On an involution on the set of Littlewood-Richardson tableaux and the hidden commutativity

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Abstract

The original definition of the Littlewood-Richardson (LR) rule for composing partitions is exclusively considered, *i.e.*, the classical combinatorial device for calculating the Littlewood-Richardson coefficients. The main result is an explicit involution on the set of LR tableaux which transforms an LR tableau of type $[a, b, c]$ into one of type $[b, a, c]$. On the basis of the involution definition it is a projection of LR tableaux of order r into those of order $r-1$, for $r \geq 1$. The main feature of this projection is the decomposition of an LR tableau of order r and type $[a, b, c]$ into a nested sequence of LR tableaux of order s and type $[a^{(s)}, (b_1, \ldots, b_s); (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, where $(a^{(s)})_{s=1}^{r}$ is a sequence of interlacing partitions which defines a decomposition of an LR tableau of type $[b, a, c]$ into a nested sequence of LR tableaux of order s and type $[(b_1, \ldots, b_s); a^{(s)}; (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$. This projection is accomplished introducing a combinatorial deletion and insertion operation on a LR tableau preserving the LR conditions. This involution yields a self-contained and direct combinatorial interpretation of the well-known commutative property of the original LR rule, as well as of the symmetry of the Littlewood-Richardson coefficients given by the equality $N_{ab}^{c} = N_{ba}^{c}$. It is known that the LR rule describes the Smith invariants of a product of integral matrices. It has been proven that this rule is also describing the eigenvalues of a sum of Hermitian matrices [13, 14, 17]. With the present involution we aim to a deeper understanding of the structure the LR rule and its relationship with these two problems in matrix theory.

Keywords: Young tableaux; Littlewood-Richardson rule; projection; involution

AMS Subject Classification: 05E10, 05E05, 15A18; 15A33

1 Introduction

The Littlewood-Richardson rule (LR rule for short) has many symmetries and properties [5, 6, 9, 24]. They do not seem clear from the original definition [15, 16], *i.e.*, from the combinatorial algorithm that calculates the Littlewood-Richardson coefficients. In this paper, our main goal is to present an involution on the set of LR tableaux which shows up the hidden commutativity of the LR rule. The main result is a combinatorial bijection on the set of LR tableaux transforming

[1] Work supported by FCT, CMUC/FCT, project Praxis 2/2.1/MAT/458/94, and Fundação Luso-Americana para o Desenvolvimento, project 574/94.

an LR tableau of type $[a, b, c]$ into one of type $[b, a, c]$, which is self-inverse. This bijection stresses that the combinatorial algorithm defining the LR rule for composing two partitions a and b is completely symmetric with respect to a and b. Our combinatorial approach relying on the notion of a projection of an LR tableau of order r into one of order $r-1$, with $r \geq 1$, is analogous to the behaviour of the triple of Smith invariants corresponding to a triple of square nonsingular matrices A, B, C, with $C = AB$, over a local principal ideal domain, under one row and column deletion, when a special form of the matrices A, B and C is considered, as that one established by R. C. Thompson in [19]. In [19] it is shown that we may consider A lower triangular, B diagonal, and C lower triangular with the Smith invariants along the diagonal.

We look mainly at the LR rule as a combinatorial object for composing partitions and focus our attention on the structure of that combinatorial object. Our motivation comes from the recent unified answer [13, 14, 17], given by the LRrule, for a long standing analogy between two problems in matrix theory: the description of the invariant factors (Smith invariants) of a product of matrices over a principal domain, and the eigenvalues of a sum of complex Hermitian matrices. We aim to a better understanding of the relation between the structure of this combinatorial object with these two problems in matrix theory.

In [9, 23] other descriptions of the Littlewood-Richardson coefficients are given, from which commutativity and other properties for these numbers also follow. Also in [22], increasing LR tableaux (or sequences) are used to point out some symmetries of the LR rule in the classical setting. For instance, it is shown that a bijection exists between the set of LR tableaux of type $[a, b, c]$ and the set of *increasing* LR tableaux of type $[b, a, c]$.

On the basis of [6], we consider LR *rectangular tableaux* and *LR rectangular triples.* Given partitions a, b and c (nonnegative integral vectors by weakly decreasing order) with length $\leq r$, an LR *rectangular tableau* of type [a,b,c] is an LR tableau of type (a, b, c^*) [16], where $c^* = (m - c_{r-i+1})_{i=1}^{r}$ for some nonnegative integer $m \geq c_1$, called dual partition of c. We call $[a, b, c]$ an LR rectangular triple. Therefore, $[a, b, c]$ is an LR *rectangular triple* iff (a, b, c^*) is an LR triple. Let $N_{a\,b}^{c}$ be the Littlewood-Richardson coefficient, *i.e.*, the number of LR tableaux of type (a, b, c). The number of LR rectangular tableaux of type $[a, b, c]$, written $N_{a\,b\,c}$, is precisely $N_{a\,b}^{c^*}$. Let V_a, V_b, V_c be irreducible finite dimensional SL_r-modules with highest weights a, b and c. In [6, 24], $N_{a\,b\,c}$ is the triple multiplicity, that is, the dimension of the space of SL_r-invariants in the triple tensor product $V_a \otimes V_b \otimes V_c$ and $N_{a\,b\,c} = N_{a\,b}^{c^*}$, where c^* is the highest weight of the module V_c^* dual to V_c. In the matrix setting LR rectangular tableaux, may be interpreted as follows. Let A, B and C r-square nonsingular matrices over a local principal ideal domain, with invariant factors (Smith invariants) $p^{a_1}, \ldots, p^{a_r}$, $p^{b_1}, \ldots, p^{b_r}$, and $p^{c_1}, \ldots, p^{c_r}$ respectively, where the exponents of the p-powers are considered by decreasing order, such that $ABC = p^{|c|} I_r$, with $|c| = \sum_{i=1}^{r} c_i$. Then, there is one and only one LR rectangular tableau $\mathcal{T}$ of type $[a, b, c]$ which $ABC = p^{|c|} I_r$ *realizes.* That is, $AB = C^*$ *realizes* one LR tableau of type (a, b, c^*) [1, 2, 12], and $\mathcal{T}$ is the corresponding LR rectangular tableau of type $[a, b, c]$, where C^* is the transpose of the adjugate of C and whose exponents of the invariant factors are given by $c_i^* = |c| - c_{r-i+1}$, for $i = 1, \ldots, r$.

The starting point for exhibiting the commutativity of the LR rule is to consider the following algebraic formulation of the LR rule. (For a similar presentation see [11, 20].) There exists an LR rectangular tableau of order r and type $[a, b, c]$ with $|a| + |b| + |c| = rm$, iff there exists a sequence of partitions $b^{(s)} = (b_1^{(s)}, \ldots, b_s^{(s)}, 0^{r-s})$, $s = 0, 1, ..., r$, with $b^{(r)} = b$, satisfying the interlacing inequalities

$$b_{i+1}^{(s)} \leq b_i^{(s-1)} \leq b_i^{(s)}, \text{ for } s = 1, 2, ..., r, \; i = 1, ..., r-1, \quad (i)$$

and the system of linear inequalities

$$a_{s-1}+\sum_{j=1}^{k-1}(b_j^{(s-1)}-b_j^{(s-2)})\geq a_s+\sum_{j=1}^{k}(b_j^{(s)}-b_j^{(s-1)}),\ k=1,\dots,s-1,\ s=2,\dots,r,\quad (ii)$$

$$a_s+\sum_{j=1}^{r}(b_j^{(s)}-b_j^{(s-1)})=m-c_{r-s+1},\ s=1,\dots,r.\quad (iii)$$

Given an LR rectangular tableau $\mathcal{T}$ of order r and type $[a,b,c]$ with $|a|+|b|+|c|=rm$, we may associate, for each $s\in\{1,2,\dots,r-1\}$, by deleting the $r,\dots,(s+1)$-th rows of $\mathcal{T}$, an LR rectangular tableau of order s and type $[(a_1,\dots,a_s);(b_1^{(s)},\dots,b_s^{(s)});\ (c_{r-s+1},\ \dots,c_r)]$. Indeed, this sequence of nested LR rectangular triples or LR rectangular tableaux of type $[(a_1,\dots,a_s);(b_1^{(s)},\dots,b_s^{(s)});(c_{r-s+1},\dots,c_r)]$, $s=1,\dots,r$, is such that the sequence of partitions $b^{(s)}=(b_1^{(s)},\dots,\ b_s^{(s)},\ 0^{r-s})$, $s=0,1,...,r$, with $b^{(r)}=b$, satisfy the previous linear inequalities (i), (ii) and (iii). We call it a b-decomposition of the LR triple $[a,b,c]$, and the b-decomposition of $\mathcal{T}$. The number of b-decompositions of $[a,b,c]$ is the Littlewood-Richardson number N_{ab}^c.

The bijection to be exhibited between LR rectangular tableaux of type $[a,b,c]$ and $[b,a,c]$ is based on a projection $\mathbf{P}$ of an LR rectangular tableau of order r into one of order $r-1$, for $r\geq 1$, which defines an a-decomposition of an LR tableau of type $[a,b,c]$. Given an LR rectangular tableau $\mathcal{T}$ of order r and type $[a,b,c]$ with $|a|+|b|+|c|=rm$, the projection $\mathbf{P}$ defines a nested sequence of LR rectangular tableaux $\mathcal{T}^{(0)}\subseteq\mathcal{T}^{(1)}\subseteq\ldots\subseteq\mathcal{T}^{(r)}=\mathcal{T}$, with $\mathcal{T}^{(0)}$ the empty tableau and $\mathcal{T}^{(s)}$ of order s and type $[(a_1^{(s)},\dots,a_s^{(s)});(b_1,\dots,b_s);(c_{r-s+1},\ \dots,c_r)]$, $s=1,...,r$, and such that the sequence of partitions $a^{(s)}=(a_1^{(s)},\dots,a_s^{(s)},0^{r-s})$, $s=0,1,...,r$, with $a^{(r)}=a$, satisfies

$$a_{i+1}^{(s)}\leq a_i^{(s-1)}\leq a_i^{(s)},\quad s=1,\dots,r,\quad i=1,\dots,r-1,\qquad (iv)$$

$$b_{s-1}+\sum_{j=1}^{k-1}(a_j^{(s-1)}-a_j^{(s-2)})\geq b_s+\sum_{j=1}^{k}(a_j^{(s)}-a_j^{(s-1)}),\ k=1,...,s-1,\quad s=2,\dots,r,\quad (v)$$

$$b_s+\sum_{j=1}^{r}(a_j^{(s)}-a_j^{(s-1)})=m-c_{r-s+1},\quad s=1,\dots,r.\quad (vi)$$

This decomposition defined by $\mathbf{P}$ is called the a-decomposition of $\mathcal{T}$ and an a-decomposition of $[a,b,c]$. On the other hand, regarding inequalities (iv), (v) and (vi), $[(b_1,\dots,b_s);(a_1^{(s)},\dots,a_s^{(s)});(c_{r-s+1},\dots,c_r)]$, $s=1,...,r$, is also an a-decomposition of $[b,a,c]$. Thus we conclude that, given an LR rectangular tableau $\mathcal{T}$ of type $[a,b,c]$, we may associate, by means of the projection $\mathbf{P}$, an LR rectangular tableau of type $[b,a,c]$, defined by the a-decomposition of $\mathcal{T}$.

Considering [19] we may assert:

Let A, B and C be r-square non-singular matrices with entries in a local principal ideal domain, and with $ABC=p^{|c|}I_r$. Let $p^{a_1},\dots,p^{a_r}$, $p^{b_1},\dots,p^{b_r}$, and $p^{c_1},\dots,p^{c_r}$ be the invariant factors of A, B and C, respectively, where $a_1\geq\ldots\geq a_r$, $b_1\geq\ldots\geq b_r$ and $c_1\geq\ldots\geq c_r$. We may assume that:

(i) A is lower triangular;

(ii) B is diagonal, $B=diag(p^{b_1},\dots,p^{b_r})$;

(iii) C^* is lower triangular and $C^*=[\gamma_{ij}]$ with $\gamma_{ii}=p^{c_i^*}$, $p^{c_i^*}|\gamma_{ij}$ for $i>j$, $1\leq i\leq r$ (the symbol "$|$" denotes divisibility).

Let $A^{(r)} := A$, $B^{(r)} := B$ and $C^{(r)} := C$. Now consider the sequence of product of matrices $A^{(s)}B^{(s)}C^{(s)} = p^{|c|}I_s$, $s = 1, \ldots, r-1$, obtained by deleting the $(s+1)$-th rows and columns of $A^{(s+1)}$, $B^{(s+1)}$ and $C^{(s+1)}$. That is, $A^{(s)}$, $B^{(s)}$ and $C^{(s)}$ are the s-leading submatrices in the first s rows of $A^{(s+1)}$, $B^{(s+1)}$ and $C^{(s+1)}$ respectively, for $s = 1, 2, \ldots, r-1$. Since A is in the triangular form, by the interlacing property relating the invariant factors of a matrix with those of a submatrix, we obtain, for each $s \in \{1, 2, \ldots, r\}$, one LR rectangular tableau of type $[(a_1^{(s)}, \ldots, a_s^{(s)}); (b_1, \ldots, b_s); (c_{r-s+1}, \ldots, c_r)]$ *realized* by $A^{(s)}B^{(s)}C^{(s)} = p^{|c|}I_s$, where the sequence $a^{(s)} = (a_1^{(s)}, \ldots a_s^{(s)}, 0^{r-s})$, $s = 0, 1, \ldots, r$, satisfies (iv).

We point out the analogy between the sequence of LR rectangular triples $[(a_1^{(s)}, \ldots, a_s^{(s)}); (b_1, \ldots, b_s); (c_{r-s+1}, \ldots, c_r)]$ where $a^{(s-1)}$ interlaces with $a^{(s)}$, for $s = 1, 2, \ldots, r$, produced by the matrix sequence $A^{(s)}B^{(s)}C^{(s)} = p^{|c|}I_s$, of s-leading submatrices in the first s rows of $ABC = p^{|c|}I_r$, $s = 1, \ldots, r$, and that one produced by the sequence of projections $\mathcal{T}^{(s)}$, $s = 1, \ldots, r$, of the LR rectangular tableau $\mathcal{T}$ of type $[a, b, c]$ *realized* by $ABC = p^{|c|}I_r$. This sequence of projections $\mathcal{T}^{(s)}$, $s = 1, \ldots, r$, is achieved by means of a combinatorial deletion operation defining $\mathbf{P}$, which decomposes a into a sequence of interlacing partitions. A matrix translation of this combinatorial deletion operation in the matrix product $ABC = p^{|c|}I_r$ is not explored here, that is, the answer to the question whether the sequences of partitions $(a^{(s)})_{s=1}^r$ produced by the matrix sequence $A^{(s)}B^{(s)}C^{(s)} = p^{|c|}I_s$ and the sequence of LR rectangular tableaux $\mathcal{T}^{(s)}$, $s = 1, \ldots, r$, coincide or not. If the answer is affirmative, since by transposition, the s-leading submatrices in the first s rows of $B^tA^tC^t = p^{|c|}I_r$, produce the sequence of LR rectangular triples $[(b_1, \ldots, b_s); (a_1^{(s)}, \ldots, a_s^{(s)}); (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, we obtain an a-decomposition of $[b, a, c]$, and the matrix meaning of our combinatorial involution in the context of the invariant factors is the transposition.

In the case of a sum of Hermitians matrices $A+B = C$, except when A, B and C are diagonal matrices, our combinatorial projection $\mathbf{P}$ does not translate the behaviour of the sequence of the triples of eigenvalues of the sum $A^{(s)} + B^{(s)} = C^{(s)}$, $s = 1, \ldots, r$, where $A^{(s)}$, $B^{(s)}$ and $C^{(s)}$ are, respectively, the s-leading submatrices in the first s rows of the square Hermitian matrices A, B and C, for $s = 1, \ldots, r$. In the Hermitian case, other projections have to be searched.

The paper is organized as follows. In section 2 we introduce some combinatorics related with LR rectangular tableaux as well as some polyhedral properties when looking at the set of LR rectangular tableaux of a fixed order, as a rational polyhedral cone. We also exhibit a bijection between the set of LR rectangular tableaux and the set of column LR rectangular tableaux [2], [12]. This section may have independent interest.

In section 3 we introduce a combinatorial deletion operation in an LR rectangular tableau with the aim to define a projection map $\mathbf{P}$. This deletion operation acts on an LR rectangular tableau $\mathcal{T}$ of order r and type $[a, b, c]$, by deleting boxes in the diagram of a in order to decompose $\mathcal{T}$ into a nested sequence of LR rectangular tableaux $\mathcal{T}^{(s)}$ of type $[(a_1^{(s)}, \ldots, a_s^{(s)}); (b_1, \ldots, b_s); (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, where $(a^{(s)})_{s=0}^r$ is a sequence of interlacing partitions.

In section 4, we introduce a combinatorial insertion operation in an LR rectangular tableau. The insertion operation acts on an LR rectangular tableau $\mathcal{F}$ of type $[(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)}); (b_1, \ldots, b_{r-1}); (c_2, \ldots, c_r)]$ by inserting boxes in the rows of the diagram of $(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)}, 0)$ in order to make a prolongation to a new LR tableau $\mathcal{T}$ of type $[(a_1, \ldots, a_r); (b_1, \ldots, b_r); (c_1, \ldots, c_r)]$ with $|a| + |b| + |c| = rm$, satisfying $\mathbf{P}(\mathcal{T}) = \mathcal{F}$. The number of inserted boxes in each row of $(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)}, 0)$ defines an *insertion sequence* of $\mathcal{F}$.

Deletion and insertion operations are reverse of each other. Let $\mathcal{F}^{(s)}$ be of type $[(a_1^{(s)}, \ldots, a_s^{(s)}); (b_1, \ldots, b_s); (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r-1$, be the decomposition of $\mathcal{F}$ defined by the projection map $\mathbf{P}$. It is shown, in theorems 6 and 7, that a sequence of non-negative integeres $(y_1, \ldots, y_{r-1}, a_r)$ is an inserting sequence *modulo* b_r, with $b_{r-1} \geq b_r \geq 0$, of $\mathcal{F}$ with prolongation $\mathcal{T}$ iff

$$a_{i+1} \leq a_i^{(r-1)} \leq a_i, \ \ i = 1, \ldots, r-1,$$

$$b_{r-1} + \sum_{j=1}^{k-1} (a_j^{(r-1)} - a_j^{(r-2)}) \geq b_r + \sum_{j=1}^{k} y_j, \ k = 1, ..., r-1, \ \ (*)$$

$$b_r + \sum_{j=1}^{r-1} y_i + a_r = m - c_1.$$

Since, $(a_i^{(s)} - a_i^{(s-1)})_{i=1}^{s}$ is an insertion sequence *modulo* b_s of $\mathcal{F}^{(s-1)}$, for $s = 1, \ldots, r-1$, considering the linear inequalities $(*)$, we conclude that the projection $\mathbf{P}$ decomposes an LR rectangular tableau $\mathcal{T}$ of type $[a, b, c]$ into a sequence of LR rectangular tableaux of order s and type $[(a_1^{(s)}, \ldots, a_s^{(s)})\ ; (b_1, \ldots, b_s);\ (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, where $a^{(s)}$, $s = 0, 1, \ldots, r$, with $a^{(r)} = a$, satisfies the linear inequalities (iv), (v) and (vi). We call $[(a_1^{(s)}, \ \ldots, a_s^{(s)});$ $(b_1, \ldots, b_s);\ (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, an a-decomposition of the LR triple $[a, b, c]$, and the a-decomposition of $\mathcal{T}$.

The main result of section 4 is theorem 7. This theorem asserts under what conditions, given LR rectangular tableaux $\mathcal{T}$ and $\mathcal{F}$ of types $[(a_1 \ldots, a_r);\ (b_1, \ \ldots, b_r);\ (c_1, \ldots, c_r)]$ and $[(a'_1 \ldots, a'_{r-1});\ (b_1, \ \ldots, b_{r-1});\ (c_2, \ldots, c_r)]$ respectively, one has $\mathcal{F} \in \mathbf{P}^{-1}(\mathcal{T})$.

In section 5, we define a bijection ϕ between LR rectangular tableaux of type $[a, b, c]$ and $[b, a, c]$, and we show that ϕ is an involution. Given an LR rectangular tableau $\mathcal{T}$ of order r and type $[a, b, c]$, we calculate, using projection $\mathbf{P}$, the a-decomposition of $\mathcal{T}$, $[(b_1, \ldots, b_s)$ $; (a_1^{(s)}, \ldots, a_s^{(s)});\ (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$. We define ϕ by transforming $\mathcal{T}$ into the LR rectangular tableau of type $[b, a, c]$ whose a-decomposition is the a-decomposition of $\mathcal{T}$. Clearly, ϕ is an injection. On the other hand, if we are given an LR rectangular $\mathcal{H}$ of type $[b, a, c]$ with a-decomposition $[(a_1^{(s)}, \ldots, a_s^{(s)})\ ; (b_1, \ldots, b_s);\ (c_{r-s+1}, \ldots, c_r)]$, $s = 1, \ldots, r$, then putting $\mathcal{T}^{(0)}$ the empty tableau, $(a_i^{(s)} - a_i^{(s-1)})_{i=1}^{s}$ is an insertion sequence *modulo* b_s of $\mathcal{T}^{(s-1)}$, with prolongation $\mathcal{T}^{(s)}$ of type $[(a_1^{(s)}, \ldots, a_s^{(s)}); (b_1, \ldots, b_s);\ (c_{r-s+1}, \ldots, c_1)]$, for $s = 1, \ldots, r$. This means that $\phi(\mathcal{T}^{(r)}) = \mathcal{H}$ and ϕ is also a surjection.

Finally, we give a recursive algebraic definition of the a-decomposition of $\mathcal{T}$ of type $[a, b, c]$. With this, we are able to prove, in theorem 10 and corollary 5, that ϕ is an involution: $\mathcal{T}$ of type $[a, b, c]$ has a-decomposition defined by $(a_s^{(s)})_{s=1}^{r}$, iff the b-decomposition of $\phi(\mathcal{T})$ of type $[b, a, c]$ with a-decomposition defined by $(a_s^{(s)})_{s=1}^{r}$ is the b-decomposition of $\mathcal{T}$. We close this section with an example of this involution.

2 LR rectangular tableaux and LR rectangular triples

2.1 Combinatorics and polyhedral properties

By partition a we mean any finite sequence $a = (a_1, \ldots, a_r)$ of nonnegative integers by (weakly) decreasing order. The *weight* of a, written $|a|$, is the sum of of the components. The partition of weight zero is denoted by 0. By *length* of a partition a we mean the number of non zero entries of a.

Let $m \geq 0$ and $r > 0$ be integers. Let $\mathcal{P}_r = \{a \in \mathbb{Z}^r : 0 \leq a_r \leq \ldots \leq a_1\}$ be the set of all partitions with r components. We write (x^r) to mean the constant partition of $\mathcal{P}_r$ with all components equal to x. We define $\mathcal{P}_{r,m} = \{a \in \mathcal{P}_r : 0 \leq a_r \leq \ldots \leq a_1 \leq m\}$. ($\mathcal{P}_{r,0} = \{0\}$.) Notice that, $\mathcal{P}_r = \bigcup_{m \geq 0} \mathcal{P}_{r,m}$.

Given $a \in \mathcal{P}_{r,m}$, $a^* := (m - a_{r-i+1})_{i=1}^r \in \mathcal{P}_{r,m}$ is called the *dual* partition of a in $\mathcal{P}_{r,m}$.

Consider the rectangular Young diagram of (m^r), *i.e.*, a sequence of r rows of boxes with row lengths m. If $a \in \mathcal{P}_{r,m}$ then $a \subseteq (m^r)$. (We identify a partition with its Young diagram.) Graphically, a^* is the partition defined by the complement of a in the Young diagram of (m^r). For example, if $r = 5$, $m = 6$ and $a = (5,5,4,4,2)$ we have $a^* = (4,2,2,1,1)$ (reading from bottom to top) represented by the blank boxes:

·	·	·	·	·	
·	·	·	·	·	
·	·	·	·		
·	·	·	·		
·	·				

Clearly, $(a^*)^* = a$.

