i_l j}$-block in the current state and let N be the number of occurrences of $x_{i_l j}$ coming from the $\eta_{i_l j 11}$-block just added. The resulting state allows us to do the same construction once again and we can repeat it arbitrary many times. In every semi-equality obtained in this way, the number of occurrences of $x_{i_l j}$ added by any block of $(\eta_{i_l j 11}, \eta_{i_l j 12})$ is at least the same as the number added by the previous one. Since the ratio of the number of occurrences of $x_{i_l j}$ coming from different blocks is preserved by a power adjustment, at latest during the $\lceil (M+N)/N \rceil$th iteration $x_{i_l j}$ vanishes from the state. Finally we add the pairs $(\eta_{i_l j 31}, \eta_{i_l j 32})$ and $(\eta_{i_l j 21}, \eta_{i_l j 22})$ prior to ϑs to remove the rest of the block of $p_{i_l j}$ and to replace $e_{i_l j}$ by $p_{i_l j}$ again.

If (12) is not valid, we have to increase the power of certain $\overline{x}_{ik}$ in the block corresponding to the block of $x_{i_l j}$ under consideration until (12) holds, thus reducing this case to the previous one. This can be done in an analogous way as in the previous paragraph. To see that it really gives the desired result, notice that we introduce letters using $\overline{p}_{ik}$ only when corresponding blocks of letters from Ξ are being adjusted which cannot happen behind this block of $\overline{x}_{ik}$. The other letters, whose ratios of powers are changed by additions of $\overline{\eta}$s, never affect adjustments. Thus when including the new iteration into the proper place in the semi-equality, the lengths of all blocks from the original semi-equality are multiplied by the same number.

The current state is (in the same notation as above)

$$\left(l_1 l_2 c(s) a_{i_l j+1}, r_1 r_2 d(s) g_{i_l j}\right), \qquad \left(u'_{i_0} \cdots u'_{i_{l-1}} \cdot u'_{i_l}[j], v'_{i_0} \cdots v'_{i_{l-1}}\right).$$

To remove $g_{i_l j}$ we can employ the same method as for the previous matching of corresponding blocks using $\xi_{i_l j+1}$s ($\xi_{i_l (i_l)_u+1} = \overline{\xi}_{i_l 1}$). We get either the next state (9) or (10) if $j = (i_l)_u$.

Obviously, in this way we can construct all required semi-equalities. $\square$

As an immediate consequence of the proposition, we get a result whose stronger version appeared already in [6] as Theorem 22.

Corollary 2 ([6])**.** *The weak trace code problem is undecidable.*

7. Content Fixation

In this section we show that in some special cases the conditions given by mappings μ and ν can be provided by additional structures in the alphabets.

Proposition 6. *Let (Σ, I) and (Σ', I') be two independence alphabets and $\mu, \nu : \Sigma \to 2^{\Sigma'}$ mappings such that I is transitive, $\Sigma' = \Sigma'_0 \cup \Gamma$, $\Sigma'_0 \cap \Gamma = \emptyset$, $\Gamma \times \Sigma' \subseteq I'$, for $x \in \Sigma$*

$$\mu(x) \subseteq \Sigma'_0, \quad \Gamma = \bigcup \{\{y\} \times \mu(y) \times (\Sigma \setminus \{y\}) \mid y \in \Sigma\}, \tag{13}$$

$$\nu(x) \cap \Gamma = \bigcup \{\{z\} \times \mu(z) \times \{y\} \mid yIx,\ z \in \Sigma \setminus \{y\}\}, \tag{14}$$

and for all $x, y \in \Sigma$

$$xIy,\ x \neq y \implies \mu(x) = \mu(y) = \emptyset, \tag{15}$$

$$\left(\nu(x) \times \nu(x) \subseteq I'\right) \,\&\, \left(xIy \implies \nu(x) = \nu(y)\right). \tag{16}$$

Define new independence alphabets (Σ_1, I_1) and (Σ'_1, I'_1) and $\nu_1 : \Sigma_1 \to 2^{\Sigma'_1}$ as follows. Set $\Theta = \bigcup \{\{x\} \times \mu(x) \mid x \in \Sigma\}$ and

$$\Sigma_1 = \Sigma \cup \Theta, \quad \Sigma'_1 = \Sigma' \cup \Theta,$$

$$I_1 \cap \Sigma^2 = I, \quad I_1 \cap (\Theta \times \Sigma_1) = id_\Theta.$$

As for I'_1, $I'_1 \cap (\Sigma' \times \Sigma') = I'$ and for all $x, y, z \in \Sigma$, $y \neq z$, $a \in \mu(x)$, $b \in \mu(y)$, $c \in \Sigma'_0$:

$$(x, a)\, I'_1 c \iff a \neq c, \tag{17}$$

$$(x, a)\, I'_1 (y, b, z) \iff (x, a) \neq (y, b), \tag{18}$$

$$(x, a)\, I'_1 (y, b) \iff (x, a) = (y, b). \tag{19}$$

For $x \in \Sigma$ and $a \in \mu(x)$, $\nu_1(x) = \nu(x)$ and $\nu_1((x,a)) = \{(x,a)\}$. Then I_1 is again transitive and ν_1 satisfies the condition (16)*. Assume in addition that whenever there is a (μ,ν)-weak coding $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$, there is also such a coding satisfying $\mathsf{c}(\varphi(x)) \subseteq \Sigma'_0$ for every $x \in \Sigma$. Then there is a (μ,ν)-weak coding $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ if and only if there is a ν_1-weak coding $\psi : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}(\Sigma'_1, I'_1)$.*

Proof. "$\Longrightarrow$" Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a (μ,ν)-weak coding. We can assume that $\mathsf{c}(\varphi(x)) \subseteq \Sigma'_0$ for every $x \in \Sigma$. For $x \in \Sigma$ and $a \in \mu(x)$, set

$$\psi(x) = \varphi(x) \cdot \prod_{\substack{y \in \Sigma \setminus \{x\} \\ a \in \mu(y)}} (y, a, x)\,, \quad \psi((x,a)) = (x,a)\,.$$

It is clear that this defines a ν_1-weak morphism $\psi : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}(\Sigma'_1, I'_1)$. In order to prove that ψ is injective, let $u, v \in (\Sigma_1)^+$ satisfy $\psi(u) \sim_{I'_1} \psi(v)$. $\pi_\Sigma(u) \sim_{I_1} \pi_\Sigma(v)$ since $\varphi(\pi_\Sigma(u)) = \pi_{\Sigma'_0}(\psi(u)) \sim_{I'} \pi_{\Sigma'_0}(\psi(v)) = \varphi(\pi_\Sigma(v))$. Also $\pi_\Theta(u) = \pi_\Theta(\psi(u)) = \pi_\Theta(\psi(v)) = \pi_\Theta(v)$. For $x, y \in \Sigma$, $x \neq y$, $a \in \mu(x)$, we can obtain $\pi_{(x,a),y}(u)$ from $\pi_{(x,a),(x,a,y)}(\psi(u))$ by substituting y for (x,a,y). As the same holds also for v and $(x,a)\,D'_1\,(x,a,y)$, we have $\pi_{(x,a),y}(u) = \pi_{(x,a),y}(v)$. To get $u \sim_{I_1} v$, it remains to show $\pi_{(x,a),x}(u) = \pi_{(x,a),x}(v)$ for $x \in \Sigma$, $a \in \mu(x)$. We have $\pi_{(x,a),a}(\psi(u)) = \pi_{(x,a),a}(\psi(v))$ and from $\pi_{(\Sigma \setminus \{x\}) \cup \{(x,a)\}}(u) \sim_{I_1} \pi_{(\Sigma \setminus \{x\}) \cup \{(x,a)\}}(v)$ shown above we get

$$\pi_{(x,a),a}\left(\psi\left(\pi_{\Sigma_1 \setminus \{x\}}(u)\right)\right) = \pi_{(x,a),a}\left(\psi\left(\pi_{\Sigma_1 \setminus \{x\}}(v)\right)\right).$$

As $a \in \mathsf{c}(\psi(x))$, these equalities imply $\pi_{(x,a),x}(u) = \pi_{(x,a),x}(v)$. Hence ψ is a coding.

"$\Longleftarrow$" Let $\psi : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}(\Sigma'_1, I'_1)$ be a ν_1-weak coding. Then the restriction φ of ψ to $\mathbb{T}(\Sigma, I)$ is clearly a ν-weak coding into $\mathbb{T}(\Sigma', I')$. Let $x \in \Sigma$ and $a \in \mu(x)$. As $\mathsf{c}(\psi((x,a))) = \{(x,a)\}$, $xD_1(x,a)$ implies that there is $\alpha \in \mathsf{c}(\psi(x))$ such that $\alpha D'_1(x,a)$. But $\mathsf{c}(\psi(x)) \subseteq \nu(x) \subseteq \Sigma'_0 \cup \bigcup \{\{y\} \times \mu(y) \times \{x\} \mid y \in \Sigma \setminus \{x\}\}$ due to (15). By (17) and (18), $\alpha = a$. Thus $a \in \mathsf{c}(\varphi(x))$. So φ is a (μ,ν)-weak coding. $\square$