Given $a, b \in \mathcal{P}_r$, we say that a and b are *congruent*, written $a \equiv b$, if $b = a + (M^r)$, for some integer $M \geq 0$. Clearly, $a \equiv (a_1 - a_r, \ldots a_{r-1} - a_r, 0) + (a_r^r)$. Therefore, when we write a^* without mentioning an upper bound for the largest component, we mean a partition congruent to $(a_1 - a_{r-i+1})_{i=1}^r$. Moreover, if a and b are congruent, a^* and b^* are congruent. Clearly, $a^* \in \mathcal{P}_{r,k}$, for all $k \geq a_1$.

Given a, b, $c \in \mathcal{P}_r$, we say that (a,b,c) is an *LR triple* if there is an LR tableau of type (a,b,c) [8]. Folowing [11], we identify an LR tableau of type (a,b,c) filled with x_{ij} symbols j in row i, for $r \geq i \geq j \geq 1$, with the element $(a,b,c,X) \in \mathbb{Z}^{3r+r^2}$, where $X = [x_{ij}]$ is an, $r \times r$, integral lower triangular matrix, such that the following system of linear inequalities is satisfied [8, 11, 17]:

$$
\begin{aligned}
x_{ij} &\geq 0, \quad 1 \leq i,j \leq r. && (1)\\
\sum_{i=1}^r x_{ij} &= b_j, \quad j = 1, \ldots, r. && (2)\\
\sum_{j=1}^r x_{ij} &= c_i - a_i, \quad i = 1, \ldots, r. && (3)\\
\sum_{i=1}^k x_{ij} &\geq \sum_{i=1}^{k+1} x_{i,j+1}, \quad 1 \leq k, j \leq r-1. && (4)\\
a_i + \sum_{j=1}^{k-1} x_{ij} &\geq a_{i+1} + \sum_{j=1}^k x_{i+1,j}, \quad {\scriptstyle k=1,\ldots,r-1 \text{ and } i=1,\ldots,r-1}. && (5)
\end{aligned}
$$

The *Littlewood-Richardson number*, $N_{a\,b}^c$, is the number of lower triangular matrices $X \in \mathbb{Z}^{r,r}$ whose entries satisfy this system of linear inequalities for fixed partitions a, b and c.

We may easily extend the *LR* rule to finite sequences of nonnegative real numbers. Given $\alpha = (\alpha_1, ..., \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_r)$ and $\gamma = (\gamma_1, \ldots, \gamma_r)$ sequences of nonnegative real numbers by weakly decreasing order, we say (α, β, γ) is a *real LR* triple if there is a lower triangular matrix $X = [x_{ij}] \in \mathbb{R}^{r,r}$ such that $(\alpha, \beta, \gamma, X) \in \mathbb{R}^{3r+r^2}$ satisfies the system of linear inequalities above (replacing a by α, b by β and c by γ). We call $(\alpha, \beta, \gamma, X)$ an *LR design* of order r [11]. When α, β, γ and X are integral, we have an integral *LR* design or, equivalently, an *LR* tableau of order r.

For $r \geq 1$, let $LRD_r^{\mathbf{R}}$ be the set of elements $(\alpha, \beta, \gamma, X) \in \mathbb{R}_{\geq 0}^{3r+r^2}$ such that the following conditions hold: $\alpha_1 \geq \ldots \geq \alpha_r \geq 0$, $\beta_1 \geq \ldots \geq \beta_r \geq 0$, $\gamma_1 \geq \ldots \geq \gamma_r \geq 0$ and $(\alpha, \beta, \gamma, X)$ satisfy linear inequalities (1) – (5). Let $LRD_r := LRD_r^{\mathbf{R}} \cap \mathbb{Z}^{3r+r^2}$ be the set of integral LR tableaux of order r.

$LRD_r^{\mathbf{R}}$ is a pointed rational polyhedral cone in $\mathbb{R}^{3r+r^2}$. Therefore, $LRD_r^{\mathbf{R}}$ has an integral Hilbert basis [18] and LRD_r is a finitely generated (additive) semigroup. Notice that $(a, b, c, X) + (a', b', c', X') = (a + a', b + b', c + c', X + X')$, with componentwise sum.

Let $LR_r = \{(a, b, c) \in (\mathcal{P}_r)^3 : (a, b, c, X) \in LRD_r, \text{for some}, r \times r, \text{ integral matrix X}\}$ be the set of LR triples of order r. Clearly, LR_r is also a finitely generated (additive) semigroup, called the *Littlewood -Richardson semigroup of order* r [24].

Let $LR_r^{\mathbf{R}}$ be the set of *real* LR triples of order r. $LR_r^{\mathbf{R}}$ is also a pointed rational polyhedral cone, finitely generated by the indecomposable elements of LR_r with respect to the sum. In [14] it is shown that LR_r is saturated in $LR_r^{\mathbf{R}}$.

Let $a, b, c \in \mathcal{P}_r$. A *rectangular tableau* of type $[a, b, c]$ is a tableau of type (a, b, c^*).

Notice that rectangular tableaux are symmetric in some sense relatively to a and c. Reading a rectangular tableau from right to left and from bottom to top we obtain an *opposite* (or *increasing*) rectangular tableau of type $[c, b, a]$, replacing each symbol i by $r - i + 1$ (see [4]).

Example 1 *Graphically, the following*

						1	1
					1		
		1	2	2			
1	2	2	3				

,

						4	4
					4		
		4	3	3			
4	3	3	2				

are, respectively, a rectangular LR tableau of type $[a, b, c]$ *and the corresponding increasing LR rectangular tableau of type* $[c, b, a]$*, where* $a = (6, 5, 2, 0)$, $b = (4410)$, e $c = (4, 3, 2, 0)$.

For rectangular tableaux and rectangular triples, we define, respectively,

$$\overline{LRD}_r = \{[a, b, c, X] : (a, b, c^*, X) \in LRD_r\},$$

and

$$\overline{LR}_r = \{[a, b, c] : (a, b, c^*) \in LR_r\}.$$

Clearly, if $[a, b, c, X] \in \overline{LRD}_r$ then $|a| + |b| + c| = rm$, for some non negative integer m.

Graphically, $[a, b, c, X] \in \overline{LRD}_r$ may be represented as follows:

<table><tr><td colspan="4">a_1</td><td>x_{11}</td><td>c_3</td></tr><tr><td colspan="2">a_2</td><td>x_{21}</td><td>x_{22}</td><td colspan="2">c_2</td></tr><tr><td>a_3</td><td>x_{31}</td><td>x_{32}</td><td>x_{33}</td><td colspan="2">c_1</td></tr></table>

an *LR rectangular* tableau of type $[a, b, c]$, where x_{ij} denotes the number of symbols j in row i, for $r \geq i \geq j \geq 1$.

Let $[a,b,c,X] \in \overline{LRD}_r$ with $|a|+|b|+c| = rm$. Notice that $[a,b,c,X] \in \overline{LRD}_r$ if $X = [x_{ij}]$ satisfy the system of linear inequalities defined by (1), (2), (4), (5) and (3) replaced by

$$\sum_{j=1}^{r} x_{ij} = m - a_i - c_{r-i+1},\ i = 1,\dots,r. \quad (**)$$

Denoting by $N_{a,b,c}$ the number of matrices X satisfying the conditions $(1)-(5)$ of the above system, with (3) replaced by $(**)$, it is clear that $N_{a,b,c} = N_{a,b}^{c^*}$. Hence, studying LR_r and LRD_r is the same as studying $\overline{LR}_r$ and $\overline{LRD}_r$, respectively, with the advantage that this triples and these tableaux are more symmetrical (see [6]). Extending the definitions of LR rectangular tableau and LR rectangular triple to nonnegative real numbers, we define $\overline{LRD}_r^{\mathbb{R}}$ the set of LR rectangular designs and $\overline{LR}_r^{\mathbb{R}}$ the set of real LR rectangular triples which are also pointed rational polyhedral cones in $\mathbb{R}^{3r+r^2}$. The integral vectors of these cones, $\overline{LRD}_r$ and $\overline{LR}_r$ respectively, are finitely generated semigroups.

2.2 Column LR rectangular tableaux

Column LR tableaux were firstly introduced in [12]. A reformulation of this rule in terms of indexing sets was given in [2].

In this subsection we exhibit a bijection between LR rectangular tableaux and column rectangular tableaux of the same type. The basic facts for this bijection are theorem 1 and algorithm 1 below. Algorithm 1 establishes a bijection between LR rectangular tableaux of type $[a,b,c]$ and column LR rectangular tableaux of type $[c,b,a]$; then theorem 1, (a) establishes a bijection between these tableaux and the opposite column LR rectangular tableaux of type $[a,b,c]$, and, finally, theorem 1, (b) establishes a bijection between the latter ones and the column LR rectangular tableaux of type $[a,b,c]$

Definition 1 *[2]Let $J = \{x_1,...,x_s\}$ and $K = \{y_1,...,y_m\}$ be finite sets of integers, where we are assuming that $x_1 > ... > x_s$ and $y_1 > ... > y_m$. Then we write $J \geq K$ (or $K \leq J$) whenever $s \geq m$ and $x_i \geq y_i$, for $i = 1,...,m$.*

Definition 2 *[2, 3, 4] Let J and K be the finite sets of integers defined above, where we are assuming that $x_1 < ... < x_s$ and $y_1 < ... < y_m$. We write $J \geq_{op} K$ (or $K \leq_{op} J$) whenever $s \leq m$ and $x_i \geq y_i$, for $i = 1,...,s$.*

Definition 3 *[2] Let $\mathcal{T}$ be a column rectangular tableau of type $[a,b,c]$ with indexing sets $J_1,...,J_t$. We say that $\mathcal{T}$ is a column Littlewood-Richardson rectangular tableau or a column Littlewood-Richardson sequence if*

$$J_1 \geq ... \geq J_t.$$

Definition 4 *[3] Let $\mathcal{T}$ be a column rectangular tableau of type $[a,b,c]$ with indexing sets $J_1,...,J_t$. We say that $\mathcal{T}$ is an opposite column Littlewood-Richardson rectangular tableau or an opposite column Littlewood-Richardson sequence if*

$$J_1 \geq_{op} ... \geq_{op} J_t.$$

Theorem 1 (a) *There is a bijection between the set of column LR rectangular tableaux of type* $[a, b, c]$ *and the set of opposite column LR rectangular tableaux of type* $[c, b, a]$.

(b) *There is a bijection between the set of column LR rectangular tableaux* $[a, b, c]$ *and the set of column opposite LR rectangular tableaux of type* $[a, b, c]$.

Proof : (a) See [3], Theorem 2.15.
(b) See [4], pp. $75, 79, 80$. ■

In [22] an opposite LR sequence is called increasing LR sequence.

Let us denote by $\overline{LRD}_r^{col}$ the set of column rectangular LR tableaux of degree r. If $\mathcal{T}$ is a column rectangular tableau of type $[a, b, c]$ with indexing sets $J_1, ..., J_t$, we write $\mathcal{T} = [a, b, c, J_1, ..., J_t]$.

Let $J \subseteq \{1, \ldots, r\}$. We define the *characteristic function* χ^J as $(\chi^J)_i = 1$ if $i \in J$, and $(\chi^J)_i = 0$ otherwise. If b is a partition and $\tilde{b} = (y_1, \ldots, y_t)$ is the conjugate partition of b, then $b = \sum_{i=1}^{t}(\chi^{J_i})$, where $J_i = \{1, \ldots, y_i\}$, for $i = 1, \ldots, t$.

The next algorithm defines a bijection between $\overline{LRD}_r$ and $\overline{LRD}_r^{col}$ transforming rectangular tableaux of type $[a, b, c]$ into those of type $[c, b, a]$.

Algorithm 1 *Let* $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$ *and* $\tilde{b} = (y_1, \ldots, y_t)$.

1. *Let* $k = 0$, $\mathcal{T}^{(0)} := \mathcal{T}$, $b^{(0)} := b$, $c^{(0)} := c$ *and* $X^{(0)} := X$.
   Let $\mathcal{F}^{(0)} := [c, 0, c^*, \emptyset, \ldots, \emptyset]$.

2. *Do* $k := k + 1$. *Consider the upper right most symbols* $1, 2, \ldots, y_k$ *in* $\mathcal{T}^{(k-1)}$ *and replace each of them by* k. *Let* $1 \leq i_1 < \ldots < i_{y_k}$ *be, respectively, the indices of the rows of these symbols. Define*
$$J_k := \{r - i_j + 1 : j = 1, \ldots, y_k\},$$
$$b^{(k)} := b^{(k-1)} - (1^{y_k}),$$
$$c^{(k)} := c^{(k-1)} + \chi^{J_k},$$
$$X^{(k)} = [x_{ij}^{(k)}], \ \textit{with}\ x_{ij}^{(k)} := x_{ij}^{(k-1)} - \chi^{\{i_1,\ldots,i_{y_k}\}},$$
$$\mathcal{T}^{(k)} := [a, b^{(k)}; c^{(k)}; X^{(k)}],$$
   and
$$\mathcal{F}^{(k)} := [c, \sum_{j=1}^{k}(1^{y_j}); (c^{(k)})^*; J_1, \ldots, J_k].$$

3. *If* $k = t$, *stop and write* $\mathcal{F} = [c, b, a; J_1, \ldots, J_t]$. *Otherwise, go to* 2. ■

Clearly $\mathcal{F} = [c, b, a; J_1, \ldots, J_t]$ is a column LR rectangular tableau, since by construction $J_1 \geq \ldots \geq J_t$.

Example 2
$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 & \ & \ \\ \hline \ & \ & 1 & 2 & 2 & \ & \ & \ \\ \hline 1 & 2 & 2 & 3 & \ & \ & \ & \ \\ \hline \end{array} \quad \longrightarrow \quad \mathcal{F} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 2 & 1 \\ \hline \ & \ & \ & \ & \ & 3 & \ & \ \\ \hline \ & \ & 4 & 2 & 1 & \ & \ & \ \\ \hline 5 & 4 & 3 & 1 & \ & \ & \ & \ \\ \hline \end{array}$$

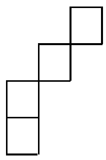
$\mathcal{F}$ *is obtained reading the second tableau from right to left, along rows, and from up to down, along columns.*

3 Deletion, deletion sequences and projection of LR rectangular tableaux

The main goal of this section is to define a deletion operation acting on an LR rectangular tableau preserving the LR conditions. We shall see that this operation acting on an LR rectangular tableau defines a projection map on $\overline{LRD}_r$ by transforming an LR tableau of order r and type $[(a_1,\ldots,a_r);(b_1,\ldots,b_r);(c_1,\ldots,c_r)]$ into one of order $r-1$, and type $[(a'_1,\ldots,a'_{r-1});(b_1,\ldots,b_{r-1});\ (c_2,\ \ldots,c_r)]$, where $a_{i+1}\le a'_i\le a_i$, for $i=1,\ldots,r-1$. This combinatorial operation aims to decompose an LR rectangular tableau of order r and type $[a,b,c]$ into a sequence of LR rectangular tableaux of order s and type $[(a_1^{(s)},\ldots,a_s^{(s)});(b_1,\ldots,b_s);\ (c_{r-s+1},\ \ldots,c_r)]$, $s=1,\ldots,r-1$, such that $a^{(s)}=(a_1^{(s)},\ldots,\ a_s^{(s)},\ 0^{r-s})$, $s=0,1,\ldots,r$, is a sequence of interlacing partitions.

3.1 Paths, path chains and deletion path chains. The poset of interior paths of length at least $r-k$ of an LR rectangular tableau of order r

A skew-diagram is called a vertical k-*strip*, where $k>0$, if it has k boxes and at most one box in each row. We let the vertical 0-strip equals to the empty set. For example,



is a vertical 4-*strip*.

For convenience, in what follows, we assume that the blank boxes of rectangular tableaux are numbered by 0.

Definition 5 *Let $r\ge 1$, $\mathcal{T}=[a,b,c,X]\in\overline{LRD}_r$ and $k\in\{1,\ldots,r\}$. An (r,k)-path $\mathcal{Z}$ of $\mathcal{T}$ is a numbered vertical k-strip of $\mathcal{T}$ containing exactly one nonzero numbered box of each row $i\in\{r-k+1,\ldots,r\}$, such that*

1. *if $(z_r,\ldots,z_{r-k+1})$ is the k-tuple of positive integers numbering the vertical k-strip, then*
$$r\ge z_r>\ldots>z_{r-k+1}\ge 1.$$
2. *if $(j_r,\ldots,j_{r-k+1})$ with $j_r,\ldots,j_{r-k+1}\in\{a_r+1,\ldots,a_{r-k}\}$ is the k-tuple of column indices of the vertical k-strip, from bottom to top and from left to right, then*
$$a_i+1\le j_i\le \min\{a_{r-k},a_i+\sum_{t=1}^{i}x_{it}\},\ \textit{for}\ i=r-k+1,\ldots,r.$$

(When $k=r$, we convention that $a_0:=a_1+x_{11}$.)

We let the $(r,0)$-path equals to the empty set $\emptyset$, and we call it the empty path

The strictly decreasing sequence $(z_r, \ldots, z_{r-k+1})$ of positive integers is called the *numbering sequence* of $\mathcal{Z}$, and $(j_r, \ldots, j_{r-k+1})$, where $a_r < j_r \leq \ldots \leq j_{r-k+1} \leq a_{r-k}$, is called the *column indexing sequence*.

An (r, k)-path $\mathcal{Z}$ is completely identified by its *numbering sequence* and its *column indexing sequence*. We write

$$\mathcal{Z} = ((z_r, j_r), \ldots, (z_{r-k+1}, j_{r-k+1})) \in \mathbb{Z}^{2k}.$$

All (r, k)-paths $\mathcal{Z}$ of $\mathcal{T}$, for $k \in \{1, \ldots, r\}$, have the bottom cell in the r-th row, and the top cell in the $(r - k + 1)$-th row of $\mathcal{T}$. Condition 2 of the previous definition means that the top cell of $\mathcal{Z}$ is always under a cell numbered by 0 when $r > k \geq 1$. Clearly, $1 \leq k \leq z_r \leq r$ and $1 \leq z_{r-k+1} \leq z_r - k + 1 \leq r - k + 1$.

Two paths are said *disjoint* if the supporting strips do not have cells in common.

The maximum number of pairwise disjoint (r, r)-paths of $\mathcal{T} = [a, b, c, X]$ is $b_r = x_{r,r}$. That is, the maximum number of pairwise disjoint vertical strips with numbering sequences of the form $(r, r - 1, \ldots, 2, 1)$ is b_r. These paths are called the *border* paths of $\mathcal{T}$.

Clearly, if the bottom cell of an (r, k)-path is numbered with $z_r < r$, then $k < r$ and $z_i < i$, for all $i = r - k + 1, \ldots, r$. We have also $1 \leq z_{r-k+1} \leq z_r - k + 1 < r - k + 1$.

Definition 6 *Let $r \geq 1$ and $k \in \{0, 1, \ldots, r - 1\}$. An (r, k)-path $\mathcal{Z}$ of $\mathcal{T}$ is called an interior (r, k)-path of $\mathcal{T}$ if its numbering sequence $(z_r, \ldots, z_{r-k+1})$ satisfies $r > z_r$.*

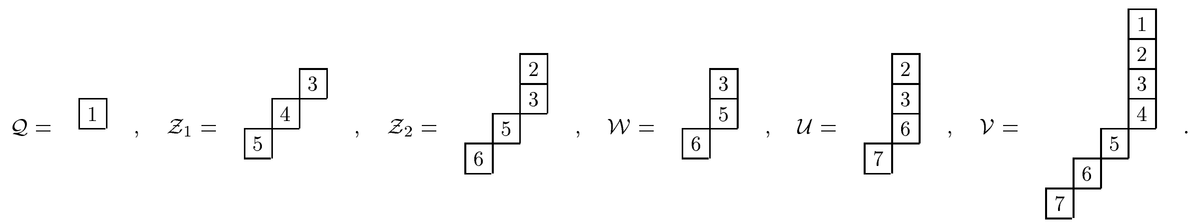
If $k > 0$, an *interior* (r, k)-path is an (r, k)-path whose bottom cell is numbered with $z_r < r$. Therefore, an interior path has length $k < r$ and does not have the top cell in the first row of $\mathcal{T}$.

The set of interior paths of $\mathcal{T}$ may be reduced to the empty path. This happens when $x_{r1} = \ldots = x_{r,r-1} = 0$.

Example 3 *Let*

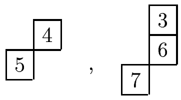
$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline
 & & & & & & & & & 1 & 1 & 1 & 1 & 1 \\\hline
 & & & & & & & 1 & 1 & 2 & 2 & 2 & 2 & \\\hline
 & & & & & & & 2 & 3 & 3 & 3 & 3 & & \\\hline
 & & & & & & 2 & 4 & 4 & 4 & 4 & & & \\\hline
 & & & 1 & 2 & 3 & 3 & 5 & 5 & & & & & \\\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & & & & & & \\\hline
1 & 3 & 4 & 5 & 6 & 7 & & & & & & & & \\\hline
\end{array}$$

The following numbered vertical strips of $\mathcal{T}$



are paths of $\mathcal{T}$: the first four $\mathcal{Q}, \mathcal{Z}_1, \mathcal{Z}_2$, $\mathcal{W}$ are interior ones, $\mathcal{V}$ is a border path and $\mathcal{U}$ is neither interior or border.

The vertical strips



are not paths of $\mathcal{T}$.

We shall omit "r" in the prefix $(r,k)-$ of these notations since "r" is always the order of the rectangular LR tableau, and we write only (interior) *path* or *k-path* if we want to stress the length of the strip.

Definition 7 *Let $r \geq 1$, $\mathcal{T} \in \overline{LRD}_r$ and $k, s \in \{1,\dots,r\}$. Let $\mathcal{Z}$ be a k-path and $\mathcal{W}$ be an s-path of $\mathcal{T}$. We say that $\mathcal{Z}$ is less than $\mathcal{W}$, written $\mathcal{Z} < \mathcal{W}$, if $k \leq s$ and the supporting strip of $\mathcal{W}$ is strictly to the right of the supporting strip of $\mathcal{Z}$. That is, for all $i \leq min\{k,s\}$, the cell of $\mathcal{W}$ in the $(r-i+1)$-th row is strictly to the right of the cell of $\mathcal{Z}$ in the $(r-i+1)$-th row. We write $\emptyset < \mathcal{W}$, where $\emptyset$ is the empty path.*

If $\mathcal{Z} < \mathcal{W}$, then $\mathcal{Z}$ and $\mathcal{W}$ are disjoint. In particular, distinct paths with cells in common are not comparable.

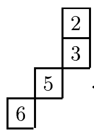
We say that $\mathcal{Z}$ is *less than or equal* to $\mathcal{W}$, written $\mathcal{Z} \leq \mathcal{W}$, if either $\mathcal{Z} < \mathcal{W}$ or $\mathcal{Z} = \mathcal{W}$.

Let P be a non empty subset of the set of (interior) paths of $\mathcal{T}$ then $(P, \leq)$ is a finite *poset*.

Definition 8 *Let $r \geq 1$, $\mathcal{T} \in \overline{LRD}_r$ and $k, s \in \{1,\dots,r\}$. Let $\mathcal{Z}$ be a k-path and $\mathcal{W}$ be an s-path of $\mathcal{T}$. We say that $\mathcal{W}$ is to the right of $\mathcal{Z}$, written $\mathcal{Z} \preceq \mathcal{W}$, if $k \leq s$ and the supporting strip of $\mathcal{W}$ is to the right of the supporting strip of $\mathcal{Z}$ possibly with cells in common. We write $\emptyset \preceq \emptyset$ and $\emptyset \preceq \mathcal{W}$.*

Clearly, if $\mathcal{Z} \leq \mathcal{W}$ then $\mathcal{Z} \preceq \mathcal{W}$. We write $\mathcal{Z} \prec \mathcal{W}$ when $\mathcal{Z} \preceq \mathcal{W}$ and $\mathcal{Z} \neq \mathcal{W}$. Clearly, if $\mathcal{Z} \prec \mathcal{W}$ and $\mathcal{Z}, \mathcal{W}$ do not have cells in common, then $\mathcal{Z} < \mathcal{W}$. Let P as before then $(P, \preceq)$ is a finite *poset*. $(P, \preceq)$ has maximum (minimum) if there is the right (left) most path of P, and, in this case, it is the left (right) most minimal (maximal) element of $(P, \leq)$.

Example 4 *In example 3, the least interior path and greatest interior path with respect to " $\preceq$ ", are $\mathcal{Q}$ and $\mathcal{Z}_2$ respectively,*

$\mathcal{Q} = \boxed{1}$, $\mathcal{Z}_2 =$  .

Definition 9 *[7, 10] Let $(P, \leq)$ be a poset.*

(*a*) *A chain C in P is a non empty subset, which, as a subposet is a chain.*

(*b*) *The length of a finite chain C is $\#C - 1$.*

(*c*) *A chain C in P is maximal if for any chain D in P, $C \subseteq D$ implies $C = D$.*

(# stands for the cardinal of a set.)

We say that $C = \{x_0 < x_1 < \ldots < x_d\}$ is a connected chain in P if x_i covers x_{i-1} for all i. If C is maximal then C is connected.

Definition 10 *[10] Let $(P, \leq)$ be a finite poset. The dimension of P, written $d[P]$, is*

$$max\,\{q : x_0 < x_1 < \ldots < x_q \text{ is a connected chain in } P\}.$$

Let $s \geq 0$ and $\mathcal{Z}_0 < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$, $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_s$ two s-chains of paths of $\mathcal{T}$ with respect to " $\leq$ ".. We say that $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_s$ is to the right of $\mathcal{Z}_0 < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$, written $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s) \preceq (\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_s)$, if $\mathcal{Z}_i \preceq \mathcal{W}_i$, for $i = 0, 1, \ldots, s$

If $\mathcal{C}$ is a non empty set of s-chains $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ of $\mathcal{T}$, then $(\mathcal{C}, \preceq)$ is a poset. A chain $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$ is called the right most s-chain of $\mathcal{C}$ if, for all s-chains, $\emptyset < \mathcal{W}_1 < \ldots < \mathcal{W}_s$ in $\mathcal{C}$ it holds $\mathcal{W}_i \preceq \mathcal{Z}_i$, for $i = 1, \ldots, s$.

Definition 11 *Let $r \geq 1$ and $\mathcal{T} \in \overline{LRD}_r$. For each $k \in \{0, 1, \ldots, r-1\}$, let $\mathcal{I}_{k+1}$ be the set of all interior paths of $\mathcal{T}$ having one cell in the $(k+1)$-th row, plus the 0-path. Equivalently, the set of all interior paths whose top cells are in the first $k+1$ rows, plus the 0-path.*

For $k \in \{0, 1, \ldots, r-1\}$, $(\mathcal{I}_{k+1}, \leq)$ is a finite *poset*. $(\mathcal{I}_{k+1}, \leq)$ is the poset of interior paths of length at least $r - k$ of $\mathcal{T}$, plus the 0-path. Clearly, $\{\emptyset\} = \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \ldots \subseteq \mathcal{I}_r$ and $0 = d[\mathcal{I}_1] \leq d[\mathcal{I}_2] \leq \ldots \leq d[\mathcal{I}_r]$.

Let $\emptyset < \mathcal{Z} < \mathcal{W}$ in $\mathcal{I}_{k+1}$. Then $\mathcal{W}$ covers $\mathcal{Z}$ iff $\mathcal{Z}$ and $\mathcal{W}$ have at least two adjacent cells, and $\mathcal{Z}$ covers $\emptyset$ iff $\emptyset < \mathcal{X} < \mathcal{Z}$ implies $\mathcal{X} \notin \mathcal{I}_{k+1}$.

Let $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_d$ be a chain in $\mathcal{I}_{k+1}$. Then, $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_d$ is a maximal chain only if, $\mathcal{W}_0 = \emptyset$, and, for each $i \in \{1, \ldots, d-1\}$, $\mathcal{W}_i$, $\mathcal{W}_{i+1}$ have at least two adjacent cells. In particular, if $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_d$ is a connected chain such that $\mathcal{W}_0 = \emptyset$, and the bottom symbols of the paths $\mathcal{W}_1, \ldots, \mathcal{W}_d$ are the d-right most symbols $< r$ of the r-th row, then the chain is maximal and it is the longest right most one.

In the following we determine the dimension of the poset $(\mathcal{I}_{k+1}, \leq)$ and we characterize the right most longest chain in order to define a deletion operation on an LR tableau.

If $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_q$ is a chain in $(\mathcal{I}_{k+1}, \leq)$, then $\{\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_q\}$ is a subset of $\mathcal{I}_{k+1}$ of pairwise disjoint interior paths. In general, if $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_q$ is a maximal chain in $\mathcal{I}_{k+1}$, it is not true that $\{\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_q\}$ is a maximal subset of pairwise disjoint paths of $\mathcal{I}_{k+1}$ with respect to set inclusion (see example 5). But if $\mathcal{W}_0 < \mathcal{W}_1 < \ldots < \mathcal{W}_q$ has maximum length next proposition shows that $\{\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_q\}$ is a maximal subset of pairwise disjoint paths of $\mathcal{I}_{k+1}$ with respect to set inclusion. Furthermore, we show that $d[\mathcal{I}_{k+1}]$ is the maximum cardinal of a maximal subset of $\mathcal{I}_{k+1} \setminus \{\emptyset\}$, with respect to set inclusion. Equivalently, $d[\mathcal{I}_{k+1}]$ is the maximum number of pairwise disjoint interior paths of $\mathcal{I}_{k+1} \setminus \{\emptyset\}$.

Proposition 1 *Let $r \geq 1$, $\mathcal{T} \in \overline{LRD}_r$ and $k \in \{0, 1, \ldots, r-1\}$. Let $D \subseteq \mathcal{I}_{k+1}$ be a maximal subset of $\mathcal{I}_{k+1}$ of pairwise disjoint interior paths, with respect to set inclusion,* *that is,* $\mathcal{X} \in \mathcal{I}_{k+1} \setminus \{\emptyset\}$ *only if* $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$*, for some* $\mathcal{Y} \in D$*. Then, there exists* $D' \subseteq \mathcal{I}_{k+1}$ *such that*

(a) $\#D' = \#D$*;*

(b) D' *is a maximal chain of* $(\mathcal{I}_{k+1}, \leq)$*;*

(c) *the set of cells defining the paths of* D *and the set of cells defining* D' *respectively, are the same.*

Proof: Clearly, if there exists a chain D' satisfying (a) and (c), then D' is a maximal chain of $(\mathcal{I}_{k+1}, \leq)$. Let $d := \#D$. We prove by induction on d that there exists a chain D' satisfying (a) and (c).

If $d = 1, 2$, then $D' = D$. In particular, $D' = D = \{\emptyset\}$, if $d = 1$.

Let $d > 2$ and suppose the claim true for $d - 1$. Let $D = \{\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_d\}$ and $f_1^{(r)} \leq \ldots \leq f_d^{(r)}$ the bottom symbols of $\mathcal{W}_1, \ldots, \mathcal{W}_d$, respectively.

1st case. $\mathcal{W}_1, \ldots, \mathcal{W}_{d-1}$ are comparable with $\mathcal{W}_d$ with respect to " $\leq$ ". Since $f_1^{(r)} \leq \ldots \leq f_d^{(r)}$, this means $\mathcal{W}_i < \mathcal{W}_d$, for $i = 1, \ldots, d-1$. Let $\tilde{\mathcal{D}} := \{\mathcal{W}_1, \ldots, \mathcal{W}_{d-1}\}$. By induction hypothesis, there exists $D'' = \{\emptyset < \mathcal{W}_1'' < \ldots < \mathcal{W}_{d-1}''\}$ where the cells of D'' and $\tilde{D}$ are the same. Therefore $D' = D'' \cup \{\mathcal{W}_d\}$ satisfies the required conditions.

2nd case. Let $\mathcal{W}_\alpha, \ldots, \mathcal{W}_\gamma$ with $\alpha < \ldots < \gamma$, be the elements of D which are not comparable with $\mathcal{W}_d$ with respect to " $\leq$ " (that is, $\mathcal{W}_i < \mathcal{W}_d$ iff $i \notin \{d, \alpha, \ldots, \gamma\}$). Suppose the numbering sequence of $\mathcal{W}_\gamma$ and $\mathcal{W}_d$ are $f_\gamma^r < \ldots < f_\gamma^u$ and $f_d^r < \ldots < f_d^v$ respectively. To fix ideas let $u \geq v$. We define the interior path $\mathcal{W}_d \preceq \mathcal{W}_d'$ with numbering sequence $(max\{f_\gamma^r, f_d^r\}, \ldots, max\,\{f_\gamma^v, f_d^v\}, f_\gamma^{v+1}, \ldots, f_\gamma^u)\}$ and the interior path $\mathcal{W}_\gamma' \preceq \mathcal{W}_\gamma$ with numbering sequence $(min\,\{f_\gamma^r, f_d^r\}, \ldots, min\,\{f_\gamma^v, f_d^v\})$. Clearly, $\mathcal{W}_\gamma' < \mathcal{W}_d'$ and $\mathcal{W}_\gamma' \cap \mathcal{W}_i = \mathcal{W}_d' \cap \mathcal{W}_i = \emptyset$, for $i \neq d, \gamma$, since the sets of cells involved in the paths of $\mathcal{W}_\gamma'$, $\mathcal{W}_d'$ and $\mathcal{W}_\gamma$, $\mathcal{W}_d$ respectively, are the same. We have therefore $\mathcal{W}_i < \mathcal{W}_d'$, for $i \notin \{d, \alpha, \ldots, \gamma\}$, and $\mathcal{W}_\gamma' < \mathcal{W}_d'$.

Now, we repeat the same reasoning for $\mathcal{W}_d'$ and $\{\mathcal{W}_\alpha, \ldots, \mathcal{W}_\gamma\} \setminus \{\mathcal{W}_\gamma\}$, etc. At the end of the process we obtain: $\mathcal{W}_i < \mathcal{W}_d' \preceq \tilde{\mathcal{W}}_d$, for $i \in \{1, \ldots, d-1\} \setminus \{\alpha, \ldots, \gamma\}$, $\mathcal{W}_\alpha' \preceq \mathcal{W}_\alpha, \ldots, \mathcal{W}_\gamma' \preceq \mathcal{W}_\gamma$ and $\mathcal{W}_\alpha', \ldots, \mathcal{W}_\gamma' < \tilde{\mathcal{W}}_d$. Moreover, the cells involved in $\{\mathcal{W}_\alpha, \ldots, \mathcal{W}_\gamma, \mathcal{W}_d\}$ and $\{\mathcal{W}_\alpha', \ldots, \mathcal{W}_\gamma', \tilde{\mathcal{W}}_d\}$ respectively, are the same.

Let $\tilde{\mathcal{D}} := \{\mathcal{W}_\alpha', \ldots, \mathcal{W}_\gamma', \tilde{\mathcal{W}}_d\} \cup (\{\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_{d-1}\} \setminus \{\mathcal{W}_\alpha, \ldots, \mathcal{W}_\gamma\})$. $\tilde{\mathcal{D}}$ is also a set of disjoint interior paths and the set of cells involved in $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are the same. We get reduced to the first case.■

Corollary 1 *Let* $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_q$ *be a chain of* $(\mathcal{I}_{k+1}. \leq)$*.*

(a) *If* $q = d[\mathcal{I}_{k+1}]$*, then* $\{\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_q\}$ *is a maximal subset of* $\mathcal{I}_{k+1}$ *of pairwise disjoint paths,*

$$\mathcal{X} \in \mathcal{I}_{k+1},\ \mathcal{X} \neq \emptyset \ \textit{ only if } \ \mathcal{X} \cap \mathcal{Z}_j \neq \emptyset, \textit{ for some } \ j \in \{1, \ldots, q\}.$$

(b) $q = d[\mathcal{I}_{k+1}]$ *iff* $\{\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_q\}$ *is a maximal subset of* $\mathcal{I}_{k+1}$ *of pairwise disjoint paths, with maximum cardinal.*

Proof: (a) Suppose that $\mathcal{X} \in \mathcal{I}_{k+1}$, $\mathcal{X} \neq \emptyset$ and $\{\mathcal{X}, \emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_q\}$ is a subset of pairwise disjoint paths of $\mathcal{I}_{k+1}$. By previous proposition, there would be a chain in $\mathcal{I}_{k+1}$ of length at least $q + 1$, wich contradicts $q = d[\mathcal{I}_{k+1}]$.

(b) Follows from (a) and the previous proposition.■.

Corollary 2 *Suppose $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_q$ is the right most q-chain of $(\mathcal{I}_{k+1}, \leq)$. Then, the following conditions are equivalent*

(a) $q = d[\mathcal{I}_{k+1}]$.

(b) $\{\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_q\}$ is a maximal subset of $\mathcal{I}_{k+1}$ of pairwise disjoint paths,

$$\mathcal{X} \in \mathcal{I}_{k+1},\ \mathcal{X} \neq \emptyset \ \text{ only if }\ \mathcal{X} \cap \mathcal{Z}_j \neq \emptyset, \text{ for some } j \in \{1, \ldots, q\}.$$

(c) $\mathcal{X} \neq \emptyset$, $\mathcal{X} < \mathcal{Z}_1$ only if $\mathcal{X} \notin \mathcal{I}_{k+1}$.

Proof: $(a) \Rightarrow (b)$. It is corollary $1, (a)$.

$(b) \Rightarrow (c)$. Suppose that $\emptyset < \mathcal{X} < \mathcal{Z}_1$ and $\mathcal{X} \in \mathcal{I}_{k+1}$. Then $\{\emptyset, \mathcal{X}, \mathcal{Z}_1, \ldots, \mathcal{Z}_q\}$ is a set of pairwise disjoint paths of $\mathcal{X} \in \mathcal{I}_{k+1}$, which contradicts (b).

$(c) \Rightarrow (a)$. Suppose $d[\mathcal{I}_{k+1}] = t > q$ and let $\emptyset < \mathcal{W}_1 < \ldots < \mathcal{W}_{t-q+1} < \ldots < \mathcal{W}_t$ be a chain of maximum length of $\mathcal{I}_{k+1}$. In particular, $\emptyset < \mathcal{W}_{t-q+1} < \ldots < \mathcal{W}_t$ is a q-chain, then $\mathcal{W}_{t-q+1} \preceq \mathcal{Z}_1 \ldots \mathcal{W}_t \preceq \mathcal{Z}_q$. Since $\mathcal{W}_1 < \ldots < \mathcal{W}_{t-q+1}$ this implies $\mathcal{W}_1 < \mathcal{Z}_1$ with $\emptyset < \mathcal{W}_1$ and $\mathcal{W}_1 \in \mathcal{I}_{k+1}$. This contradicts (c). ■

The last corollary says that to find a chain of maximum length in $(\mathcal{I}_{k+1}, \leq)$ it is enough to choose among the right most q-chains $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_q$ in $\mathcal{I}_{k+1}$ that one satisfying $\mathcal{Z}_1$ covers $\emptyset$.

Example 5 *Let*

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \ & & & & & & 1 & 1 & 1\\\hline & & & & & 1 & 2 & 2 & \\\hline & & 1 & 1 & 2 & 3 & & & \\\hline & 2 & 3 & 3 & 4 & & & & \\\hline\end{array}.$$

The following paths of $\mathcal{T}$

$\left(\emptyset,\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)$ *form a maximal subset of pairwise disjoint paths of $\mathcal{I}_3$ and therefore a maximal chain of $\mathcal{I}_3$ but not a maximal chain of maximum length. On the other hand,*

$(\emptyset, \mathcal{F}_1, \mathcal{F}_2) = \left(\emptyset\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array},\ \begin{array}{ccc}\cline{3-3} & & \multicolumn{1}{|c|}{1}\\\cline{2-3} & \multicolumn{1}{|c|}{2} & \\\cline{1-2} \multicolumn{1}{|c|}{3} & & \\\cline{1-1}\end{array}\right)$ *form a a maximal chain of $\mathcal{I}_3$ but does not form a maximal subset of pairwise disjoint paths of $\mathcal{I}_3$. $(\emptyset, \mathcal{F}_1, \mathcal{F}_2)$ is a maximal chain but not of maximum length. In this case,*

$\left(\emptyset,\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{1}\\\cline{1-2} \multicolumn{1}{|c|}{2} & \\\cline{1-1}\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{1}\\\cline{1-2} \multicolumn{1}{|c|}{3} & \\\cline{1-1}\end{array},\ \begin{array}{ccc}\cline{3-3} & & \multicolumn{1}{|c|}{1}\\\cline{2-3} & \multicolumn{1}{|c|}{2} & \\\cline{1-2} \multicolumn{1}{|c|}{3} & & \\\cline{1-1}\end{array}\right)$ *is a maximal chain of maximum length of $\mathcal{I}_3$ and $d[\mathcal{I}_3] = 3$. This means that not all connected chains between fixed end points have the same length.*

The following definition aims characterizing the right most maximal chain of maximum length.

Let $r \geq 1$ and $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$. Let $1 \leq \alpha < \beta < \ldots < \gamma < r$, and $0 \leq u \leq x_{r,\alpha}$, $0 \leq v \leq x_{r,\beta}, \ldots, 0 \leq w \leq x_{r,\gamma}$ We say that $\Lambda = \{\alpha_1, \ldots, \alpha_u,\ \beta_1, \ldots, \beta_v,\ \ldots,\ \gamma_1, \ldots, \gamma_w\}$ is a *list of symbols* $< r$ in the r-th row of $\mathcal{T}$, if $\alpha_1, \ldots, \alpha_u$, $\beta_1, \ldots, \beta_v, \ldots,$

$\gamma_1, \ldots, \gamma_w$ are, respectively, the *right most* u symbols α, v symbols $\beta, \ldots$, and w symbols γ of the r-th row of $\mathcal{T}$. If $u = v = \ldots = w = 0$, we have $\Lambda = \emptyset$, the empty list of symbols. When $u + \ldots + v + w = \sum_{j=1}^{r-1} x_{rj}$, we have the *full list of symbols* $< r$ of the r-th row of $\mathcal{T}$.

Definition 12 *Let $\Lambda = \{z_1 \leq \ldots \leq z_s\}$, $s \geq 0$, be a list of symbols, not necessarily the full list, of the r-th row of $\mathcal{T}$. We denote by $\mathcal{C}_\Lambda$ (or $\mathcal{C}_{z_1,\ldots,z_s}$ if $s > 0$) the set of all tuples $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ of s-chains $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$ of interior paths of $\mathcal{T}$, where the bottom cells of $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$ are numbered by $z_1, \ldots, z_s$, respectively. When, $s = 0$, $\Lambda = \emptyset$, and we have $\mathcal{C}_\Lambda = \{(\emptyset)\}$.*

For every list of symbols Λ in the r-th row of $\mathcal{T}$, $(\mathcal{C}_\Lambda, \preceq)$ is a *poset*.

Let $\Lambda = \{z_1, \ldots, z_s\}$, $s \geq 0$. The maximum of $\mathcal{C}_\Lambda$ is characterized in the following

Proposition 2 *The maximum $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ of $(\mathcal{C}_\Lambda, \preceq)$ is such that $\mathcal{Z}_0 = \emptyset$, $\mathcal{Z}_s$ is the right most path with bottom symbol z_s, and for $1 \leq j < s$, $\mathcal{Z}_j$ is the right most path with bottom symbol z_j, satisfying $\mathcal{Z}_j < \mathcal{Z}_{j+1}$.*

Proof: If $\Lambda = \{\emptyset\}$, the maximum of $\mathcal{C}_\Lambda$ is $(\emptyset)$.

We assume $\mathcal{T} \in \overline{LRD}_r$, with $r > 1$. We recall that the cells with nonzero symbols of a tableau are numbered along columns, from down to up, by strictly decreasing order.

The maximum element of $(\mathcal{C}_\Lambda, \preceq)$ may be determined as follows:

We start the process in the r-th row of $\mathcal{T}$. Write $z_i^{(r)} := z_i$, for $i = 1, \ldots, s$. Let $0 \leq z_1^{(r-1)} \leq \ldots \leq z_s^{(r-1)} < r - 1$ be the right most s symbols (possibly zeros) of the $(r-1)$-th row of $\mathcal{T}$ such that

$$0 \leq z_1^{(r-1)} < z_1^{(r)}, \ldots, z_s^{(r-1)} < z_s^{(r)} < r.$$

Delete the symbols 0 among $0 \leq z_1^{(r-1)} \leq \ldots \leq z_s^{(r-1)} < r - 1$. If they are all 0, then the maximum is the chain $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$ of interior 1-paths containing the symbols $z_1^{(r)}, \ldots, z_s^{(r)}$, respectively. Otherwise, nonzero symbols are left, let us say, $0 < z_{u+1}^{(r-1)} < \ldots < z_s^{(r-1)} < r - 1$, where $s > u \geq 0$.

Let $0 \leq z_{u+1}^{(r-2)} \leq \ldots \leq z_s^{(r-2)} < r - 2$ be the right most $s - u$ symbols of the $(r-2)$-th row of $\mathcal{T}$ such that

$$0 \leq z_{u+1}^{(r-2)} < z_{u+1}^{(r-1)}, \ldots, z_s^{(r-2)} < z_s^{(r-1)} < r - 1.$$

Delete the symbols 0 among $0 \leq z_{u+1}^{(r-2)} \leq \ldots \leq z_s^{(r-2)} < r - 2$. If they are all 0, then the maximum element of $\mathcal{C}_\Lambda$ is the chain $\emptyset < \mathcal{Z}_1 < \ldots < \mathcal{Z}_u < \mathcal{Z}_{u+1} < \ldots < \mathcal{Z}_s$, where $\mathcal{Z}_1 < \ldots < \mathcal{Z}_u$ is the chain of u interior 1-paths containing the symbols $z_1^{(r)}, \ldots, z_u^{(r)}$ and $\mathcal{Z}_{u+1} < \ldots < \mathcal{Z}_s$ is the chain of $s-u$ interior 2-paths containing the symbols $z_{u+1}^{(r)}, \ldots, z_s^{(r)}$. Otherwise, nonzero symbols are left, and, as before, proceed up, to the next rows, until nothing is left. The process will end up in the first row or before since the initial step starts in the r-th row with symbols $< r$ and, in each row, the symbols are at least one unity lesser than in the previous row. At the end of the process we arrive at the chain $(\mathcal{Z}_1, \ldots, \mathcal{Z}_s)$.

Let $(\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_s) \in \mathcal{C}_\Lambda$. We claim that $(\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_s) \preceq (\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$. Clearly, $\mathcal{W}_s \preceq \mathcal{Z}_s$. By construction, $\mathcal{Z}_{s-1}$ is the right most path $< \mathcal{Z}_s$, therefore $\mathcal{W}_{s-1} \preceq \mathcal{Z}_{s-1}$. Let $1 \leq j < s$, and suppose that $\mathcal{W}_{j+1} \preceq \mathcal{Z}_{j+1}$. Since $\mathcal{W}_j < \mathcal{W}_{j+1}$ it follows that $\mathcal{W}_j < \mathcal{Z}_{j+1}$. By construction, $\mathcal{Z}_j$ is the right most path $< \mathcal{Z}_{j+1}$. Therefore, $\mathcal{W}_j \preceq \mathcal{Z}_j$. The claim is proved. ■

Definition 13 *Let $s \geq 0$ and $\Lambda = \{z_1, \ldots, z_s\}$ be a list of symbols. Let $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ be the maximum of $\mathcal{C}_\Lambda$. $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ is called the deletion path chain of $\mathcal{T}$ generated by the list of symbols $\Lambda = \{z_1, \ldots, z_s\}$, or the symbols $z_1, \ldots, z_s$, with $s \geq 0$. (If $s = 0$, $\Lambda = \emptyset$ and $\mathbf{Z} = (\emptyset)$.)*

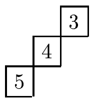
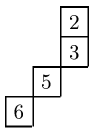
For $i = 1, \ldots, r-1$, let $y^{\mathbf{Z}}_{r,i}$ be the *number of interior paths* in $\mathbf{Z} = (\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ with length $r - i$. Equivalently, the number of interior paths in $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ with top cells in the $(i+1)-th$ row of $\mathcal{T}$. Clearly, $\sum_{i=1}^{r-1} y^{\mathbf{Z}}_{ri} = s$ and, for $k = 1, \ldots, r-1$, $\sum_{i=1}^{k} y_{rj}$ is the number of interior paths of $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ with top cells in the first $k+1$ rows of $\mathcal{T}$.