Proposition 7. *Let (Σ, I) and (Σ', I') be two independence alphabets with I transitive and $|\Sigma'| \geq 2$ and $\nu : \Sigma \to 2^{\Sigma'}$ a mapping satisfying* (16)*. Let $\star \notin \Sigma$ be a new letter and $\tau : \Sigma \cup \{\star\} \to \{1, \ldots, n\}$ a bijection such that $\tau(\star) = 1$. Define new independence alphabets (Σ_1, I_1) and (Σ'_1, I'_1) as follows. Set*

$$\Theta = \big\{(x,i) \mid x \in \Sigma \cup \{\star\}\,,\ i \in \{1, \ldots, 2\tau(x)|\Sigma'|\}\big\}$$

and let $\Sigma_1 = \Sigma \cup \Theta$ and $\Sigma'_1 = \Sigma' \cup \Theta$. Next $I_1 \cap \Sigma^2 = I$, $I'_1 \cap (\Sigma')^2 = I'$ and for $(x,i), (y,j), (s,k), (\star,l) \in \Theta$, $x, y \in \Sigma \cup \{\star\}$, $s, t \in \Sigma$, $a \in \Sigma'$:

$$(x,i)\, I_1\, (y,j) \iff (x,i)\, I'_1\, (y,j) \iff x = y,$$

$$(s,i)\, I_1 t \iff sIt, \quad (s,i)\, I'_1 a \iff a \in \nu(s)\,,$$

$$(\star,l)\, I_1 t, \quad (\star,l)\, I'_1 a.$$

Then the following statements are equivalent.

1. *There is a ν-weak coding* $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$.
2. *There is a weak coding* $\psi : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}(\Sigma_1', I_1')$.
3. *There is a weak coding* $\sigma : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}\left((\Sigma_1')_{\Sigma_1}, (I_1')_{\Sigma_1}\right)$.

Proof. $1 \Longrightarrow 2$. Let $\varphi : \mathbb{T}(\Sigma, I) \to \mathbb{T}(\Sigma', I')$ be a ν-weak coding. Extend φ to $\psi : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}(\Sigma_1', I_1')$ taking it identical on Θ. When verifying that ψ is a weak morphism, everything is clear except the case $(x,i)\, I_1 y$ with $x, y \in \Sigma$; then xIy and $\mathsf{c}(\psi(y)) = \mathsf{c}(\varphi(y)) \subseteq \nu(y) = \nu(x)$ since φ is ν-weak and ν satisfies (16), but $\nu(x) \times \{(x,i)\} \subseteq I_1'$.

Let us prove that ψ is a coding. Suppose $u, v \in (\Sigma_1)^+$, $\psi(u) \sim_{I_1'} \psi(v)$. It is easy to see that $\pi_\Sigma(u) \sim_{I_1} \pi_\Sigma(v)$ and $\pi_\Theta(u) \sim_{I_1} \pi_\Theta(v)$ since both φ and the identity are injective. Take $(x,i) \in \Theta$ with $x \in \Sigma$. Consider the weak morphisms

$$\rho : \mathbb{T}\left(\Sigma \cup \{(x,i)\}, I_1 \cap (\Sigma \cup \{(x,i)\})^2\right) \to \mathbb{T}(\Sigma, I)$$

identical on Σ and mapping (x,i) to x and

$$\rho' : \mathbb{T}\left(\Sigma' \cup \{(x,i)\}, I_1' \cap \left(\Sigma' \cup \{(x,i)\}\right)^2\right) \to \mathbb{T}\left(\Sigma', I'\right)$$

identical on Σ' and mapping (x,i) to $\varphi(x) = \psi(x)$. Notice that ρ' is really a weak morphism because $(x,i)\, I_1' a$, where $a \in \Sigma'$, gives $a \in \nu(x)$, by (16) we get $\{a\} \times \nu(x) \subseteq I'$ and $\{a\} \times \mathsf{c}(\psi(x)) \subseteq I'$ as φ is ν-weak. Then the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{T}\left(\Sigma \cup \{(x,i)\}, I_1 \cap (\Sigma \cup \{(x,i)\})^2\right) & \xrightarrow{\rho} & \mathbb{T}(\Sigma, I) \\
{\scriptstyle\psi}\big\downarrow & & {\scriptstyle\varphi}\big\downarrow \\
\mathbb{T}\left(\Sigma' \cup \{(x,i)\}, I_1' \cap (\Sigma' \cup \{(x,i)\})^2\right) & \xrightarrow{\rho'} & \mathbb{T}(\Sigma', I')
\end{array}$$