Notice that $y^{\mathbf{Z}}_{r,i}$ is precisely the number of zeros deleted in the i-th row of the diagram of a during the process of calculation of the maximum element $\mathbf{Z}$ described in the previous proposition. Thus, we call $(y^{\mathbf{Z}}_{r,1}, y^{\mathbf{Z}}_{r,2}, ..., y^{\mathbf{Z}}_{r,r-1})$ the $\mathbf{Z}$-*deletion sequence* of $\mathcal{T}$ or the *deletion sequence generated* by the symbols $z_1, \ldots, z_s$. When there is no ambiguity we omit the supraindexation "$\mathbf{Z}$" of each $y^{\mathbf{Z}}_{r,i}$ in the notation of the $\mathbf{Z}$-*deletion sequence*, and writing only $(y_{r1}, y_{r2}, ..., y_{r,r-1})$.

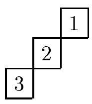
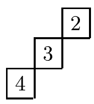
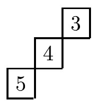
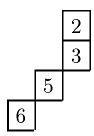
Definition 14 *Let $\mathcal{T} \in \overline{LRD}_r$ with $r \geq 1$ and $\sum_{i=1}^{r-1} x_{ri} = s \geq 0$. Let $\Lambda = \{z_1 \leq \ldots \leq z_s\}$, $s \geq 0$, be the full list of symbols $< r$ in the r-th row, and $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ the maximum of $\mathcal{C}_\Lambda$. Let $(y_{r1}, y_{r2}, ..., y_{r,r-1})$ be the deletion sequence generated by the symbols $z_1 \leq \ldots \leq z_s$, $s \geq 0$. We call $(y_{r1}, y_{r2}, ..., y_{r,r-1})$ the r-deletion sequence of $\mathcal{T}$, and $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ the r-deletion path chain of $\mathcal{T}$.*

Note that if $\Lambda = \emptyset$, $\underbrace{(0, \ldots, 0)}_{r-1}$ is the r-deletion sequence of $\mathcal{T}$.

Example 6 *In example 3, the maximum element of $\mathcal{C}_{56}$ is*

$\mathbf{Z} = (\emptyset, \mathcal{Z}_1, \mathcal{Z}_2) = (\emptyset,$  , ),

and the $\mathbf{Z}$-deletion sequence is $(0, 0, 1, 1, 0, 0)$. The 7-deletion path chain of $\mathcal{T}$ is

$(\emptyset,$  ,  ,  ,  , ), *and the deletion sequence is* $(0, 0, 1, 3, 1, 1)$.

Considering proposition 2 and the definition of $\mathbf{Z}$-deletion sequence, we have

Observation 1 *Let* $r \geq 1$, $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$. *Let* $\Lambda = \{z_1, \ldots, z_s\}$, $s \geq 0$, *be a list of symbols and* $\Lambda_j = \{z_j, \ldots, z_s\}$, *for* $j = 1, \ldots, s$.

1. $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s) = max\, \mathcal{C}_\Lambda$ *iff* $(\emptyset, \mathcal{Z}_j, \ldots, \mathcal{Z}_s) = max\, \mathcal{C}_{\Lambda_j}$, *for* $j = 1, \ldots, s$.

2. *Let* $\mathbf{Z}$ *be the maximum of* $\mathcal{C}_\Lambda$ *with* $\mathbf{Z}$*-deletion sequence* $(y_{r1}, \ldots, y_{r,r-1})$. *Let* $(\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_s) \in \mathcal{C}_\Lambda$, *and* m_{ri} *the number of interior paths in* $(\emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_s)$ *with top cells in the* $(i+1)$*-th row of* $\mathcal{T}$, *for* $i = 1, \ldots, r-1$. *Then,*

   (a) $a_{i+1} \leq a_i - y_{ri} \leq a_i$, *for* $i = 1, \ldots, r-1$.

   (b) $(y_{r1}, y_{r2}, ..., y_{r,r-1})$ *majorizes* $(m_{r1}, m_{r2}, ..., m_{r,r-1})$, *that is,*

   $$\sum_{j=1}^{k} y_{rj} \geq \sum_{j=1}^{k} m_{rj}, \quad \textit{for } k = 1, \ldots, r-1,$$

   with $\sum_{j=1}^{r-1} y_{rj} = \sum_{j=1}^{r-1} m_{rj} = s$. *Among the chains in* $\mathcal{C}_\Lambda$, $\mathbf{Z}$ *is the one which have more top cells in the first* $k+1$ *rows, for* $k = 1, \ldots, r-1$.

Lemma 1 *Let* $r \geq 1$, $\mathcal{T} \in \overline{LRD}_r$. *Let* $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ *be the* r*-deletion path chain and* $(y_{r1}, \ldots, y_{r,r-1})$ *the* r*-deletion sequence of* $\mathcal{T}$. *Let* $k \in \{0, 1, \ldots, r-1\}$ *and* $m = \sum_{j=1}^{k} y_{rj}$. *Then*

1. *For* $i = k+1, \ldots, r$, $\sum_{j=1}^{k} y_{rj} \leq \sum_{j=1}^{i-1} x_{i,j}$ *and* $\sum_{j=1}^{k} y_{rj} + b_r \leq \sum_{j=1}^{i} x_{i,j}$. *In particular, when* $k = r-1$, $\sum_{j=1}^{r-1} y_{rj} = \sum_{j=1}^{r-1} x_{r,j}$.

2. $d[\mathcal{I}_r] = \sum_{j=1}^{r-1} y_{rj}$ *and* $\mathcal{Z}_0 < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$ *is the right most longest chain of* $\mathcal{I}_r$.

3. $\mathcal{Z}_j \in \mathcal{I}_{k+1}$ *iff* $j \in \{s-m+1, \ldots, s\}$ *or* j*=0.*

4. *If* $\mathcal{X} \in \mathcal{I}_{k+1}$ *and* $\mathcal{X} \neq \emptyset$, *then* $\mathcal{X} \preceq \mathcal{Z}_s$ *and there exists* $j \in \{s-m+1, \ldots, s\}$ *such that* $\mathcal{X} \cap \mathcal{Z}_j \neq \emptyset$. *That is,* $\mathcal{X} \neq \emptyset$, $\mathcal{X} < \mathcal{Z}_{s-m+1}$ *only if* $\mathcal{X} \notin \mathcal{I}_{k+1}$.

Proof: (1) and (2) are obvious.

(3) Let $\mathcal{Z}_j \neq \emptyset$. By definition of r-deletion sequence and m, if $j < s-m+1$, $\mathcal{Z}_j \notin \mathcal{I}_{k+1}$.

(4) Suppose that $\mathcal{X} \neq \emptyset$, $\mathcal{X} < \mathcal{Z}_{s-m+1}$ and $\mathcal{X} \in \mathcal{I}_{k+1}$. Since $\mathcal{Z}_{s-m} < \ldots < \mathcal{Z}_s$ is the right most $(m+1)$-chain of $\mathcal{I}_r$, it holds $\mathcal{X} \preceq \mathcal{Z}_{s-m} < \mathcal{Z}_{s-m+1}$. But then $\emptyset < \mathcal{Z}_{s-m} < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ is a chain in $\mathcal{I}_{k+1}$. This is a contradiction with (3). ■

Theorem 2 *Let* $r \geq 1$, $\mathcal{T} \in \overline{LRD}_r$ *and* $k \in \{0, 1, \ldots, r-1\}$. *Let* $\mathbf{Z} = (\mathcal{Z}_\mathbf{0}, \mathcal{Z}_\mathbf{1}, \ldots, \mathcal{Z}_s)$ *be the* r*-deletion path chain of* $\mathcal{T}$ *with* r*-deletion sequence* $(y_{r1}, \ldots, y_{r,r-1})$. *Let* $m = \sum_{j=1}^{k} y_{rj}$. *Then*

1. $(\mathcal{I}_{k+1}, \leq)$ *is a poset of dimension* $\sum_{j=1}^{k} y_{rj}$, *and* $\emptyset < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ *is the right most longest chain of* $\mathcal{I}_{k+1}$.

2. If $a_k - a_{k+1} > 0$, $d[\mathcal{I}_{k+1}] < a_k - a_{k+1}$ only if $d[\mathcal{I}_r] < a_k - a_r$.

3. $\mathcal{I}_{k+1} = \{\emptyset\}$ only if $d[\mathcal{I}_r] \leq a_{k+1} - a_r$.

Proof: (1) By construction, $\emptyset < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ is the right most m-chain of $\mathcal{I}_{k+1}$ and satisfies condition (4) of the previous lemma. By corollary 2, this means that $\emptyset < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ is the right most maximal chain of $\mathcal{I}_{k+1}$ of maximum length.

(2) Let $q := a_k - a_{k+1} > 0$. If $a_k \leq a_r + \sum_{j=1}^{r-1} x_{rj}$, there is a chain $\mathcal{W}_1 < \ldots < \mathcal{W}_q \in \mathcal{I}_{k+1}$ with top cells exactly in row $k+1$ of $\mathcal{T}$. Now the conclusion follows from (1).

(3) If $a_r + \sum_{j=1}^{r-1} x_{rj} > a_{k+1}$ there is a path in $\mathcal{I}_{k+1}$ with the top cell in some row $i \in \{2, \ldots, k+1\}$. ■

3.2 Deletion, projection of LR rectangular tableaux and interlacing conditions

Based on the notion of deletion path chain and deletion sequence of an LR rectangular tableau introduced in the previous subsection, we are led to a deletion operation in LR rectangular tableaux. Given an LR rectangular tableau $\mathcal{T}$ of type $[a, b, c]$ with deletion sequence $(y_1, \ldots, y_{r-1})$ and deletion path sequence $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$, the deletion operation is defined as follows: moving one step up the cells along each path $\mathcal{Z}_j$, $j > 0$, of the deletion path sequence, insert in each row i of the diagram of a, the y_i top cells of the y_i paths in $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ whose top cells are in row $i+1$ of $\mathcal{T}$, and remove equal number of cells in the row i of a; eventually, delete the last rows of a, b and the first row of c. This combinatorial operation defines a projection $\mathbf{P}$ of $\overline{LRD}_r$ on $\overline{LRD}_{r-1}$, for $r \geq 1$. The map $\mathbf{P}$ on $\overline{LRD}_r$ transforms an LR tableau $\mathcal{T}$ of type $[(a_1, \ldots, a_r); (b_1, \ldots, b_r); (c_1, \ldots, c_r)]$ with $|a| + |b| + |c| = rm$ into one of type $[(a'_1, \ldots, a'_{r-1}, 0); (b_1, \ldots, b_{r-1}, 0); (m, c_2, \ldots, c_r)]$, such that the partition a' interlaces with a, that is, $a_{i+1} \leq a'_i \leq a_i$, for $i = 1, \ldots, r-1$.

Lemma 2 *(Elementary deletion) Let $r \geq 1$ and $\mathcal{T} = [(a_1, ..., a_r), (b_1, ..., b_r), (c_1, ..., c_r), X] \in \overline{LRD}_r$. Let Λ be a list of symbols such that $0 \leq \#\Lambda \leq 1$ and $\mathbf{Z}$ the maximum of $\mathcal{C}_\Lambda$. If $\Lambda = \emptyset$, let $X^{\mathbf{Z}} = X^{(\emptyset)} = X$. If $\Lambda \neq \emptyset$, let $\mathbf{Z} = (\emptyset, \mathcal{Z}_1)$ with $(k_r, k_{r-1}, ..., k_{r-s+1})$ the numbering sequence of $\mathcal{Z}_1$, and $X^{\mathbf{Z}} = (X^{(\mathcal{Z}_1)})^{(\emptyset)}$, where $X^{(\mathcal{Z}_1)} = [x'_{ij}] \in \mathbb{Z}^{r,r}$ is such that $x'_{i,k_{i+1}} = x_{i,k_{i+1}} + 1$, $r - 1 \leq i \leq r - s$, $x'_{i,k_i} = x_{i,k_i} - 1$, $r \leq i \leq r - s + 1$, and $x'_{ij} = x_{ij}$, otherwise. Then $\mathcal{T}^{\mathbf{Z}} = [(a_1, \ldots, a_{r-s} - \#\Lambda, \ldots, a_{r-1}, a_r), (b_1, ..., b_r), (c_1 + \#\Lambda, ..., c_r), X^{\mathbf{Z}}] \in \overline{LRD}_r$.*

Proof: If $\Lambda = \emptyset$, there is nothing to prove. Let $r > 1$ and $\Lambda = \{k_r\}$. The proof imitates the calculation of the maximum element $\mathbf{Z}$ of $\mathcal{C}_{k_r}$ in proposition 2. Recall that $r > k_r > k_{r-1} > \ldots > k_{r-s+1}$ and that k_i is the right most symbol of the i-th row such that $k_{i+1} > k_i$, for $i \in \{r - s + 1, \ldots, r\}$. (We let $k_{r+1} := k + 1$.) Insert the symbol k_r of the r-th row in the $(r-1)$-th row by shoving the symbol k_{r-1}. Then insert k_{r-1} in the $(r-2)$-th row by shoving the symbol k_{r-2}. We shall get $x'_{r,k_r} = x_{r,k_r} - 1$, $x'_{r-1,k_r} = x_{r-1,k_r} + 1$ and $x'_{r-1,k_{r-1}} = x_{r-1,k_{r-1}} - 1$, $x'_{r-2,k_{r-1}} = x_{r-2,k_{r-1}} + 1$. Proceed up, in this way, until reaching the $(r-s)$-th row, where the symbol k_{r-s+1} is inserted by deleting the right most symbol zero in the $(r-s)$-th row of the diagram of a. Eventualy,

$x'_{r-s,k_{r-s+1}} = x_{r-s,k_{r-s+1}} + 1$. It is clear, that the output is a rectangular tableau $\mathcal{T}^{\mathbf{Z}} = [(a_1, \ldots, a_{r-s} - 1, \ldots, a_{r-1}, a_r), (b_1, ..., b_r), (c_1 + 1, ..., c_r), X^{\mathbf{Z}}]$, that is the nonzero symbols strictly increase along columns, from up to down, and do not decrease along rows, from left to right.

Let us prove that $\mathcal{T}^{\mathbf{Z}}$ is in $\overline{LRD}_r$. We have only to check that $X^{\mathbf{Z}}$ satisfies inequality (4), in the definition of LR rectangular tableau given in section 2, for $k_r \geq j \geq k_{r-s+1}$ and $r \geq i \geq r - s$.

Let $t \in \{r - s + 1, \ldots, r - 1\}$ and $t + 1 > u = k_{t+1} > v = k_t$. The basic operation in the previous process is as follows: the right most symbol u of the $(t+1)$-th row is inserted in the t-th row by shoving the right most symbol v, which on its turn, is inserted in the $(t-1)$-th row by shoving a symbol $< v$. Therefore $x'_{tu} = x_{tu} + 1$, $x'_{t+1,u} = x_{t+1,u} - 1 \geq 0$ and $x'_{iu} = x_{iu}$, for $i \neq t, t+1$.

If $v < u - 1$, $x'_{i,u-1} = x_{i,u-1}$, for all i, and, in particular, $x'_{t,u-1} = x_{t,u-1} = 0$.

If $v = u - 1$, $x'_{t,u-1} = x_{t,u-1} - 1 \geq 0$ and $x'_{t-1,u-1} = x_{t-1,u-1} + 1$.

It is sufficient to prove that

$\sum_{i=1}^{t-1} x'_{i,u-1} \geq \sum_{i=1}^{t} x'_{iu}$.

Case 1: $v < u - 1 \Leftrightarrow x_{t,u-1} = 0$.

In this case,

$\sum_{i=1}^{t-1} x'_{i,u-1} = \sum_{i=1}^{t-1} x_{i,u-1} = \sum_{i=1}^{t} x_{i,u-1} \geq \sum_{i=1}^{t+1} x_{i,u} = \sum_{i=1}^{t-1} x_{i,u} + x_{tu} + 1 + x_{t+1,u} - 1 \geq \sum_{i=1}^{t-1} x_{i,u} + x_{tu} + 1 = \sum_{i=1}^{t} x'_{i,u}$.

Case 2 : $v = u - 1 \Leftrightarrow x_{t,u-1} > 0$.

In this case, $\sum_{i=1}^{t-1} x'_{i,u-1} = \sum_{i=1}^{t-1} x_{i,u-1} + 1 \geq \sum_{i=1}^{t} x_{i,u} + 1 = \sum_{i=1}^{t} x'_{i,u}$.$\blacksquare$

We say that $\mathcal{T}^{\mathbf{Z}}$ is obtained from $\mathcal{T}$ by $\mathbf{Z}$-*deleting zeros* or, by abuse of language, by $\mathcal{Z}_1$-*deleting zeros* when $\mathbf{Z} = (\emptyset, \mathcal{Z}_1)$.

Remark 1 *In the previous lemma, let $\Lambda \neq \emptyset$ be the full list of symbols and k the right most symbol $< r$ in the r-th row. If $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ is the r-deletion path chain of $\mathcal{T}$ then $(\emptyset, \mathcal{Z}_s)$ is the maximum element of $\mathcal{C}_k$. Suppose the top cell of $\mathcal{Z}_s$ is in the $(t+1)$-th row of $\mathcal{T}$, then the r-deletion sequence is $(0, \ldots, 0, y_{rt} \neq 0, \ldots, y_{r,r-1})$. If $\mathcal{T}^{\mathcal{Z}_s}$ is obtained from $\mathcal{T}$ by $\mathcal{Z}_s$-deleting zeros, then the r-deletion path chain of $\mathcal{T}^{\mathcal{Z}_s}$ is $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_{s-1})$ and the r-deletion sequence is $(0, \ldots, 0, y_{rt} - 1, \ldots, y_{r,r-1})$.*

Example 7 *Let $\mathcal{C}_6$ of $\mathcal{T}$ (example 3) with maximum $(\emptyset, \mathcal{Z}_2)$ and $\mathcal{Z}_2 = ((k_7, j_7); (k_6, j_6); (k_5, j_5); (k_4, j_4)) = ((6,6); (5,6); (3,7); (2,7))$. The $(\emptyset, \mathcal{Z}_2)$-deletion sequence is $(0,0,1,0,0,0)$.*

$\mathcal{T} =$

									1	1	1	1	1
							1	1	2	2	2	2	
							2	3	3	3	3		
						2	4	4	4	4			
			1	2	3	3	5	5					
	1	2	3	4	5	6	6						
1	3	4	5	6	7								

$\mathcal{T}' =$

									1	1	1	1	1
							1	1	2	2	2	2	
						2	2	3	3	3	3		
						3	4	4	4	4			
			1	2	3	5	5	5					
	1	2	3	4	6	6	6						
1	3	4	5	7									

$\mathcal{T}'$ is obtained from $\mathcal{T}$ by $\mathcal{Z}_2$-deleting zeros. X' is such that $x'_{76} = x_{76} - 1, x'_{65} = x_{65} - 1$; $x'_{53} = x_{53} - 1$; $x'_{42} = x_{42} - 1$; $x'_{66} = x_{66} + 1$; $x'_{55} = x_{55} + 1$; $x'_{43} = x_{43} + 1$; $x'_{32} = x_{32} + 1$.

Theorem 3 *Let $r \geq 1$ and $\mathcal{T} = [(a_1, ..., a_r); (b_1, ..., b_r); (c_1, ..., c_r); X] \in \overline{LRD}_r$. Let $\Lambda = \{\alpha_1 \leq \ldots \leq \alpha_p\}$, $p \geq 0$, be a list of symbols of the r-th row of $\mathcal{T}$ and $\mathbf{Z} = (\mathcal{Q}_\mathbf{0}, \mathcal{Q}_1, \ldots, \mathcal{Q}_p)$ the maximum of $\mathcal{C}_\Lambda$ with $\mathbf{Z}$-deletion sequence $(q_{r1}, \ldots, q_{r,r-1})$. Let $X^{(\mathcal{Q}_p)}$ be as in lemma 1, and $X^{(\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_p)} \in \mathbb{Z}^{r,r}$ defined inductively by $X^{(\mathcal{Q}_{p-i},\ldots,\mathcal{Q}_p)} = (X^{(\mathcal{Q}_{p-i+1},\ldots,\mathcal{Q}_p)})^{(\mathcal{Q}_{p-i})}$, for $i = 1, \ldots, p$. Then, $\mathcal{T}^{\mathbf{Z}} = [(a_1 - q_{r1}, \ldots, a_{r-1} - q_{r,r-1}, a_r), (b_1, ..., b_r), (c_1 + p, c_2, \ldots, c_r), X^{(\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_p)}] \in \overline{LRD}_r$ and $a_{i+1} \leq a_i - q_{ri} \leq a_i$, for $i = 1, \ldots, r-1$.*

Proof: Observe that the maximum $(\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_p)$ of $\mathcal{C}_\Lambda$ is such that: $(\emptyset, \mathcal{Q}_p)$ is the maximum of $\mathcal{C}_{\alpha_p}$ of $\mathcal{T}$; if $\mathcal{T}^{\mathcal{Q}_p}$ is the LR rectangular tableau obtained, according to lemma 1, by $\mathcal{Q}_p$-deleting zeros on $\mathcal{T}$, and $X^{\mathcal{Q}_p}$ is the matrix defined as in lemma 1, then $(\emptyset, \mathcal{Q}_{p-1})$ is the maximum of $\mathcal{C}_{\alpha_{p-1}}$ in $\mathcal{T}^{\mathcal{Q}_p}$; again, if $\mathcal{T}^{\mathcal{Q}_{p-1},\mathcal{Q}_p}$ is the LR rectangular tableau obtained, according to lemma 1, by $\mathcal{Q}_{p-1}$-deleting zeros on $\mathcal{T}^{\mathcal{Q}_p}$, and $X^{(\mathcal{Q}_{p-1},\mathcal{Q}_p)} = (X^{\mathcal{Q}_p})^{(\mathcal{Q}_{p-1})}$, then $(\emptyset, \mathcal{Q}_{p-2})$ is the maximum of $\mathcal{C}_{\alpha_{p-2}}$ in $\mathcal{T}^{\mathcal{Q}_{p-1},\mathcal{Q}_p}$. Therefore, applying successively lemma 1 p-times, we obtain $\mathcal{T}^{\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_p}$. ∎

We say that $\mathcal{T}^{\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_p}$ is obtained from $\mathcal{T}$ by $(\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_p)$-*deleting zeros.*

Remark 2 (*a*) *In the theorem above, let $\alpha_1, \ldots, \alpha_p$ be the p right most symbols $< r$. Let $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ be the r-deletion path chain of $\mathcal{T}$ and $(y_{r1}, \ldots, y_{r,r-1})$ the r-deletion sequence. Let $\mathcal{T}^{\mathcal{Z}_0,\mathcal{Z}_{s-p+1},\ldots,\mathcal{Z}_s}$ be the rectangular LR tableau obtained from $\mathcal{T}$ by $(\emptyset, \mathcal{Z}_{s-p+1}, \ldots, \mathcal{Z}_s)$-deleting zeros. Then $(\emptyset, \mathcal{Z}_1, \ldots, \mathcal{Z}_{s-p})$ is the r-deletion path chain of $\mathcal{T}^{\mathcal{Z}_0,\mathcal{Z}_{s-p+1},\ldots,\mathcal{Z}_s}$ and the r-deletion sequence is $(0, \ldots, 0, y'_{rt}, y_{r,t+1}, \ldots, y_{r,r-1})$ where $t \in \{1, \ldots, r-1\}$ is such that $p - \sum_{j=1}^{t-1} y_{rj} = y'_{rt}$ and $\sum_{j=1}^{t-1} y_{rj} \leq p < \sum_{j=1}^{t} y_{rj}$.*

(*b*) *Let $\Lambda_j := \{\alpha_j, \ldots, \alpha_p\}$, $j = 1, \ldots, p$, $\Lambda_{p+1} := \emptyset$ and $\Lambda_0 := \Lambda_1$. The lemma and theorem above give another characterization of $(\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_p)$ the maximum of $\mathcal{C}_\Lambda$, namely: $(\emptyset, \mathcal{Q}_p)$ is the maximum of $\mathcal{C}_{\Lambda_p \setminus \Lambda_{p+1}}$ in $\mathcal{T}$; and, if $\mathcal{T}^{\mathcal{Q}_p}$ is the LR rectangular tableau obtained by $\mathcal{Q}_p$-deleting zeros, then, for $i \geq 1$, $(\emptyset, \mathcal{Q}_{p-i})$ is the maximum of $\mathcal{C}_{\Lambda_{p-i} \setminus \Lambda_{p-i+1}}$ in $\mathcal{T}^{\mathcal{Z}_{p-i+1},\ldots,\mathcal{Z}_p}$ obtained from $\mathcal{T}^{\mathcal{Z}_{p-i+2},\ldots,\mathcal{Z}_p}$ by $\mathcal{Q}_{p-i+1}$-deleting zeros.*

Example 8 *Let $\mathcal{C}_{5,6}$ of $\mathcal{T}$, defined in examples 3, 4, with maximum element $\mathbf{Z} = (\emptyset, \mathcal{Z}_\mathbf{1}, \mathcal{Z}_\mathbf{2})$, as defined in example 4, and with $\mathbf{Z}$-deleting sequence $(0, 0, 1, 1, 0, 0)$.*

Applying theorem above to $\mathcal{T}$, we obtain

$\mathcal{T}$" =

									1	1	1	1	1
							1	1	2	2	2	2	
						2	2	3	3	3	3		
					3	3	4	4	4	4			
			1	2	4	5	5	5					
	1	2	3	5	6	6	6						
1	3	4	7										

by $(\emptyset, \mathcal{Z}_1, \mathcal{Z}_2)$-deleting zeros on $\mathcal{T}$.

Theorem 4 *Let $r \geq 0$ and $\mathcal{T} = [(a_1, ..., a_{r+1}), (b_1, ..., b_{r+1}), (c_1, ..., c_{r+1}), X] \in \overline{LRD}_{r+1}$ with $(r+1)$-deleting path chain $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$. Let $a' = (a'_1, ..., a'_r) \in \mathcal{P}_r$ such that $a_{i+1} \leq a'_i \leq a_i$, for $i = 1, ..., r$. If $(a_1 - a'_1, \ldots, a_r - a'_r)$ is the $(r+1)$-deleting sequence of $\mathcal{T}$, then $\mathcal{T}^{(r)} = [a', (b_1, ..., b_r), (c_2, ..., c_{r+1}), X'] \in \overline{LRD}_r$, where $X^{(\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_s)} = X' \oplus [b_r]$. (When $r = 0$, we let $\mathcal{T}^{(0)}$ be the empty tableau.)*

Proof: Applying theorem 3, by $(r+1)$-deleting zeros on $\mathcal{T}$, we obtain the rectangular LR tableau $\mathcal{T}^{\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_s} = [a' = (a'_1, \ldots, a'_r, a_{r+1});\ (b_1, ..., b_r, b_{r+1}), (c_1 + s, c_2, ..., c_{r+1}), X^{(\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_s)}] \in \overline{LRD}_{r+1}$. After deleting the $(r+1)$-th row of $\mathcal{T}^{\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_s} \in \overline{LRD}_{r+1}$, we obtain $\mathcal{T}^{(r)} \in \overline{LRD}_r$ as desired. ■

$\mathcal{T}^{(r)}$ is obtained by $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ *deleting* zeros and the $(r+1)$-th row of $\mathcal{T}$.

If $(y_{r+1,1}, \ldots, y_{r+1,r})$ is the $(r+1)$-deletion sequence of $\mathcal{T} = [(a_1, ..., a_{r+1}), (b_1, ..., b_{r+1}), (c_1, ..., c_{r+1}), X] \in \overline{LRD}_{r+1}$, we shall call $(y_{r+1,1}, \ldots, y_{r+1,r}, a_{r+1})$ the $(r+1)$-*full deletion sequence* of $\mathcal{T}$. In particular, when $r = 0$, the 1-full deletion sequence is (a_1).

We say that $\mathcal{T}^{(r)}$ is obtained by $(r+1)$-*full deletion* .

Corollary 3 *Let $r \geq 0$. $[(a_1, \ldots, a_{r+1}), (b_1, \ldots, b_{r+1}),\ (c_1, \ldots, c_{r+1})] \in \overline{LR}_{r+1}$ only if there exists a partition $(a'_1, \ldots, a'_r) \in \mathcal{P}_r$ satisfying $a_{i+1} \leq a'_i \leq a_i$, for $i = 1, \ldots, r$, such that $[a', (b_1, \ldots, b_r), (c_2, \ldots, c_{r+1})] \in \overline{LR}_r$.*

Proof: Let $(a'_1, \ldots, a'_r) \in \mathcal{P}_r$ such that $(a_1 - a'_1, \ldots, a_r - a'_r, a_{r+1})$ is the $(r+1)$-full deletion sequence of some LR rectangular tableau of type $[(a_1, \ldots, a_{r+1}),\ (b_1, \ldots, b_{r+1}), (c_1, \ldots, c_{r+1})]$. ■

Example 9 *Applying to $\mathcal{T}$ the full deletion operation we obtain $\mathcal{T}'$.*

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 & 1 \\ \hline & & & & & 1 & 2 & 2 & \\ \hline & & 1 & 2 & 2 & 3 & & & \\ \hline 1 & 2 & 2 & 3 & 4 & & & & \\ \hline \end{array} \implies \mathcal{T}' = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 1 & 1 \\ \hline & & & & 1 & 2 & 2 & 2 & \\ \hline 1 & 2 & 2 & 2 & 3 & 3 & & & \\ \hline \end{array}$$

In particular, the rectangular LR triple $[(6, 5, 2, 0); (5, 4, 1, 0); (4, 3, 2, 0)]$ of degree 4 is transformed into the rectangular LR triple $[(5, 4, 0); (5, 4, 1); (3, 2, 0)]$ of degree 3.

A map $\mathbf{P}$ on a set A is called a projection if $\mathbf{P}^2 = \mathbf{P}$.

If $[(a_1, \ldots, a_{r-1}); (b_1, \ldots, b_{r-1}); (c_1, \ldots, c_{r-1}); X'] \in \overline{LRD}_{r-1}$, with $|a|+|b|+|c| = rm$, then $[(a_1, \ldots,\ a_{r-1}, 0);\ (b_1, \ldots, b_{r-1}, 0);\ (m,\ c_1, \ldots,\ c_{r-1}); X' \oplus [0]] \in \overline{LRD}_r$. So, we may look at $\overline{LRD}_{r-1}$ as a subset of $\overline{LRD}_r$ and the r-full deletion operation as a projection of $\overline{LRD}_r$ on $\overline{LRD}_{r-1}$, $r \geq 1$, as shown in the next theorem.

Theorem 5 *Let $r \geq 1$ and $\mathbf{P} : \overline{LRD}_r \longrightarrow \overline{LRD}_r$ defined by $\mathbf{P}([a, b, c, X]) = \mathcal{T}'$, where $\mathcal{T}' = [(a_1 - y_{r,1}, \ldots, a_{r-1} - y_{r,r-1}, 0); (b_1, \ldots, b_{r-1}, 0); (m, c_2, \ldots, c_r); X' \oplus [0]]$, is such that $rm = |a| + |b| + |c|$, $X^{(\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_s)} = X' \oplus [b_r]$, with $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$ and $(y_{r,1}, \ldots, y_{r,r-1})$ respectively, the r-deletion path chain and the r-deletion sequence of $[a, b, c, X]$. Then, $\mathbf{P}$ is a projection on $\overline{LRD}_{r-1}$.*

Proof: It is a consequence of the previous theorem. Since the r-deletion sequence of $\mathcal{T}'$ is $(\underbrace{0, \ldots, 0}_{r-1})$, $\mathbf{P}(\mathcal{T}') = \mathcal{T}'$ and $\mathbf{P}$ is a projection on $\overline{LRD}_{r-1}$. Therefore, in $\overline{LRD}_{r-1}$ we may identify $\mathcal{T}^{(r-1)}$ with $\mathbf{P}(\mathcal{T})$.■

The next remark shows that given an LR rectangular tableau $\mathcal{T}$ of type $[a, b, c]$ and order r, not all sequences of nonnegative integers $(y_{r1}, \ldots, y_{r,r-1})$ satisfying the conditions $a_i - a_{i+1} \geq y_{ri}$, for $i = 1, \ldots, r-1$, and such that $[(a_1 - y_{r1}, \ldots, a_{r-1} - y_{r,r-1}); (b_1, \ldots, b_{r-1}); (c_2, \ldots, c_r)] \in \overline{LR}_{r-1}$ are r-deletion sequences of $\mathcal{T}$. In the next section we shall see that a deletion sequence has to satisfy further conditions.

Remark 3 *(a) Let* $[(a'_1,\ldots,a'_r),\ (b_1,\ldots,b_r),\ (c_2,\ldots,c_{r+1})] \in \overline{LR}_r$ *and* $[(a_1,\ldots,a_{r+1}),$ $(b_1,\ \ldots,b_{r+1}),(c_1,\ldots,c_{r+1})] \in \overline{LR}_{r+1}$ *such that* $a_{i+1} \le a'_i \le a_i$, $i=1,\ldots,r$. *In general, it is not true that there exist* $X' \in \mathbb{Z}^{r,r}$ *and* $X \in \mathbb{Z}^{r+1,r+1}$ *such that* $\mathcal{T}' = [(a'_1,\ldots,a'_r),$ $(b_1,\ldots,b_r),\ (c_2,\ \ldots,c_{r+1}),X'] \in \overline{LRD}_r$ *may be obtained from* $\mathcal{T} = [(a_1,\ldots,a_{r+1}),\ (b_1,$ $\ldots,b_{r+1}),\ (c_1,\ldots,c_{r+1}),X] \in \overline{LRD}_{r+1}$ *by the* $(r+1)$*-deletion operation, that is,* $\mathbf{P}(\mathcal{T}) = \mathcal{T}'$. *So, if we are given* $\mathcal{T} \in \overline{LRD}_{r+1}$, *interlacing conditions are not sufficient for* $\mathcal{T}'$ *to be a projection of* $\mathcal{T}$ *by* $\mathbf{P}$. *For example, consider* $[(4,3,0);(4,1,1);(3,2,0)] \in LR_3$. *There is only one LR tableau of this type which is*

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & 2 & \ & \ \\ \hline 1 & 1 & 3 & \ & \ & \ \\ \hline \end{array}$$

Let $[(3,2);(4,1);(2,0)] \in LR_2$. *We have* $(a_1,a_2,a_3) = (4,3,0)$ *and* $(a'_1,a'_2) = (3,2)$ *such that* $a_3 \le a'_2 \le a_2 \le a'_1 \le a_1$. *On the other hand, the projection of* $\mathcal{T}$ *is*

$$\mathcal{T}' = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 \\ \hline \ & 1 & 1 & 2 & \ & \ \\ \hline \end{array}$$

which is of type $[(4,1);(4,1);(2,0)]$. *In fact* $(1,1)$ *is not the deletion sequence of* $\mathcal{T}$ *and therefore of none LR tableau of type* $[(4,3,0);(4,1,1);(3,2,0)]$.

(b) Also $[(4,2,0);(5,4,0);(6,3,0)] \in \overline{LR}_3$ *but* $[(2,2);(5,4);(3,0)] \notin \overline{LR}_2$, *with* $(a_1,a_2,$ $a_3) = (4,2,0)$ *and* $(a'_1,a'_2) = (2,2)$ *stisfying* $a'_2 \le a_2 \le a'_1 \le a_1$.

Let $r \ge 1$, $[(a_1,\ldots,a_r,a_{r+1});\ (b_1,\ldots,b_r,b_{r+1});\ (c_1,\ldots,c_r,c_{r+1})] \in \overline{LR}_{r+1}$ and $[(a'_1,$ $\ldots,a'_r);\ (b_1,\ldots,b_r);\ (c_2,\ldots,c_r,c_{r+1})] \in \overline{LR}_r$ such that $a_{i+1} \le a'_i \le a_i$, for $i=1,\ldots,r$. Given $\mathcal{T} \in \overline{LRD}_{r+1}$ of type $[(a_1,\ldots,a_r,\ a_{r+1});\ (b_1,\ldots,b_r,\ b_{r+1});\ (c_1,\ldots,c_r,c_{r+1})]$ and $\mathcal{T}' \in \overline{LRD}_r$ of type $[(a'_1,\ldots,a'_r);\ (b_1,\ \ldots,b_r);\ (c_2,\ldots,c_r,c_{r+1})]$, under what conditions do we have $\mathbf{P}(\mathcal{T}) = \mathcal{T}'$?

This question will be answered in the next section.

4 Insertion and insertion sequences of an LR rectangular tableau

The main goal of this section is to answer the question addressed at the end of the previous section. Since every elementary deletion operation on an LR rectangular tableau can be reversed downwards, we are led to insertion path chains and insertion sequences of an LR rectangular tableau. Here we introduce an insertion operation on LR rectangular tableaux preserving the LR property.

Given an LR tableau $\mathcal{T}' \in \overline{LRD}_r$ of type $[a',b,c]$, the insertion operation is defined by inserting zeros (cells numbered with zeros) on the diagram of a, and by adjoining one bottom row to a and b, and one top row to c such that the obtained tableau $\mathcal{T} \in \overline{LRD}_{r+1}$ of type $[(a_1,\ldots,a_r,\ a_{r+1});\ (b_1,\ldots,b_r,b_{r+1}),\ (c_1,\ldots,c_{r+1})]$ with $a_{i+1} \le a'_i \le a_i$, $1 \le i \le r$, satisfies $\mathbf{P}(\mathcal{T}) = \mathcal{T}'$.

Recall that if $k \in \{0,1,\ldots,r-1\}$, $\mathcal{I}_{k+1}$ is the set of interior paths which have the top cells in the first $k+1$ rows of $\mathcal{T}$, plus the empty path, and that any maximal subset of pairwise disjoint interior nonempty paths has cardinal less or equal than $\sum_{j=1}^{k} y_{rj}$, where $(y_{r1},\ldots,y_{r,r-1})$ is the deleting sequence of $\mathcal{T}$. Therefore, the number of zeros which may be inserted in each row of a will depend on the deletion sequence of $\mathcal{T}$. More precisely, the length of a maximal chain (with respect to " $\leq$ ") of interior paths with the top cells in row $k+1$ of $\mathcal{T}$ should be lesser or equal than the dimension of the poset $(\mathcal{I}_{k+1}, \leq)$.

Theorem 6 gives necessary and sufficient conditions for a sequence of non negative integers to be an insertion sequence. These conditions relate insertion sequence with the deletion sequence of the given LR tableau. Finally, since every insertion operation can be reversed upwards, every deletion sequence is an insertion sequence of the projected LR tableau and our question is answered.

4.1 The poset of the paths of length $r-k$ of an LR rectangular tableau of order r

Definition 15 *Let $r \geq 1$ and $\mathcal{T} \in \overline{LRD}_r$. For each $k \in \{0, 1, \ldots, r-1\}$, let $\mathcal{C}^{k+1}$ be the set of interior paths whose top cells are in row $k+1$ of $\mathcal{T}$, plus the empty path. That is,*

$$\mathcal{C}^1 = \mathcal{I}_1 = \{\emptyset\}, \text{ and } \mathcal{C}^{k+1} = \mathcal{I}_{k+1} \setminus \mathcal{I}_k \cup \{\emptyset\}, \text{ for } k > 0.$$

Definition 16 *Let $r \geq 1$ and $\mathcal{T} \in \overline{LRD}_r$. For each $k \in \{0, 1, \ldots, r-1\}$, let $\check{\mathcal{C}}^{k+1}$ be the set of paths of $\mathcal{T}$ with top cells in row $k+1$ whose bottom cells are numbered with r, plus the empty path, and $\overline{\mathcal{C}}^{k+1} = \mathcal{C}^{k+1} \cup \check{\mathcal{C}}^{k+1}$, the set of paths of $\mathcal{T}$ with top cells in row $k+1$, plus the empty path.*

$\mathcal{C}^{k+1} \subseteq \mathcal{I}_{k+1}$, and so $(\mathcal{C}^{k+1}, \leq)$ is a subposet of $(\mathcal{I}_{k+1}, \leq)$ with $d[\mathcal{C}^{k+1}] \leq d[\mathcal{I}_{k+1}]$. $(\check{\mathcal{C}}^{k+1}, \leq)$ is finite poset of dimension $\leq b_r$ and, therefore, $(\bar{\mathcal{C}}^{k+1}, \leq)$ is a finite poset of dimension $\leq d[\mathcal{I}_{k+1}] + b_r$.

In the following, we determine the dimensions of $(\mathcal{C}^{k+1}, \leq)$ and $(\overline{\mathcal{C}}^{k+1}, \leq)$ and relate them with the dimension of $(\mathcal{I}_{k+1}, \leq)$. Since we want to determine the left most longest chains of those posets in order to define insertion paths chains and insertion sequences of an LR rectangular tableau, we also consider the posets$(\mathcal{C}^{k+1}, \preceq)$ and $(\bar{\mathcal{C}}^{k+1}, \preceq)$.

Lemma 3 *Let $r \geq 1$ and $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$ with r-deletion path chain $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$. Let $k \in \{0, 1, \ldots, r-1\}$ and $m = d[\mathcal{I}_{k+1}]$. Let $(\mathcal{C}^{k+1}, \preceq)$ and $(\bar{\mathcal{C}}^{k+1}, \preceq)$. Then*

1. *$\mathcal{C}^{k+1} \neq \{\emptyset\}$ iff $\min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}]\} > 0$.*
2. *$\check{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $\min\{a_k - a_{k+1}, a_k - a_r - d[\mathcal{I}_r], b_r\} > 0$. In particular, when $\mathcal{C}^{k+1} = \{\emptyset\}$, it holds $\check{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $\min\{a_k - a_{k+1}, b_r\} > 0$.*
3. *$\overline{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $\min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}] + b_r\} > 0$.*
4. *If $\overline{\mathcal{C}}^{k+1} \neq \{\emptyset\}$, the numbering sequence $(v_r > \ldots > v_{k+1})$ of $\min(\bar{\mathcal{C}}^{k+1} \setminus \{\emptyset\})$ is defined as follows: v_{k+1} is the left most nonzero symbol in the $(k+1)$-th row of $\mathcal{T}$, and, for $i \geq k+2$, v_i is the left most nonzero symbol in the i-th row of $\mathcal{T}$ such that $v_i > v_{i-1}$. We have $v_r < r$ if $\mathcal{C}^{k+1} \neq \{\emptyset\}$ and $v_r = r$ if $\mathcal{C}^{k+1} = \{\emptyset\}$. Moreover, if $\mathcal{C}^{k+1} \neq \{\emptyset\}$, $\min(\bar{\mathcal{C}}^{k+1} \setminus \{\emptyset\}) = \min(\mathcal{C}^{k+1} \setminus \{\emptyset\}) = \min\{\mathcal{Z} \in \mathcal{C}^{k+1} : \emptyset < \mathcal{Z} \preceq \mathcal{Z}_{s-m+1}\}$; and if $\mathcal{C}^{k+1} = \{\emptyset\}$, $\min(\check{\mathcal{C}}^{k+1} \setminus \{\emptyset\}) = \min(\bar{\mathcal{C}}^{k+1} \setminus \{\emptyset\})$.*

Proof: (1) By definition $\mathcal{C}^{k+1} = \mathcal{I}_{k+1} \setminus \mathcal{I}_k \cup \{\emptyset\}$. We claim that $\mathcal{I}_{k+1} \setminus \mathcal{I}_k \neq \emptyset$ iff $a_k - a_{k+1} > 0$ and $\mathcal{I}_{k+1} \neq \{\emptyset\}$. The *"only if"* part is obvious. If $a_k - a_{k+1} > 0$ and $\mathcal{I}_{k+1} \neq \{\emptyset\}$, consider the $(r-k)$-path $\mathcal{B}$ with numbering sequence $(v_r > \ldots > v_{k+1})$ such that v_{k+1} is the left most nonzero symbol in the $(k+1)$-th row of $\mathcal{T}$, and, for $i \geq k+2$, v_i is the left most nonzero symbol in the i-th row of $\mathcal{T}$ with $v_i > v_{i-1}$. By construction $\mathcal{B} \preceq \mathcal{W}$, for all path $\mathcal{W}$ of length at least $(r-k)$ in $\mathcal{T}$. Since $\mathcal{Z}_0 < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ is the right most longest chain of $\mathcal{I}_{k+1}$, then, in particular, $\mathcal{B} \preceq \mathcal{Z}_{s-m+1}$. Therefore, $v_r < r$, and $\mathcal{Z} \in \mathcal{I}_{k+1} \setminus \mathcal{I}_k$. Hence, $\mathcal{C}^{k+1} \neq \{\emptyset\}$ iff $a_k - a_{k+1} > 0$ and $\mathcal{I}_{k+1} \neq \{\emptyset\}$.

(2) The *"if"* part. If $b_r > 0$, $a_k - a_{k+1} > 0$ and $a_r + \sum_{j=1}^{r-1} x_{rj} < a_k$, we may define an $(r-k)$ -path with column indexing $(j_r, \ldots, j_{k+1})$ where $j_i = \min\{a_i + \sum_{j=1}^{i} x_{ij}, a_k\}$, for $i = k+1, , \ldots, r$. Since $b_r > 0$, then $x_{ii} > 0$, for all i, and it follows that $j_i \in \{a_i + 1, \ldots, a_k\}$, for $i = k+1, , \ldots, r$. The bottom cell of this path is numbered with r because $a_r + \sum_{j=1}^{r-1} x_{rj} < j_r \leq a_k$. So, $\breve{\mathcal{C}}^{k+1} \neq \{\emptyset\}$.

The *"only if"* part. If $\breve{\mathcal{C}}^{k+1} \neq \emptyset$, by definition there exists an $(r-k)$-path with bottom cell numbered with r. So, $b_r > 0$, and considering the definition of path (definition 5), $a_k - a_{k+1} > 0$, $a_r + \sum_{j=1}^{r-1} x_{rj} < a_k$.

Notice that theorem 2, (3) implies $a_r + \sum_{j=1}^{r-1} x_{rj} \leq a_{k+1}$, when $\mathcal{I}_{k+1} = \{\emptyset\}$. Therefore, when $\mathcal{C}^{k+1} = \{\emptyset\}$, $\min\{a_k - a_{k+1}, a_k - a_r - \sum_{j=1}^{r-1} x_{rj}, b_r\} = \min\{a_k - a_{k+1}, b_r\}$.

(3) By definition $\bar{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $\mathcal{C}^{k+1} \neq \{\emptyset\}$ or $\breve{\mathcal{C}}^{k+1} \neq \{\emptyset\}$. We claim that $\mathcal{C}^{k+1} \neq \{\emptyset\} \vee \breve{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $a_k - a_{k+1} > 0 \wedge (\mathcal{I}_{k+1} \neq \{\emptyset\} \vee b_r > 0)$.

From (1), $\mathcal{C}^{k+1} \neq \{\emptyset\}$ iff $a_k - a_{k+1} > 0$ and $\mathcal{I}_{k+1} \neq \{\emptyset\}$. From (2), $\mathcal{C}^{k+1} = \{\emptyset\}$ and $\breve{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $a_k - a_{k+1} > 0$, $\mathcal{I}_{k+1} = \{\emptyset\}$ and $b_r > 0$.