Thus $(\varphi \circ \rho)\left(\pi_{\Sigma \cup \{(x,i)\}}(u)\right) \sim_{I'} (\varphi \circ \rho)\left(\pi_{\Sigma \cup \{(x,i)\}}(v)\right)$ and from the injectivity of φ we deduce $\rho\left(\pi_{\Sigma \cup \{(x,i)\}}(u)\right) \sim_I \rho\left(\pi_{\Sigma \cup \{(x,i)\}}(v)\right)$. If $y \in \Sigma$, $yD_1(x,i)$, then yDx and $\pi_{x,y}\left(\rho\left(\pi_{\Sigma \cup \{(x,i)\}}(u)\right)\right)$ is obtained from $\pi_{x,y,(x,i)}(u)$ by replacing all occurrences of (x,i) with x. As the same fact holds also for v and $\pi_{x,y}(u) = \pi_{x,y}(v)$, one can see that $\pi_{y,(x,i)}(u) = \pi_{y,(x,i)}(v)$. This proves $u \sim_{I_1} v$.

$2 \Longrightarrow 3$. is trivial.

$3 \Longrightarrow 1$. Let $\sigma : \mathbb{T}(\Sigma_1, I_1) \to \mathbb{T}\left((\Sigma_1')_{\Sigma_1}, (I_1')_{\Sigma_1}\right)$ be a weak coding. Let K be a clique in the graph $\left((\Sigma_1')_{\Sigma_1}, (I_1')_{\Sigma_1}\right)$ of at least $2|\Sigma'|$ elements. If we choose for each $\alpha \in K$ an arbitrary element of $\overline{\mathsf{c}}(\alpha)$, we get a clique K' in (Σ_1', I_1') of the same size by Lemma 14. But K' must contain at least 2 elements of Θ. Moreover, all of these elements are of the form (x,i) for a unique $x \in \Sigma \cup \{\star\}$ and all other elements of K' lie in $\nu(x)$ (for $x = \star$, consider $\nu(\star) = \Sigma'$). We can also see that, for any choice of K', this x is the same.

Thus, using (16) we obtain either $K \subseteq \{(x,i) \mid i \in \{1, \ldots, 2\tau(x)|\Sigma'|\}\} \cup \nu(x)$ for some $x \in \Sigma$ or $K \subseteq \{(\star,i) \mid i \in \{1, \ldots, 2|\Sigma'|\}\} \cup \Sigma'_{\Sigma_1}$.

Since, for $x \in \Sigma \cup \{\star\}$, the cardinality of the clique

$$\bigcup \{\mathsf{c}(\sigma((x,i))) \mid i \in \{1, \ldots, 2\tau(x)|\Sigma'|\}\}$$

is at least $2\tau(x)|\Sigma'|$, we can deduce (in sequence, starting from the largest $\tau(x)$ and using the fact that elements of different cliques are dependent)

$$\mathsf{c}(\sigma((x,i))) \subseteq \{(x,j) \mid j \in \{1, \ldots, 2\tau(x)|\Sigma'|\}\} \cup \nu(x)$$

for $(x,i) \in \Theta$, $x \in \Sigma$, and

$$\mathsf{c}(\sigma((\star,i))) \subseteq \{(\star,j) \mid j \in \{1, \ldots, 2|\Sigma'|\}\} \cup \Sigma'_{\Sigma_1}$$

for $(\star,i) \in \Theta$. In particular, there exist $(x,i_x), (x,j_x) \in \Theta$ such that $(x,i_x) \in \mathsf{c}(\sigma((x,j_x)))$ for all $x \in \Sigma \cup \{\star\}$. But for every $y \in \Sigma : yI_1(\star,j_\star)$ and $yI_1(y,j_y)$. This implies that $\mathsf{c}(\sigma(y)) \times \{(\star,i_\star),(y,i_y)\} \subseteq (I'_1)_{\Sigma_1}$, consequently $\bar{\mathsf{c}}(\sigma(y)) \subseteq \nu(y)$ and $\mathsf{c}(\sigma(y)) \subseteq \nu(y)$ by (16). Hence, we obtain a required ν-weak coding $\varphi : \mathbb{T}(\Sigma,I) \to \mathbb{T}(\Sigma',I')$ as the restriction of σ. □

8. Conclusions

Theorem 2 ([1]). *The trace coding problem is decidable when the first semigroup is a product of free semigroups.*

Theorem 3. *The trace coding problem is decidable when the first dependence alphabet is a forest.*

Proof. Both theorems immediately follow from Propositions 3 and 4. □

Theorem 4. *The weak trace coding problem and the trace coding problem are undecidable.*

Proof. Notice that (Σ,I), (Σ',I'), μ and ν from Proposition 5 (with Γ defined by (13)) satisfy all assumptions of Proposition 6. Using Proposition 7 on (Σ_1,I_1), (Σ'_1,I'_1) and ν_1 obtained there, we get by Proposition 3 an instance of the trace coding problem equivalent to $\mathcal{P}$. Thus we have an effective reduction of the PCP to both problems. □

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