(4) Follows from (1) and (3). ■

Lemma 4 *(Elementary insertion) Let $r \geq 1$, $\mathcal{T} = [(a_1, \ldots, a_r), (b_1, ..., b_r), (c_2, ..., c_{r+1}), X] \in \overline{LRD}_r$, with $|a| + |b| + |c| = rm$, and $k \in \{1, \ldots, r\}$. Suppose $\bar{\mathcal{C}}^k \neq \{\emptyset\}$ with left most non empty path $\mathcal{B} = ((v_r, j_r), \ldots, (v_k, j_k))$. Let ${}^{\mathcal{B}}X = [x'_{ij}] \in \mathbb{Z}^{r+1,r+1}$ be such that $x'_{i,v_i} = x_{i,v_i} - 1$, $x'_{i+1,v_i} = x_{i+1,v_i} + 1$, $k \leq i \leq r-1$, $x'_{r,v_r} = x_{r,v_r} - 1$, $x'_{r+1,v_r} = 1$, $x_{r+1,j} = 0$, for $j \neq v_r$ and $x'_{ij} = x_{ij}$, otherwise. Then ${}^{\mathcal{B}}\mathcal{T} = [(a_1, \ldots, a_k + 1, \ldots, a_r, 0), (b_1, ..., b_r, 0), (m-1, c_2, ..., c_{r+1}), {}^{\mathcal{B}}X] \in \overline{LRD}_{r+1}$. Moreover, the $(r+1)$-deletion sequence of ${}^{\mathcal{B}}\mathcal{T}$ is $(0, \ldots, y_{r+1,k} = 1, \ldots, 0)$ and the $(r+1)$-deletion path chain is $((v_r, 1), (v_{r-1}, j_r), \ldots, (v_k, j_{k+1}))$.*

Proof: Let $v_r > v_{r-1} > \ldots > v_k$ be the numbering sequence of $\mathcal{B}$. Recall that $v_r > v_{r-1} > \ldots > v_k$ and that v_k is the left most nonzero symbol of the k-th row and, for $i \geq k+1$, v_i is the left most symbol in the i-th row such that $v_i > v_{i-1}$, for $i \in \{k+1, \ldots, r\}$. Insert one symbol zero in the k-th row by shoving the symbol v_k. Then, insert v_k in the $(k+1)$-th row by shoving the symbol v_{k+1} which, on its turn, is inserted in the $(k+2)$-th row by shoving v_{k+3}. We shall get $x'_{k,v_k} = x_{k,v_k} - 1$, $x'_{k+1,v_k} = x_{k+1,v_{k+1}} + 1$ and $x'_{k+1,v_{k+1}} = x_{k+1,v_k} - 1$, $x'_{k+2,v_{k+1}} = x_{k+2,v_{k+1}} + 1$. Proceed down, in this way, until reaching the r-th row, where the symbol v_{r-1} is inserted by shoving v_r, which, on its turn, is inserted in the row $r+1$. We shall get $x'_{r+1,v_r} = 1$. It is clear, that the output is a rectangular tableau ${}^{\mathcal{B}}\mathcal{T} = [(a_1, \ldots, a_k + 1, \ldots, a_r, 0), (b_1, ..., b_r, 0), (c_1, c_2, ..., c_{r+1}), {}^{\mathcal{B}}X]$, that is the nonzero symbols strictly increase along columns, from up to down, and do not decrease along rows, from left to right.

Let us prove that ${}^{\mathcal{B}}\mathcal{T}$ is in $\overline{LRD}_{r+1}$. We have only to check that ${}^{\mathcal{B}}X$ satisfy inequality (4), in the definition of rectangular LR tableau give in section 2, for $v_k \leq j \leq v_r$ and $k \leq i \leq r$.

Let $t \in \{k, \ldots, r-1\}$ and $u = v_{t+1} > w = v_t$. The basic operation in the previous process is as follows: the nonzero symbol w of the t-th row is inserted in the $t+1$-th row by shoving the symbol u, which on its turn, is inserted in the $(t+2)$-th row by shoving a symbol $> u$. Therefore, $x'_{tw} = x_{tw} - 1 \geq 0$, $x'_{t+1,w} = x_{t+1,w} + 1$, $x'_{t+1,u} = x_{t+1,u} - 1 \geq 0$ and $x'_{iw} = x_{iw}$, for $i \neq t, t+1$.

If $u > w+1$, $x'_{i,w+1} = x_{i,w+1}$, for all i, and, in particular, $x'_{t+1,w+1} = x_{t+1,w+1} = 0$.

If $u = w+1$, $x'_{t+1,w+1} = x_{t+1,w+1} - 1$ and $x'_{i,w+1} = x_{i,w+1}$, $1 \leq i \leq t$.

It is sufficient to prove that

$\sum_{i=1}^{t} x'_{i,w} \geq \sum_{i=1}^{t+1} x'_{i,w+1}$.

Case 1: $u > w+1 \Leftrightarrow x_{t+1,w+1} = 0$.

In this case,

$\sum_{i=1}^{t} x'_{i,w} = \sum_{i=1}^{t-1} x_{i,w} + x'_{tw} = \sum_{i=1}^{t-1} x_{i,w} + x_{tw} - 1 \geq \sum_{i=1}^{t} x_{i,w+1} + x_{tw} - 1 = \sum_{i=1}^{t} x'_{i,w+1} + x'_{t+1,w+1} = \sum_{i=1}^{t+1} x'_{i,w+1}$.

Case 2 : $u = w+1 \Leftrightarrow x_{t+1,w+1} > 0$.

In this case,

$\sum_{i=1}^{t} x'_{i,w} = \sum_{i=1}^{t-1} x_{i,w} + x_{tw} - 1 \geq \sum_{i=1}^{t+1} x_{i,w+1} - 1 = \sum_{i=1}^{t} x_{i,w+1} + x_{t+1,w+1} - 1 = \sum_{i=1}^{t} x'_{i,w+1} + x_{t+1,w+1} - 1 = \sum_{i=1}^{t+1} x'_{i,w+1}$.

The remain claim it is obvious. ■

We denote by ${}^{\mathcal{B}}\mathcal{T}$ the rectangular tableau in $\overline{LRD}_{r+1}$ obtained from $\mathcal{T}$ in $\overline{LRD}_r$ by *$\mathcal{B}$-inserting zeros.* We call this operation an *elementary insertion operation.*

The insertion of one symbol 0 in the row k of the diagram of a yields to a sliding move in X described as follows: let v_k be the index of the column of the first nonzero entry in the k-th row of X, and, for $s = k+1, \ldots, r$, let v_s be the index of the column of the first nonzero entry of the s-th row of X such that $v_{s-1} < v_s$. Then, ${}^{\mathcal{B}}X$ is

$$\begin{bmatrix}
x_{1,1} & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0\ldots & 0 & x_{k,v_k}-1 & \star & \star & \star & \star & x_{k,k} & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\
\star & \star & x_{k+1,v_k}+1 & 0 & . & 0 & x_{k+1,v_{k+1}}-1 & . & x_{k+1,k+1} & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & x_{r,v_{r-1}}+1 & 0. & 0 & x_{r,v_r}-1 & \star & x_{rr} & 0 \\
0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

Note that, if $\bar{\mathcal{C}}^k \neq \{\emptyset\}$ and $\mathcal{C}^k = \{\emptyset\}$, $v_r = r$ and $x'_{rr} = x_{rr} - 1$. We say that ${}^{\mathcal{B}}X$ is obtained from X by inserting $(0^{k-1}, 1, 0^{r-k}, 0)$. By reversing the operations we say that X is obtained from ${}^{\mathcal{B}}X$ by deleting $(0^{k-1}, 1, 0^{r-k}, 0)$. When $\bar{\mathcal{C}}^k = \{\emptyset\}$, we let ${}^{\emptyset}X := X \oplus [0] \in \mathbb{Z}^{r+1,r+1}$, and ${}^{\emptyset}\mathcal{T} := [(a_1, \ldots, a_k, \ldots, a_r, 0), (b_1, ..., b_r, 0), (m, c_2, ..., c_{r+1}), {}^{\emptyset}X] \in \overline{LRD}_{r+1}$. We say that ${}^{\emptyset}X$ was obtained from X by inserting $(0^r, 0)$.

Notation Let $r \geq 1$. Given $\mathcal{T} \in \overline{LRD}_r$, we denote by $\widehat{\mathcal{T}}$ the rectangular tableau in $\overline{LRD}_{r-1}$ obtained by suppressing the last row of $\mathcal{T}$. Note that $\mathcal{T} = {}^{\emptyset}\widehat{\mathcal{T}}$.

Lemma 5 *Let $r \geq 1$ and $\mathcal{T} \in \overline{LRD}_r$ with r-deletion path chain $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_s)$. Let $k \in \{1, \ldots, r\}$ and $\bar{\mathcal{C}}^k \neq \{\emptyset\}$ with left most non empty path $\mathcal{B}_1$. Let $m = d[\mathcal{I}_k]$. Then:*

(a) *If* $\bar{\mathcal{C}}^k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}}) \neq \{\emptyset\}$ *and* $\mathcal{B}_2$ *is its left most non empty path, then* $\mathcal{B}_2 \in \bar{\mathcal{C}}^k$ *and* $\mathcal{B}_1 < \mathcal{B}_2$.

(b) *If* $m > 0$,

(i) $(\mathcal{Z}_0, \mathcal{Z}_{s-m+2}, \ldots, \mathcal{Z}_s)$ *is the right most longest chain of* $\mathcal{I}_k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}})$.

(ii) $d[\mathcal{I}_k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}})] = d[\mathcal{I}_k] - 1$, *and* $d[\mathcal{I}_u(\widehat{{}^{\mathcal{B}}_1\mathcal{T}})] = \min\{d[\mathcal{I}_u], d[\mathcal{I}_k] - 1\}$, *for* $u < k$.

Proof: (a) Let $\mathcal{B}_1 = ((v_r, j_r), (v_{r-1}, j_{r-1}), \ldots, (v_k, j_k))$ and take into account the construction of ${}^{\mathcal{B}_1}\mathcal{T}$ as shown in the previous lemma. Then, by lemma 3, $\mathcal{B}_2$ is defined as follows. Let w_k be the symbol immediately to the right of v_k in the k-th row of $\mathcal{T}$. We have $v_k \leq w_k$. Since the left most symbol in the $(k+1)$-th row of $\widehat{{}^{\mathcal{B}}\mathcal{T}}$ is v_k and $\widehat{{}^{\mathcal{B}}\mathcal{T}}$ has at least one $(r-k+1)$-path, there is a symbol w_{k+1} in the $(k+1)$-th row of $\widehat{{}^{\mathcal{B}}\mathcal{T}}$ such that $w_{k+1} > w_k$. So, $w_{k+1} > v_k$, and the cell labelled with w_{k+1} is to the right of the cell (j_{k+1}, v_k). On the other hand, the left most symbol in the $(k+1)$-th row of $\mathcal{T}$ is v_{k+1}. So, $w_{k+1} \geq v_{k+1}$. Again, since the left most symbol in the $(k+2)$-th row of ${}^{\mathcal{B}}\mathcal{T}$ is v_{k+1} and $\widehat{{}^{\mathcal{B}}\mathcal{T}}$ has at least one $(r-k+1)$-path, there is a symbol w_{k+2} in the $(k+2)$-th row of $\widehat{{}^{\mathcal{B}}\mathcal{T}}$ such that $w_{k+2} > w_{k+1} \geq v_{k+1}$. So, $w_{k+2} > v_{k+1}$, and so on. So, $\mathcal{B}_1 < \mathcal{B}_2$ and therefore $\mathcal{B}_2 \in \bar{\mathcal{C}}^k$.

(b) If $m > 0$, we have $\mathcal{C}^k \neq \{\emptyset\}$. (i) We claim that $(\mathcal{Z}_0, \mathcal{Z}_{s-m+2}, \ldots, \mathcal{Z}_s)$ is the right most m-chain of $\mathcal{I}_k$ and $\mathcal{I}_k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}})$. Let $\mathcal{X} \in \mathcal{I}_k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}})$ such that $\emptyset < \mathcal{X} < \mathcal{Z}_{s-m+2}$. Since $\mathcal{B}_2$ is its left most non empty path of $\bar{\mathcal{C}}^k(\widehat{{}^{\mathcal{B}_1}\mathcal{T}})$, it follows $\mathcal{B}_2 \preceq \mathcal{X}$. On the other hand, $\mathcal{B}_2 \in \mathcal{C}^k$ and therefore $\mathcal{X} \in \mathcal{I}_k$. But then $\emptyset < \mathcal{B}_1 < \mathcal{B}_2 \preceq \mathcal{X} < \mathcal{Z}_{s-m+2}$ in $\mathcal{I}_k$. This contradicts $m = d[\mathcal{I}_k]$.

(ii) It is a consequence of (a).■

The next propositions characterize the left most maximal chains with maximum length of $(\mathcal{C}^{k+1}, \leq)$ and $(\bar{\mathcal{C}}^{k+1}, \leq)$, respectively, and, therefore, the dimensions of $(\mathcal{C}^{k+1}, \leq)$ and $(\bar{\mathcal{C}}^{k+1}, \leq)$.

Proposition 3 *Let* $r \geq 1$ *and* $\mathcal{T} \in \overline{LRD}_r$ *with* r*-deletion path chain* $\mathcal{Z}_0 < \mathcal{Z}_1 < \ldots < \mathcal{Z}_s$. *Let* $k \in \{0, 1, \ldots, r-1\}$. *Then,*

1. $d[\mathcal{C}^{k+1}] = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}]\}$.

2. (a) *The left most longest chain* $\mathcal{B}_0 < \mathcal{B}_1 < \ldots < \mathcal{B}_l$ *of* $\mathcal{C}^{k+1}$, *where* $l = d[\mathcal{C}^{k+1}]$, *is defined inductively by setting, for* $1 \leq t \leq l$,

$$\mathcal{B}_t = \min\{\mathcal{B} \in \mathcal{C}^{k+1}(\widehat{{}^{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_{t-1}}\mathcal{T}) : \mathcal{B} > \emptyset\},$$

with $\widehat{{}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_t}\mathcal{T}} \in \overline{LRD}_r$ *obtained from* ${}^{\mathcal{B}_t}(\widehat{{}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{t-1}}\mathcal{T}}) \in \overline{LRD}_{r+1}$ *by suppressing the last row. Moreover,* $\mathcal{B}_0 < \mathcal{B}_1 < \ldots < \mathcal{B}_l$ *is such that*

$$\mathcal{B}_t = \min\{\mathcal{B} \in \mathcal{C}^{k+1} : \mathcal{B}_{t-1} < \mathcal{B} \preceq \mathcal{Z}_{s-m+t}\}, \text{ for } t = 1, \ldots, l,$$

where $m = d[\mathcal{I}_{k+1}]$.

(b) *For* $t \in \{0, 1, \ldots, l\}$ *and* $u < k+1$,

i. $d[\mathcal{I}_{k+1}] = t + d[\mathcal{I}_{k+1}(\widehat{{}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_t}\mathcal{T}})]$.

ii. $d[\mathcal{I}_u(^{\widehat{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_t}}\mathcal{T})] = \min\{d[\mathcal{I}_u], d[\mathcal{I}_{k+1}] - t\}$.

iii. $d[\mathcal{C}^{k+1}(^{\widehat{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_t}}\mathcal{T})] = d[\mathcal{C}^{k+1}] - t$.

iv. $d[\mathcal{C}^u(^{\widehat{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_t}}\mathcal{T})] = \min\{a_{u-1} - a_u,\ d[\mathcal{I}_u],\ d[\mathcal{I}_{k+1}] - t\}$.

Proof: Let $l := \min\{a_k - a_{k+1}, m\}$. The proof will proceed by induction on l.

$1, 2, (a)$. If $l = 0$, by lemma 3, $\mathcal{C}^{k+1} = \{\emptyset\}$. In this case, $d[\mathcal{C}^{k+1}] = 0$ and the longest chain is $(\mathcal{B}_0 = \emptyset)$.

Let $l > 0$ and suppose the veracity of the statement for $l-1$. Since $l > 0$, $\mathcal{C}^{k+1} \neq \{\emptyset\}$ and we may consider $\mathcal{B}_1 = \min(\mathcal{C}^{k+1} \setminus \{\emptyset\})$. Let $\widehat{^{\mathcal{B}_1}\mathcal{T}} \in \overline{LRD}_r$. By lemma 5, $\mathcal{Z}_0 < \mathcal{Z}_{s-m+1} < \ldots < \mathcal{Z}_s$ is the right most longest chain of $\mathcal{I}_{k+1}(\widehat{^{\mathcal{B}_1}\mathcal{T}})$, $d[\mathcal{I}_{k+1}(\widehat{^{\mathcal{B}_1}\mathcal{T}})] = d[\mathcal{I}_{k+1}] - 1$ and $d[\mathcal{I}_u(\widehat{^{\mathcal{B}_1}\mathcal{T}})] = \min\{d[\mathcal{I}_u], d[\mathcal{I}_{k+1}] - 1\}$, $u < k+1$. So, $\widehat{^{\mathcal{B}_1}\mathcal{T}}$ is in the conditions of induction hypothesis. By induction hypothesis we have: $d[\mathcal{C}^{k+1}(\widehat{^{\mathcal{B}_1}\mathcal{T}})] = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}]\} - 1 = l - 1$, and $d[\mathcal{C}^u(\widehat{^{\mathcal{B}_1}\mathcal{T}})] = \min\{a_{u-1} - a_u, d[\mathcal{I}_u(\widehat{^{\mathcal{B}_1}\mathcal{T}})]\} = \min\{a_{u-1} - a_u, d[\mathcal{I}_u], d[\mathcal{I}_{k+1}] - 1\}$, $u < k+1$; the left most longest chain $\mathcal{B}_0 < \mathcal{B}_2 < \ldots < \mathcal{B}_l$ of $\mathcal{C}^{k+1}(\widehat{^{\mathcal{B}_1}\mathcal{T}})$ is such that, for $2 \le t \le l$, $\mathcal{B}_t = \min\{\mathcal{B} \in \mathcal{C}^{k+1}(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_{t-1}}}\mathcal{T}) : \mathcal{B} > \emptyset\} = \min\{\mathcal{B} \in \mathcal{C}^{k+1} : \mathcal{B}_{t-1} < \mathcal{B} \preceq \mathcal{Z}_{s-m+t}\}$.

On the other hand, since $\emptyset < \mathcal{B}_2 < \ldots < \mathcal{B}_l$ is a chain in $\mathcal{C}^{k+1}$ and $\emptyset < \mathcal{B}_1 < \mathcal{B}_2 < \ldots < \mathcal{B}_l$ is the left most l-chain of $\mathcal{C}^{k+1}$, then $\emptyset < \mathcal{B}_1 < \ldots < \mathcal{B}_l$ is the left most longest chain of $\mathcal{C}^{k+1}$. Otherwise there would be a chain $\emptyset < \mathcal{Q}_1 < \ldots < \mathcal{Q}_{l+d}$, with $d > 0$, in $\mathcal{C}^{k+1}$ such that $\emptyset < \mathcal{B}_2 \preceq \mathcal{Q}_2 < \ldots < \mathcal{Q}_{l+1}$. This implies $\emptyset < \mathcal{Q}_2 < \ldots < \mathcal{Q}_{l+1}$ is a l-chain in $\mathcal{C}^{k+1}(\widehat{^{\mathcal{B}_1}\mathcal{T}})$ which is absurd. Therefore, $d[\mathcal{C}^{k+1}] = l = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}]\}$ and $\emptyset < \mathcal{B}_1 < \ldots < \mathcal{B}_l$ satisfy the required conditions.

Concerning $2, (b)$, if $l = 0$, recall that $\mathcal{I}_u \subseteq \mathcal{I}_{k+1}$ for $u < k+1$, and $\widehat{^{\mathcal{B}_0}\mathcal{T}} = \mathcal{T}$. If $l \ge 1$, the equalities $(i) - (iv)$ follow easily by induction on l. ■

Remark 4 *Let $d[\mathcal{C}^{k+1}] = l$ and $\mathcal{B}_0 < \mathcal{B}_1 < \ldots < \mathcal{B}_l$ the left most longest chain of $\mathcal{C}^{k+1}$.*

(a) $a_k - a_{k+1} > d[\mathcal{C}^{k+1}]$ and $b_r > 0$ iff $\breve{\mathcal{C}}^{k+1}(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_l}}\mathcal{T}) \neq \{\emptyset\}$.

(b) If $a_k - a_{k+1} > d[\mathcal{C}^{k+1}]$, then $d[\mathcal{C}^{k+1}] = d[\mathcal{I}_{k+1}]$ and $\mathcal{I}_u(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_l}}\mathcal{T}) = \{\emptyset\}$, $u \le k+1$.

(c) $d[\bar{\mathcal{C}}^{k+1}] > d[\mathcal{C}^{k+1}]$ iff $d[\mathcal{C}^{k+1}] = d[\mathcal{I}_{k+1}] < a_k - a_{k+1}$ and $b_r > 0$.

The following lemma relates the dimensions of $\mathcal{C}^{k+1}$ and $\bar{\mathcal{C}}^{k+1}$.

Lemma 6 *Let $r \ge 1$ and $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$. Let $k \in \{0, 1, \ldots, r-1\}$ and $d[\mathcal{I}_{k+1}] = 0$. Then*

(a) $d[\breve{\mathcal{C}}^{k+1}] = \min\{a_k - a_{k+1}, b_r\}$.

(b) The left most longest chain $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{\breve{l}})$ of $\breve{\mathcal{C}}^{k+1}$, with $\breve{l} = d[\breve{\mathcal{C}}^{k+1}]$, is such that, for $1 \le t \le \breve{l}$,

$$\mathcal{B}_t = \min\{\mathcal{B} \in \breve{\mathcal{C}}^{k+1}(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_{t-1}}}\mathcal{T}) : \emptyset < \mathcal{B}\}.$$

Proof: In this case, by lemma 2, $\breve{\mathcal{C}}^{k+1} \neq \{\emptyset\}$ iff $\min\{a_k - a_{k+1}, b_r\} \neq 0$. Since $\breve{\mathcal{C}}^{k+1} = \bar{\mathcal{C}}^{k+1}$, the claim follows from successive applications of lemma 5. ■

We are now in condition to calculate left most maximal chain of maximum length and the dimension of $(\bar{\mathcal{C}}^{k+1}, \le)$.

Proposition 4 *Let $r \ge 1$ and $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$. Let $k \in \{0, 1, \ldots, r-1\}$. Then,*

1. $d[\bar{\mathcal{C}}^{k+1}] = d[\mathcal{C}^{k+1}] + \min\{max\{0, a_k - a_{k+1} - d[\mathcal{I}_{k+1}]\}, b_r\}$.

2. $d[\overline{\mathcal{C}}^{k+1}] = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}] + b_r\}$.

3. *If* $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_l)$ *is the left most longest chain of* $\mathcal{C}^{k+1}$, *and* $(\mathcal{B}_0, \mathcal{B}_{l+1}, \dots, \mathcal{B}_{l+\breve{l}})$ *is the left most longest chain of* $\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_l}}\mathcal{T})$, *it holds*

   (a) $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_l, \mathcal{B}_{l+1}, \dots, \mathcal{B}_{l+\breve{l}})$ *is the left most longest chain of* $\bar{\mathcal{C}}^{k+1}$.

   (b) *For* $t \in \{0, 1, \dots, l + \breve{l}\}$,

      i. $d[\mathcal{I}_u({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_t}}\mathcal{T})] = \min\{d[\mathcal{I}_u], \max\{0, d[\mathcal{I}_{k+1}] - t\}\}$, $u \leq k+1$.

      ii. $d[\bar{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_t}}\mathcal{T})] = d[\bar{\mathcal{C}}^{k+1}] - t$ *and* $d[\bar{\mathcal{C}}^{u}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_t}}\mathcal{T})] = \min\{a_{u-1} - a_u, b_r + d[\mathcal{I}_u], b_r + d[\mathcal{I}_{k+1}] - t\}$, *for* $u < k+1$.

Proof: We consider two cases:

1st Case: $\mathcal{C}^{k+1} = \{\emptyset\}$.

In this case, $d[\mathcal{C}^{k+1}] = 0$ and $\bar{\mathcal{C}}^{k+1} = \breve{\mathcal{C}}^{k+1}$. Therefore, $\mathcal{I}_{k+1} = \{\emptyset\}$ or $a_k - a_{k+1} = 0$.

If $a_k - a_{k+1} = 0$, $\bar{\mathcal{C}}^{k+1} = \breve{\mathcal{C}}^{k+1} = \{\emptyset\}$ and $d[\mathcal{C}^{k+1}] = d[\bar{\mathcal{C}}^{k+1}] = d[\breve{\mathcal{C}}^{k+1}] = 0$.

If $\mathcal{I}_{k+1} = \{\emptyset\}$ it holds $d[\mathcal{I}_{k+1}] = 0$, and, by previous lemma, $d[\bar{\mathcal{C}}^{k+1}] = d[\breve{\mathcal{C}}^{k+1}] = \min\{a_k - a_{k+1}, b_r\} = \min\{max\{0, a_k - a_{k+1}\}, b_r\}$.

If $\breve{l} = d[\breve{\mathcal{C}}^{k+1}]$, then $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{\breve{l}})$ is the leftmost longest chain of $\bar{\mathcal{C}}^{k+1}$.

2nd Case: $\mathcal{C}^{k+1} \neq \{\emptyset\}$.

In this case, $l := d[\mathcal{C}^{k+1}] > 0$. Let $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_l)$ be the leftmost longest chain of $\mathcal{C}^{k+1}$.

We claim that $d[\bar{\mathcal{C}}^{k+1}] = d[\mathcal{C}^{k+1}] + d[\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_l}}\mathcal{T})]$.

For, let ${}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_l}}\mathcal{T} = [\hat{a}, \hat{b}, \hat{c}, \hat{X}]$ where $\hat{a}_{k+1} = a_{k+1} + l$, $\hat{a}_i = a_i$, for $i \neq k+1$, and $\hat{b}_r = b_r$. By proposition 3, $l = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}]\}$. So, $a_k - \hat{a}_{k+1} > 0$ iff $a_k - a_{k+1} > l$, and $a_k = \hat{a}_{k+1}$ iff $d[\mathcal{I}_{k+1}] \geq a_k - a_{k+1}$.

Since $\mathcal{C}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T}) = \{\emptyset\}$ it follows that $\bar{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T}) = \breve{\mathcal{C}}^{k+1}({}^{\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T})$.

By previous lemma, $d[\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T})] = \min\{a_k - \hat{a}_{k+1}, b_r\}$. Let $\breve{l} := d[\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T})]$. Notice that, $a_k - \hat{a}_{k+1} = max\{0, a_k - a_{k+1} - d[\mathcal{I}_{k+1}]\}$. So, $\breve{l} = \min\{max\{0, a_k - a_{k+1} - d[\mathcal{I}_{k+1}]\}, b_r\}$.

1st Subcase: *If* $\breve{l} = 0$, then $\bar{\mathcal{C}}^{k+1} = \mathcal{C}^{k+1}$ and, either $b_r = 0$ or $d[\mathcal{I}_{k+1}] \geq a_k - a_{k+1}$. In both cases, $d[\bar{\mathcal{C}}^{k+1}] = d[\mathcal{C}^{k+1}]$ and (1), (2) hold. The leftmost longest chain of $\bar{\mathcal{C}}^{k+1}$ is leftmost longest chain of $\mathcal{C}^{k+1}$.

2nd Subcase: *If* $\breve{l} > 0$, let $\mathcal{B}_0 < \mathcal{B}_{l+1} < \dots < \mathcal{B}_{l+\breve{l}}$ be the leftmost longest chain of $\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T})$. Then, $\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_t}}\mathcal{T}) \neq \{\emptyset\}$, for $l+1 \leq t < l+\breve{l}$, and $\bar{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\dots,\mathcal{B}_{l+\breve{l}}}}\mathcal{T}) = \{\emptyset\}$. Therefore, the leftmost longest chain of $\bar{\mathcal{C}}^{k+1}$ is $\mathcal{B}_0 < \mathcal{B}_1 < \dots < \mathcal{B}_l < \mathcal{B}_{l+1} < \dots < \mathcal{B}_{l+\breve{l}}$ and $d[\bar{\mathcal{C}}^{k+1}] = l + \breve{l}$. So, (1) holds.

On the other hand, if $\breve{l} > 0$, $\breve{\mathcal{C}}^{k+1}({}^{\mathcal{B}_0,\widehat{\mathcal{B}_1 \dots \mathcal{B}_l}}\mathcal{T}) \neq \{\emptyset\}$. By remark 4, it follows that $a_k - a_{k+1} > d[\mathcal{C}^{k+1}] = d[\mathcal{I}_{k+1}]$. So, $l + \breve{l} = d[\mathcal{I}_{k+1}] + \min\{a_k - a_{k+1} - d[\mathcal{I}_{k+1}], b_r\} = \min\{a_k - a_{k+1}, d[\mathcal{I}_{k+1}] + b_r\}$. We have proven $1, 2, 3(a)$.

$3, (b), (i)$ It follows from proposition 3 and remark 4.

(ii) For $t < l$, it follows from 2 and (i). For $t \geq l$, we have $l = d[\mathcal{I}_{k+1}]$ and $\check{\mathcal{C}}^u(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_t}}\mathcal{T}) = \check{\mathcal{C}}^u(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_t}}\mathcal{T})$. Now the result follows from lemma 6 taking into account that $^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\ldots,\mathcal{B}_l}}\mathcal{T} = [\hat{a}, \hat{b}, \hat{c}, \hat{X}]$ is such that $\hat{b}_r = b_r + d[\mathcal{I}_{k+1}] - t < b_r + d[\mathcal{I}_u]$. ■

As a consequence of this proposition, we obtain the following corollary, which will be useful in the next subsection.

Corollary 4 *Let $r \geq u > v > w \geq 1$, $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_p)$ the left most p-chain of $\bar{\mathcal{C}}^u$, and $(\mathcal{B}_0, \mathcal{B}_{p+1}, \ldots, \mathcal{B}_{p+q})$ the left most q-chain of $\bar{\mathcal{C}}^v(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_p}}\mathcal{T})$. Then,*

$$d[\bar{\mathcal{C}}^w(^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_{p+q}}}\mathcal{T})] = \min\{a_{w-1} - a_w, b_r + d[\mathcal{I}_w], b_r + d[\mathcal{I}_v] - q, b_r + d[\mathcal{I}_u] - p - q\}.$$

Proof: Let $\tilde{\mathcal{T}} :=^{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_p} \mathcal{T}$, where $\tilde{\mathcal{T}} = [\tilde{a}, \tilde{b}, \tilde{c}, \tilde{X}]$ is such that $\tilde{b}_r = b_r$ if $p \leq d[\mathcal{I}_u]$, and $\tilde{b}_r = b_r - d[\mathcal{I}_u] - p$ if $p > d[\mathcal{I}_u]$. Note that $^{\widehat{\mathcal{B}_0\mathcal{B}_1\ldots\mathcal{B}_{p+q}}}\mathcal{T} = {}^{\widehat{\mathcal{B}_{p+1}\ldots\mathcal{B}_{p+q}}}\tilde{\mathcal{T}}$.

By $3,(b)$ of the previous proposition, and considering the fact that $d[\mathcal{I}_w(\tilde{\mathcal{T}})] = d[\mathcal{I}_v(\tilde{\mathcal{T}})] = 0$ if $p > d[\mathcal{I}_u]$, we have

$$\begin{aligned} d[\bar{\mathcal{C}}^w(^{\widehat{\mathcal{B}_{p+1}\ldots\mathcal{B}_{p+q}}}\tilde{\mathcal{T}})] &= \min\{a_w - a_{w-1}, \tilde{b}_r + d[\mathcal{I}_w(\tilde{\mathcal{T}})], \tilde{b}_r + d[\mathcal{I}_v(\tilde{\mathcal{T}})] - q\} \\ &= \min\{a_w - a_{w-1}, b_r + d[\mathcal{I}_w], b_r + d[\mathcal{I}_v] - q, b_r + d[\mathcal{I}_u] - p - q\}. ■ \end{aligned}$$

4.2 Definition of insertion path chain and insertion sequence

Let $r \geq 1$ and $\mathcal{T} = [(a_1, \ldots, a_r); (b_1, \ldots, b_r); (c_2, \ldots, c_{r+1}), X] \in \overline{LRD}_r$, $|a| + |b| + |\sum_{i=2}^r c_i| = rm$. Let $k \in \{0, 1, \ldots, r-1\}$ and $\bar{l} = d[\bar{\mathcal{C}}^{k+1}]$. We may conclude from the previous section that the leftmost longest chain $\mathcal{B}_0 < \mathcal{B}_1 < \ldots < \mathcal{B}_{\bar{l}}$ of $\bar{\mathcal{C}}^{k+1}$ may be determined as follows.

For $u = 1, \ldots, \bar{l}$, the numbering sequence $(v_r^u > \ldots > v_{k+1}^u)$ of $\mathcal{B}_u$ is such that: $v_{k+1}^1 \leq \ldots \leq v_{k+1}^{\bar{l}}$ are the $\bar{l}$ left most nonzero symbols in the $(k+1)$-th row of $\mathcal{T}$; and, for $i = k, \ldots, r$, $v_i^1 \leq \ldots \leq v_i^{\bar{l}}$ are the $\bar{l}$ left most nonzero symbols in the i-th row of $\mathcal{T}$ satisfying $v_i^1 > v_{i-1}^1, \ldots, v_i^{\bar{l}} > v_{i-1}^{\bar{l}}$.

Let $y \in \{0, 1, \ldots, \bar{l}\}$. We may define the LR rectangular tableau $^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_y}\mathcal{T} = [(a_1, \ldots, a_{k+1} + y, \ldots, a_r, 0);\ (b_1, \ldots, b_r, 0); (m - y, c_2, \ldots, c_{r+1}); ^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_y} X] \in \overline{LRD}_{r+1}$ obtained from $\mathcal{T}$ by $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_y)$-*inserting zeros*, in the following way: In the $(k+1)$-th row of $\mathcal{T}$ insert y symbols 0 by shoving the nonzero symbols $v_{k+1}^1 \leq \ldots \leq v_{k+1}^y$. These y shoved symbols are inserted in the $(k+2)$-th row by shoving the nonzero symbols $v_{k+2}^1 \leq \ldots \leq v_{k+2}^y$. The shoved symbols $v_{k+2}^1 \leq \ldots \leq v_{k+2}^y$ are inserted in the $(k+3)$-th row as before. Proceed down, inserting and shoving, until reaching the $(r+1)$-th row.

When reaching the $(r+1)$-th row, the symbols $v_r^1 \leq \ldots \leq v_r^y$ shoved from the r-th row and inserted in the $(r+1)$-th row.

Notice that, for $i = 1, \ldots, y$, if we delete the last row of $^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_i}\mathcal{T}$ we obtain $^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\ldots,\mathcal{B}_i}}\mathcal{T}$ the rectangular LR tableau defined above by suppressing the last row of $^{\mathcal{B}_i}(^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\ldots,\mathcal{B}_{i-1}}}\mathcal{T})$.

For $i = 1, \ldots, y$, $^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_i}X$ is defined inductively by $^{\mathcal{B}_1,\ldots,\mathcal{B}_i}X := {}^{\mathcal{B}_i}(^{\mathcal{B}_0,\widehat{\mathcal{B}_1,\ldots,\mathcal{B}_{i-1}}}X) + \begin{bmatrix} 0_{r,r} & 0 \\ w_{i-1} & 0 \end{bmatrix}$, where $[w_{i-1}, 0] = [0^{k-1}, i-1, 0^{r-k+1}]$, $1 \times (r+1)$, is the $(r+1)$-th row of ${}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{i-1}}X$, and ${}^{\widehat{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{i-1}}}X$ is the r-square matrix obtained from ${}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{i-1}}X$ by deleting the $(r+1)$-th row and column. We say that ${}^{\mathcal{B}_0\mathcal{B}_1,\ldots,\mathcal{B}_i}X$ was obtained from X by inserting $(0^{k-1}, y, 0^{r-k}, 0)$.

The number of symbols r in the $(r+1)$-th row of ${}^{\mathcal{B}_1,\ldots,\mathcal{B}_y}\mathcal{T}$ is $max\,\{0, y - d[\mathcal{I}_{k+1}]\}$.

So, after inserting y symbols 0 in the row $k+1$ of $\mathcal{T}$, b_r is reduced to $\min\{b_r, b_r + d[\mathcal{I}_{k+1}] - y\}$. That is, the number of symbols r in the r-th row of ${}^{\mathcal{B}_1,\ldots,\mathcal{B}_y}\mathcal{T}$ is b_r if $y \le d[\mathcal{I}_{k+1}]$, and it is $b_r + d[\mathcal{I}_{k+1}] - y$ if $y > d[\mathcal{I}_{k+1}]$. If $(y_{r1}, \ldots, y_{r,r-1})$ is the r-deleting sequence of $\mathcal{T}$, b_r is reduced to $\min\{b_r, b_r + \sum_{j=1}^{k-1} y_{rj} - y\}$.

Given the non negative integers $a_{r+1} \le a_r$ and $b_{r+1} \le \min\{b_r, b_r + \sum_{j=1}^{k-1} y_{rj} - y\}$, by inserting a_{r+1} symbols 0 in the bottom of the diagram of a and b_{r+1} symbols $r+1$ in the $(r+1)$-th row of ${}^{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_y}\mathcal{T}$, we obtain the LR rectangular tableau $\mathcal{T}' = [(a_1, \ldots, a_{k+1} + y, \ldots, a_r, a_{r+1});\ (b_1, \ldots, b_r, b_{r+1});\ (c_1, c_2, \ldots, c_{r+1});\ {}^{\mathcal{B}_0\mathcal{B}_1,\ldots,\mathcal{B}_y}X + ([0] \oplus [b_{r+1}])]$. We say that ${}^{\mathcal{B}_0\mathcal{B}_1,\ldots,\mathcal{B}_y}X + ([0] \oplus [b_{r+1}])]$ was obtained from X by inserting $(0^{k-1}, y, 0^{r-k}, b_{r+1})$.

We can go backwards to $\mathcal{T}$ by deleting zeros on $\mathcal{T}'$, and by suppressing the last rows of $(a_1, \ldots, a_{k-1}, a_k + y, a_{k+1}, \ldots, a_r, a_{r+1})$, $(b_1, ..., b_{r+1})$ and the first row of $(c_1, ..., c_{r+1})$. Notice that if $\mathcal{B}_i = ((v_r^i, j_r^i), \ldots, (v_{k+1}^i, j_{k+1}^i), (v_k^i, j_k^i))$, for $i = 1, \ldots, y$, then, by lemma 4, the $(r+1)$-deleting path chain of $\mathcal{T}'$ is $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_y)$ where $\mathcal{Z}_i = ((v_r^i, a_{r+1} + i), (v_{r-1}^i, j_r^i), \ldots, (v_k^i, j_{k+1}^i))$, for $i = 1, \ldots, y$. We say that $\mathcal{Z}_i$ is obtained from $\mathcal{B}_i$ by one step down.

This leads us to the definition of insertion set of an LR rectangular tableau.

Let $r \ge 1$ and $\mathcal{T} = [(a_1, ..., a_r), (b_1, ..., b_r), (c_2, ..., c_{r+1}), X] \in \overline{LRD}_r$

Definition 17 *Let $k \in \{1, \ldots, r\}$, $d_k = d[\bar{\mathcal{C}}^k]$ and $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{d_k})$ the left most longest chain of $\bar{\mathcal{C}}^k$. A nonnegative integer y such that $y \le d_k$ is called a k-insertion number of $\mathcal{T}$, and $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_y)$ is the corresponding insertion path chain. We call $\{y\}$ a k-insertion set of $\mathcal{T}$.*

Definition 18 *Let $K = \{i_t, \ldots, i_2, i_1\}$ be a subset of $\{1, \ldots, r\}$, where we are assuming $r \ge i_t > \ldots > i_1 \ge 1$. Let $y = \sum_{s \in K} y_s$. A set of non negative integers $\{y_s\}_{s\in K}$ is called a K-insertion set of $\mathcal{T}$ with insertion path chain $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_y)$, if $\{y_s\}_{s \in K' = \{i_t, \ldots, i_2\}}$ is a $K' = \{i_t, \ldots, i_2\}$-insertion set of $\mathcal{T}$ with insertion path chain $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{y'})$ where $y' = \sum_{s\in K'} y_s$; and y_{i_1} is an i_1-insertion number of ${}^{\widehat{\mathcal{B}_0\mathcal{B}_1,\ldots,\mathcal{B}_{y'}}}\mathcal{T}$ with insertion path chain $(\mathcal{B}_0, \mathcal{B}_{y'+1}, \ldots, \mathcal{B}_y)$. We let the emptyset be an insertion set with insertion path chain $(\mathcal{B}_0 = \emptyset)$.*

We say that a sequence of non negative integers $(y_1, \ldots, y_r)$ is an *insertion sequence* of $\mathcal{T}$ if $\{y_r, \ldots, y_1\}$ is a $\{r, \ldots, 1\}$-insertion set of $\mathcal{T}$. Clearly, if $\{y_s\}_{s\in K}$ is a K-insertion set of $\mathcal{T}$, then $\{y_s\}_{s\in\{1,\ldots,r\}}$, with $y_s = 0$ for $s \notin K$, is also a $\{r, \ldots, 1\}$-insertion set, and, thus, $(y_1, \ldots, y_r)$ is an insertion sequence of $\mathcal{T}$.

Given $b_{r+1} \le b_r$, we say that $(y_1, \ldots, y_r)$ is an *insertion sequence mod*(b_{r+1}) of $\mathcal{T}$ if $(y_1, \ldots, y_r)$ is an insertion sequence of $\tilde{\mathcal{T}} = [(a_1, ..., a_r), (b_1, ..., b_r - b_{r+1}), (c_2 + b_{r+1}, ..., c_{r+1}), \tilde{X}]$, where $\tilde{x}_{rr} = x_{rr} - b_{r+1}$ and $\tilde{x}_{ij} = x_{ij}$, otherwise. Given $a_{r+1} \le a_r$, we say that $(y_1, \ldots, y_r, a_{r+1})$ is a full insertion sequence *mod*$(b_{r+1}$ of $\mathcal{T}$.

Theorem 6 *Let $(y_{r1}, \dots, y_{r,r-1})$ be the r-deletion sequence of $\mathcal{T}$, and $b_{r+1} \leq b_r$. Then, $(y_1, \dots, y_r)$ is an insertion sequence mod(b_{r+1}) of $\mathcal{T}$ iff*

$$a_{k-1} - a_k \geq y_k, \quad k = 1, ..., r, \tag{6}$$

$$b_r + \sum_{j=1}^{k-1} y_{rj} \geq b_{r+1} + \sum_{j=1}^{k} y_j, \quad k = 1, ..., r. \tag{7}$$

Proof: Let $\tilde{\mathcal{T}} = [(a_1, ..., a_r),\ (b_1, ...,\ b_r - b_{r+1}),\ (c_2 + b_{r+1}, ..., c_{r+1}), \tilde{X}]$, where $\tilde{x}_{rr} = x_{rr} - b_{r+1}$ and $\tilde{x}_{ij} = x_{ij}$, otherwise. Attending to definition 18, $(y_1, \dots, y_r)$ is an insertion sequence if $y_r \leq d[\bar{\mathcal{C}}^r]$ and, for $j = 1, \dots, r-1$, $y_{r-j} \leq d[\bar{\mathcal{C}}^{r-j}({}^{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{y_r+\dots+y_{r-j+1}}}\mathcal{T})]$, where $(\mathcal{B}_0, \mathcal{B}_{y_{r-j+2}+1}, \dots, \mathcal{B}_{y_r+\dots+y_{r-j+1}})$ is the left most y_{r-j+1}-chain of $\bar{\mathcal{C}}^{r-j}({}^{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{y_r+\dots+y_{r-j+2}}}\mathcal{T})$. By an inductive argument, corollary 4 says that $(y_1, \dots, y_r)$ is an insertion sequence iff for $s = 1, \dots, r$,

$$y_s \leq \min\{a_{s-1} - a_s, \min_{s \leq k \leq r}\{b_r - b_{r+1} + \sum_{j=1}^{k-1} y_{rj} - \sum_{j=s+1}^{k} y_j\}\}.$$

This is equivalent to

$$\begin{array}{ll} a_{k-1} - a_k \geq y_k, & k = 1, ..., r, \\ b_r + \sum_{j=1}^{k-1} y_{rj} \geq b_{r+1} + \sum_{j=s}^{k} y_j, & s = 1, ..., r, \quad k = s, ..., r. \end{array}$$

When $s = 1$ we obtain (7). For $s = 2, \dots, r$, we obtain particular cases of (7). ■

Let $\mathbf{P}$ be the projection of $\overline{LRD}_{r+1}$ on $\overline{LRD}_r$.

We are now in condition to show that a full deletion sequence of an LR tableau $\mathcal{T}$ is a full insertion sequence of the projected LR tableau $\mathbf{P}(\mathcal{T})$.

Let $r \geq 0$ and $\mathcal{F} \in \overline{LRD}_r$ of type $[a'; (b_1, \dots, b_r); (c_2, \dots, c_{r+1})]$ with $|a'| + \sum_{i=1}^{r}(b_i + c_{r+2-i}) = rm$. As usual, we write $\mathbf{P}^{-1}(\mathcal{F}) = \{\mathcal{T} \in \overline{LRD}_{r+1} : \mathbf{P}(\mathcal{T}) = \mathcal{F}\}$.

Theorem 7 *Let $(y_{r1}, \dots, y_{r,r-1})$ be the r-deletion sequence of $\mathcal{F}$. Let $\mathcal{T}$ of type $[a, (b_1, \dots, b_r, b_{r+1}); (c_1, c_2, \dots, c_{r+1})]$. Then, $\mathcal{T} \in \mathbf{P}^{-1}(\mathcal{F})$ iff*

1. *$a_{i+1} \leq a'_i \leq a_i$, $1 \leq i \leq r$,*
2. *For $k = 1, \dots, r$, $b_r + \sum_{j=1}^{k-1} y_{rj} \geq b_{r+1} + \sum_{j=1}^{k}(a_j - a'_j)$,*
3. *$a_{r+1} + \sum_{i=1}^{r}(a_j - a'_j) + b_{r+1} + c_1 = m$.*

Proof: When $r = 0$, $\mathcal{F}$ is the empty tableau and $\mathcal{T} = [(a_1), (b_1);\ (c_1), [b_1]]$, with $a_1 + b_1 + c_1 = m$. Let $r \geq 1$.

Proof of the *"if"* part. Let $y := |a| + |a'| - a_{r+1}$. From theorem 6, conditions (1) and (2) mean that $(a_1 - a'_1, \dots, a_r - a'_r, a_{r+1})$ is a full insertion sequence *mod*(b_{r+1}) of $\mathcal{F}$. If $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_y)$ is the corresponding insertion path chain, then we have ${}^{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_y}\mathcal{F} = [(a_1, ..., a_r\ , 0),\ (b_1, ...,\ b_r, 0), (m - y,\ c_2, ..., c_{r+1});\ {}^{\mathcal{B}_0 \mathcal{B}_1, \dots, \mathcal{B}_y}X]$.

Let $\mathcal{T}$ be obtained from ${}^{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_y}\mathcal{F}$ by inserting a_{r+1} zeros in the bottom of the diagram of a, and b_{r+1} symbols $r + 1$ in the bottom row of ${}^{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_y}\mathcal{F}$. Then, $\mathcal{T} =$

$[(a_1, ..., a_r\ , a_{r+1}),\ (b_1, ...,\ b_r, b_{r+1}),(c_1,\ c_2, ..., c_{r+1}), X'] \in \overline{LRD}_{r+1}$, where $X' =^{\mathcal{B}_0 \mathcal{B}_1, ..., \mathcal{B}_y} X + ([0] \oplus [b_{r+1}])$ and $m - y - b_{r+1} = c_1$. Moreover, $\mathcal{T}$ has $(r+1)$-full deletion sequence $(a_1 - a'_1, \dots, a_r - a'_r, a_{r+1})$ and $(r+1)$-deletion path chain the one obtained from $(\mathcal{B}_1, \dots, \mathcal{B}_y)$ by one step down. Therefore $\mathbf{P}(\mathcal{T}) = \mathcal{F}$.

Proof of the *"only if"* part. Let $\mathcal{T} = [(a_1, ..., a_{r+1}); (b_1, ..., b_{r+1});\ (c_1,\ c_2, ..., c_{r+1}), X] \in \overline{LRD}_{r+1}$ with $(r+1)$-deletion sequence $(y_1, \dots, y_r)$, such that $\mathbf{P}(\mathcal{T}) = \mathcal{F}$. Then, applying theorem 4 to $\mathcal{T}$, we obtain $\mathcal{F} = [a', (b_1, \dots, b_r); (c_2, \dots, c_{r+1}), X'] \in \overline{LRD}_r$ such that $|a'| + \sum_{i=1}^{r}(b_i + c_{r+2-i}) = m$, $a_{i+1} \leq a'_i \leq a_i$, $1 \leq i \leq r$, where $a_i - a'_i = y_i$, for $i = 1, \dots, r$.

We claim that $(y_1, \dots, y_r, a_{r+1})$ *is a full insertion sequence mod*(b_{r+1}) *of* $\mathcal{F}$.

Let $d_{r+1} := 0$ and $d_i := d_{i+1} + y_i$, $i = 1, \dots, r$. Suppose $(\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_{d_r}, \mathcal{Z}_{d_r+1}, \dots, \mathcal{Z}_{d_2}, \mathcal{Z}_{d_2+1}, \dots, \mathcal{Z}_{d_1})$ is the $(r+1)$-deletion path chain of $\mathcal{T}$, where, for $i = 1, \dots, r$, and $d_{i+1} < t \leq d_i$, $\mathcal{Z}_t = ((z^t_{r+1}, j^t_{r+1}), \dots, (z^t_{i+1}, j^t_{i+1})$.

For $i = 1, \dots, r$, and $d_{i+1} < t \leq d_i$, let $\mathcal{B}_t = ((z^t_{r+1}, j^t_r), \dots, (z^t_{i+1}, j^t_i), (z^t_i, a'_i + t))$. Then, $(\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{d_r}, \mathcal{B}_{d_r+1}, \dots, \mathcal{B}_{d_2}, \dots, \mathcal{B}_{d_2+1}, \dots, \mathcal{B}_{d_1})$ is an insertion path chain. Note that, for each $i \in \{1, \dots, r\}$, $(\mathcal{B}_0, \mathcal{B}_{d_{i+1}+1}, \dots, \mathcal{B}_{d_i})$ is the left most y_i-chain of $\bar{\mathcal{C}}^i(^{\widehat{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{d_{i+1}}}}\mathcal{T})$. Therefore, by theorem 6, conditions 1 and 2 follow. ■

5 An involution on the set of Littlewood-Richardson tableaux

5.1 The deletion-insertion matrix of an LR tableau and the commutative property of the LR rule

In section 2.1, we have defined an LR rectangular tableau of order $r \geq 1$, as being an element $[a, b, c, X] \in \mathbb{Z}^{3r+r^2}$ satisfying a certain system of linear inequalities. That is, by definition, there is an LR rectangular tableau of order r and type $[a, b, c]$ with $|a| + |b| + |c| = rm$, iff there is a lower triangular matrix $X = [x_{ij}] \in \mathbb{Z}^{r,r}$ satisfying the system of linear inequalities (1), (2), (4), (5) and $(**)$.

Now, notice that, given $a, b, c \in \mathcal{P}_r$, with $|a| + |b| + |c| = rm$, and $X = [x_{ij}] \in \mathbb{Z}^{r,r}$, $[a, b, c, X] \in \overline{LRD}_r$ iff $b^{(s)} = (\sum_{i=j}^{s} x_{ij}, 0^{r-s})_{j=1}^{s}$, for $s = 0, 1, ..., r$, is a sequence of partitions with $b^{(r)} = b$, satisfying the interlacing inequalities

$$b^{(s)}_{i+1} \leq b^{(s-1)}_i \leq b^{(s)}_i, \text{ for } s = 1, ..., r,\ i = 1, ..., r-1, \tag{8}$$

and the system of linear inequalities

$$\begin{aligned} a_{s-1} + \sum_{j=1}^{k-1}(b^{(s-1)}_j - b^{(s-2)}_j) &\geq a_s + \sum_{j=1}^{k}(b^{(s)}_j - b^{(s-1)}_j),\ k = 1, \dots, s-1,\ s = 2, \dots, r, &(9)\\ a_s + \sum_{j=1}^{r}(b^{(s)}_j - b^{(s-1)}_j) &= m - c_{r-s+1},\ s = 1, \dots, r. &(10) \end{aligned}$$

We may therefore reformulate the definition of LR rectangular tableau in the following way: There exists an LR rectangular tableau $\mathcal{T}$ of order r and type $[a, b, c]$ with $|a| +$

$|b|+|c|=rm$ iff there exists a sequence of partitions $b^{(s)}=(b_1^{(s)},\ldots,b_s^{(s)},0^{r-s})$, $s=0,1,...,r$, with $b^{(r)}=b$, satisfying the interlacing inequalities (8) and the system of linear inequalities (9) and (10). The sequence $b^{(s)}$, $s=0,1,...,r$, defines the sequence $[(a_1,\ldots,a_s);\ (b_1^{(s)},\ldots,b_s^{(s)});\ (c_{r-s+1},\ldots,c_r)]$, $s=1,\ldots,r$, of LR triples, called the b-decomposition of $\mathcal{T}$.

Applying the material developped in the previous sections, we shall see that the LR rule for composing two partitions a and b is commutative. According section 3, theorem 5, we claim:

Let $r\geq 1$. Given an LR rectangular tableau $\mathcal{T}$ of order r and type $[a,b,c]$ with $|a|+|b|+|c|=rm$, we may decompose $\mathcal{T}$ into a sequence of LR rectangular tableaux $\mathcal{T}^{(s)}$, $s=0,\ldots,r$, such that $\mathcal{T}^{(0)}$ is the empty tableau, and for $s=1,\ldots,r$, $\mathcal{T}^{(s)}$ is of order s and type $[a^{(s)},(b_1,\ldots,b_s),(c_{r-s+1},\ldots,c_r)]$, and $\mathcal{T}^{(r)}=\mathcal{T}$, where $a^{(s)}=(a_1^{(s)},\ldots,a_s^{(s)},0^{r-s})$, $s=0,1,...,r$, with $a^{(r)}=a$, satisfy

$$a_{i+1}^{(s)}\leq a_i^{(s-1)}\leq a_i^{(s)},\ s=1,2,\ldots,r,\ i=1,\ldots,r-1. \tag{11}$$

For $s=1,\ldots,r$, let $\mathcal{T}^{(s-1)}$ be obtained from $\mathcal{T}^{(s)}$ by s-full deletion, that is, $\mathcal{T}^{(s-1)}=\mathbf{P}(\mathcal{T}^{(s)})$.

For $s=1,2,\ldots,r$, let $(y_{s,1},\ldots,\ y_{s,s-1},y_{s,s})$ be the s-full deletion sequence of $\mathcal{T}^{(s)}$, then

$$a_i^{(s)}-a_i^{(s-1)}=y_{si},\ i=1,\ldots,s. \tag{12}$$

Attending to theorem 7, for $s=1,\ldots,r$, $\mathcal{T}^{(s)}$ is obtained from $\mathcal{T}^{(s-1)}$ by full insertion-$mod(b_s)$ the sequence $(a_1^{(s)}-a_1^{(s-1)},\ \ldots,a_{s-1}^{(s)}-a_{s-1}^{(s-1)},a_s^{(s)})$. So, for $s=1,\ldots,r$, $(a_1^{(s)}-a_1^{(s-1)},\ \ldots,a_{s-1}^{(s)}-a_{s-1}^{(s-1)},a_s^{(s)})$ is also a full insertion sequence $mod(b_s)$ of $\mathcal{T}^{(s-1)}$. Hence, by (11) and by theorems 6 and 7, we have,

$$\begin{aligned} a_{i+1}^{(s)}\leq a_i^{(s-1)} &\leq a_i^{(s)},\ i=1,\ldots,r-1,\ s=1,2,\ldots,r, &(13)\\ b_{s-1}+\sum_{j=1}^{k-1}(a_j^{(s-1)}-a_j^{(s-2)}) &\geq b_s+\sum_{j=1}^{k}(a_j^{(s)}-a_j^{(s-1)}),\ k=1,...,s-1,\ s=2,\ldots,r, &(14)\\ b_s+\sum_{j=1}^{r}(a_j^{(s)}-a_j^{(s-1)}) &= m-c_{r-s+1},\ s=1,\ldots,r. &(15)\end{aligned}$$

Notice that, by (12), the s-th row of the diagram $a^{(s)}$ is given by $a_s^{(s)}=a_s-\sum_{t=s+1}^r y_{t,s}$, for $s=1,\ldots,r$. When passing from $\mathcal{T}^{(s)}$ to $\mathcal{T}^{(s-1)}$ we suppress $a_s^{(s)}=a_s-\sum_{t=s+1}^r y_{t,s}$, the bottom row of $a^{(s)}$, and when passing from $\mathcal{T}^{(s-1)}$ to $\mathcal{T}^{(s)}$ we add one bottom component of length $a_s-\sum_{t=s+1}^r y_{t,s}$ to the diagram $a^{(s-1)}$.

Finally, we may conclude that if we are given an LR rectangular tableau $\mathcal{T}$ of order r and type $[a,b,c]$ with $|a|+|b|+|c|=rm$, then there exists a sequence of partitions $a^{(s)}=(a_1^{(s)},\ldots,a_s^{(s)},0^{r-s})$, $s=0,\ldots,r$, with $a^{(r)}=a$, satisfying the interlacing inequalities (13) and the system of linear inequalities (14) and (15). The sequence $a^{(s)}$, $s=0,1,...,r$, defines the sequence $[a^{(s)};\ (b_1,\ldots,b_s);\ (c_{r-s+1},\ldots,c_r)]$, $s=1,\ldots,r$, of LR triples, called the a-decomposition of $\mathcal{T}$.

Therefore, the LR rule for composing partitions is commutative.

The following $r \times r$ lower triangular matrix records the deletion/insertion steps of $\mathcal{T}$, equivalently, the a-decomposition of $\mathcal{T}$:

$$Y = \begin{bmatrix} a_1 - \sum_{t=2}^r y_{t,1} & 0 & 0 & 0 & 0 & 0 \\ y_{21} & a_2 - \sum_{t=3}^r y_{t,2} & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & a_3 - \sum_{t=4}^r y_{t3} & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \dots & 0 & 0 \\ y_{r-1,1} & y_{r-1,2} & y_{r-1,3} & \dots & a_{r-1} - y_{r,r-1} & 0 \\ y_{r1} & y_{r2} & y_{r3} & \dots & y_{r,r-1} & a_r \end{bmatrix}. \quad (16)$$

For $s = 1, \dots, r$, the first s entries $(y_{s1}, \dots, y_{s,s-1}, y_{s,s})$ of the s-th row of Y, define the full deletion sequence of $\mathcal{T}^{(s)}$/ the full insertion sequence of $\mathcal{T}^{(s-1)}$. We call Y the *deletion -insertion* matrix of $\mathcal{T}$.

By construction Y is completely determined by $\mathcal{T}$. The r-th row of Y is the r-full deletion sequence of $\mathcal{T} = \mathcal{T}^{(r)}$ which is unique (recall proposition 2), and the principal $(r-1) \times (r-1)$ matrix of Y is the *deletion -insertion* matrix of $\mathcal{T}^{(r-1)}$. So by induction on r, we may conclude that there exists *one and only one* deletion-insertion matrix.

The above discussion, namely (13), (14), (15), leads to

Theorem 8 *If $[a, b, c, X] \in \overline{LRD}_r$, and Y is its deletion-insertion matrix, then $[b, a, c, Y] \in \overline{LRD}_r$.*

On the other hand, if $[a, b, c, X] \in \overline{LRD}_r$, then $b^{(s)} = (\sum_{i=j}^s x_{ij}, 0^{r-s})_{j=1}^s$, $0 \le s \le r$, with $b^{(r)} = b$, satisfy conditions (8), (9) and (10). In particular, the sequence of partitions $b^{(s)}$, $0 \le s \le r$, satisfy the interlacing inequalities, and this implies $b_i^{(s)} - b_{i+1}^{(s)} \ge x_{si}$, for $s = 1, \dots, r$, $i = 1, \dots, s-1$. Let $\mathcal{W}^{(0)}$ be the LR tableau of order 0. Therefore, by theorem 6, for each $s = 1, \dots, r$, $(x_{s1}, \dots, x_{s,s-1}, x_{s,s})$ is an insertion sequence $mod(a_s)$ of $\mathcal{W}^{(s-1)}$ So, by theorem 7, for $s = 1, \dots, r$, we may associate an unique tableau in $\overline{LRD}_s$ of type $[b^{(s)}$, $(a_1, \dots, a_s)$, $(c_{r-s+1}, \dots, c_r)]$, by inserting $mod(a_s)$ the sequence $(x_{s1}, \dots, x_{s,s-1}, x_{s,s})$ on $\mathcal{W}^{(s-1)}$.

Given $[a, b, c, X] \in \overline{LRD}_r$, we may perform the following steps by full insertion. We start with $\mathcal{W}^{(0)}$, the LR rectangular tableau of order 0 and we pass to the LR rectangular tableau $\mathcal{W}^{(1)} = [b^{(1)} = (x_{11}),\ a^{(1)} = (a_1), c^{(1)} = (c_r), W^{(1)} = [a_1]]$ of order 1. Then we pass to the LR rectangular tableau $\mathcal{W}^{(2)}$ of order 2 and type $[b^{(2)}, (a_1, a_2)$, $(c_{r-1},\ c_r)]$ by inserting-$mod(a_2)$ the sequence (x_{21}, b_2) on $\mathcal{W}^{(1)}$. Repeating the process with $\mathcal{W}^{(2)}$, we may pass to the LR rectangular tableau $\mathcal{W}^{(3)}$ of order 3 and type $[b^{(3)}, (a_1, a_2, a_3), (c_{r-2}, c_{r-1}, c_r)]$ by inserting $mod(a_3)$ the sequence (x_{31}, x_{32}, b_3) on $\mathcal{W}^{(2)}$, and so on. Eventually, by inserting $mod(a_r)$ the sequence $(x_{r1}, \dots, x_{r,r-1}, b_r)$ in $\mathcal{W}^{(r-1)}$, we get the LR rectangular tableau $\mathcal{W}$ of type $[b, a, c]$, whose deletion-insertion matrix is X. We say that $\mathcal{W}$ was obtained by insertion of X $mod(a)$.

We have proved

Theorem 9 *If $[a, b, c, X] \in \overline{LRD}_r$, then there exists one and only one rectangular tableau in $\overline{LRD}_r$ of type $[b, a, c]$ whose deletion-insertion matrix is X.*

Let $r \geq 1$. We are now in conditions to exhibit a bijection on $\overline{LRD}_r$, transforming an LR rectangular tableau of type $[a, b, c]$ into one of type $[b, a, c]$.

We consider the map $\phi : \overline{LRD}_r \longrightarrow \overline{LRD}_r$ defined by $\phi([a, b, c, X]) = [b, a, c, Y]$, where Y is the deletion matrix of $[a, b, c, X]$. Clearly, ϕ is well defined since Y is unique. It remains to show that ϕ is a bijection.

We may define $\pi : \overline{LRD}_r \longrightarrow \overline{LRD}_r$ such that $\pi([a, b, c, X])$ is the rectangular tableau of type $[b, a, c]$ whose insertion matrix is X. That is, $\pi([a, b, c, X])$ is defined by insertion $mod(a)$ of X. Clearly, π is well defined since, by definition of insertion matrix, two LR tableaux with the same insertion matrix are equal.

Since the insertion matrix and the deletion matrix of an LR rectangular tableau are the same, it is clear that $\pi\phi = \phi\pi = id$ and thus ϕ and π are both bijections on $\overline{LRD}_r$. Therefore, these inverse bijections allows one to give a combinatorial interpretation of the equality of the Littlewood-Richardson numbers $N_{ab}^c = N_{ba}^c$. In the next subsection we shall show that $\pi = \phi$ and that we have, in fact, an involution on $\overline{LRD}_r$.

5.2 ϕ and π are the same involution

In the previous sections we have given an algorithmic characterization of the deletion-insertion matrix of an LR rectangular tableau $\mathcal{T}$. That is, we have used the combinatorial deletion and insertion operations to define the sequence $a^{(s)}$, $s = 0, 1, \ldots, r$, of the a-decomposition of $\mathcal{T}$. Here, we give a recursive algebraic characterization. With this inductive definition we are able to prove

Theorem 10 $[a, b, c, X]$ *is an LR rectangular tableau with deletion-insertion matrix* Y *iff the deletion-insertion matrix of* $[b, a, c, Y]$ *is* X*.*

Therefore,

Corollary 5 *Considering the bijections* ϕ *and* π *defined above, we have* $\phi = \pi$ *and* $\phi^2 = \pi^2 = id$*. That is,* ϕ *and* π *are the same involution on* $\overline{LRD}_r$*.*

We start by observing (*e.g.* [1]) that, if $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$, then $\mathcal{F} = [(a_i + x_{i1})_{i=2}^r;\ (b_2, \ldots, b_r);\ (c_1, \ldots, c_{r-1}); X'] \in \overline{LRD}_{r-1}$, where X' is the $(r-1) \times (r-1)$ principal submatrix of X obtained by suppressing the first row and the first column. Also observe that if $Y = [y_{ij}] \in \mathbb{Z}^{r,r}$ is the deletion-insertion matrix of $\mathcal{T}$, then the principal submatrix in the first $r-1$ rows of Y is the deletion-insertion matrix of $\mathcal{T}^{(r-1)} = [(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)}); (b_1, \ldots, b_{r-1});\ (c_2, \ldots, c_r); X^{(r-1)}] \in \overline{LRD}_{r-1}$ obtained from $\mathcal{T}$ by r-full deletion, where $a_i^{(r-1)} = a_i - y_{ri}$, $i = 1, \ldots, r-1$.

Lemma 7 *Let* $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$ *and* $\mathcal{T}^{(r-1)} = [(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)});\ (b_1, \ldots, b_{r-1});\ (c_2, \ldots, c_r);\ X^{(r-1)}] \in \overline{LRD}_{r-1}$ *obtained from* $\mathcal{T}$ *by* r*- full deletion. Let* $\mathcal{F} = [(a_i + x_{i1})_{i=2}^r,\ (b_2, \ldots, b_r),\ (c_1, \ldots, c_{r-1}), X'] \in \overline{LRD}_{r-1}$*, where* X' *is the* $(r-1) \times (r-1)$ *principal submatrix obtained from* X *by suppressing the first row and the first column, and let* $\mathcal{F}' \in \overline{LRD}_{r-2}$ *obtained from* $\mathcal{F}$ *by* $(r-1)$*-full deletion. Then*

1. $\mathcal{F}' = [(a_i^{(r-1)} + x_{i1}^{(r-1)})_{i=2}^{r-1}; (b_2, \ldots, b_{r-1}); (c_2, \ldots, c_{r-1}); X'^{(r-1)}] \in \overline{LRD}_{r-2}$, *where* $X'^{(r-1)}$ *is the* $(r-2) \times (r-2)$ *matrix obtained from* $X^{(r-1)}$ *by suppressing the first row and the first column.*

2. If $(z_{r2}, \ldots, z_{r,r})$ *is the* $(r-1)$*-full deletion sequence of* $\mathcal{F}$ *and* $\theta_i^{(r)} = \min\{x_{i1}, z_{r,i}\}$, *for* $i = 2, \ldots, r$, *it holds* $x_{11}^{(r-1)} = x_{11} + \theta_2^{(r)}$, $x_{i1}^{(r-1)} = x_{i1} - \theta_i^{(r)} + \theta_{i+1}^{(r)}$, *for* $i = 2, \ldots, r$, *and the r-full deletion sequence* $(y_{r1}, \ldots, y_{r,r})$ *of* $\mathcal{T}$ *is defined by* $y_{r1} = \theta_2^{(r)}$, $y_{r,i} = z_{ri} - \theta_i^{(r)} + \theta_{i+1}^{(r)}$, $i = 2, \ldots, r$.

Proof: Let $(z_{r,2}, \ldots, z_{r,r} = a_r + x_{r1})$ be the $(r-1)$-full deletion sequence of $\mathcal{F}$. That is, $z_{r,i}$ is the number of deleted zeros in row $i-1$ of $\mathcal{F}$, for $i = 2, \ldots, r$. Equivalently, it is the number of deleted cells in row i of the Young diagram $(a_s + x_{s1})_{s=1}^r$, for $i = 2, \ldots, r$.

Then, $a_i + x_{i1}$ is reduced to $a_i + x_{i1} - z_{ri}$, for $i = 2, \ldots, r$, and

$$\mathcal{F}' = [(a_i + x_{i1} - z_{ri})_{i=2}^{r-1}; (b_2, \ldots, b_{r-1}); (c_2, \ldots, c_{r-1}); X'^{(r-1)}] \in \overline{LRD}_{r-2}. \tag{17}$$

Let $\theta_i^{(r)} := \min\{x_{i1}, z_{ri}\}$, $i = 2, \ldots, r$, and $\theta_{r+1}^{(r)} := 0$. (Note that $\theta_r^{(r)} = x_{r,1}$.)

When we apply the $(r-1)$-full deletion operation to $\mathcal{F}$ we are applying the r-deletion operation to $\mathcal{T}$. So, if z_{ri} is the number of deleted cells in row i of the Young diagram $(a_i + x_{ri})_{i=1}^r$, then, when we consider the Young diagram $(a_s + x_{s1})_{s=1}^r$ marked, from right to left, in each row i with a_i zeros and x_{i1} symbols 1, it follows that $z_{ri} - \theta_i^{(r)} + \theta_{i+1}^{(r)}$ is the number of deleted zeros in row i of $\mathcal{T}$.

Therefore, the r-full deletion sequence $(y_{r1}, \ldots, y_{rr})$ of $\mathcal{T}$ is defined by $y_{r1} = \theta_2^{(r)}$, $y_{r,i} = z_{ri} - \theta_i^{(r)} + \theta_{i+1}^{(r)}$, $i = 2, \ldots, r$.

Thus, $\mathcal{T}^{(r-1)} = [(a_1^{(r-1)}, \ldots, a_{r-1}^{(r-1)}); (b_1, \ldots, b_{r-1});\ (c_1, \ldots, c_{r-1}); X^{(r-1)}] \in \overline{LRD}_{r-1}$ is such that $a_i^{(r-1)} = a_i - y_{ri}$, $i = 1, \ldots, r-1$, and

$$x_{11}^{(r-1)} = x_{11} + \theta_2^{(r)}, \tag{18}$$

$$x_{i,1}^{(r-1)} = x_{i1} - \theta_i^{(r)} + \theta_{i+1}^{(r)}, i = 2, \ldots, r. \tag{19}$$

Note that $x_{r1}^{(r-1)} = 0$. On the other hand, taking into account (17) and noting that, for $i = 2, \ldots, r$, $a_i^{(r-1)} + x_{i1}^{(r-1)} = a_i - y_{ri} + x_{i1}^{(r-1)} = a_i - [z_{ri} - \theta_i^{(r)} + \theta_{i+1}^{(r)}] + x_{i1} - \theta_i^{(r)} + \theta_{i+1}^{(r)} = a_i - z_{ri} + x_{i1}$, the lemma is proved. ■

Theorem 11 *Let* $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$ *and* $\mathcal{F} = [(a_i + x_{i1})_{i=2}^r;\ (b_2, \ldots, b_r);\ (c_1, \ldots, c_{r-1});\ X'] \in \overline{LRD}_{r-1}$, *where* X' *is the* $(r-1) \times (r-1)$ *matrix obtained from* X *by suppressing the first row and the first column. Let* $Z = [z_{ij}] \in \mathbb{Z}^{r,r}$ *such that the* $(r-1) \times (r-1)$ *submatrix in the last* $r-1$ *rows and columns is the deletion-insertion matrix of* $\mathcal{F}$, *and* $z_{1i} = z_{i1} = 0$, *for* $i \in \{1, \ldots, r\}$. *Then,* $Y = [y_{ij}] \in \mathbb{Z}^{r,r}$ *the deletion matrix of* $\mathcal{T}$ *is defined inductively by setting:*

$x_{i1}^{(r)} := x_{i1}$, $i = 1, \ldots, r$.

For $k = r, \ldots, 2$,

$\theta_i^{(k)} := \min\{x_{i1}^{(k)}, z_{ki}\}$, $i = 1, \ldots, k$, *and* $\theta_{k+1}^{(k)} := 0$,

$x_{i,1}^{(k-1)} := x_{i1}^{(k)} - \theta_i^{(k)} + \theta_{i+1}^{(k)}$, $i = 1, \ldots, k-1$,

$y_{k,i} := z_{ki} - \theta_i^{(k)} + \theta_{i+1}^{(k)}$, $i = 1, \ldots, k$, *and* $y_{11} = a_1 - \sum_{j=2}^r \theta_2^{(j)}$.

Proof : The proof will be handled by induction on r. For $r=1$, we have $\mathcal{T} = [(a_1),(x_{11}),(c_1);[x_{11}]]$. Then $\mathcal{F}=\emptyset$, $Z=[0]$ and $Y=[a_1]$.

For $r=2$, we have $\mathcal{T} = [(a_1,a_2);(x_{11}+x_{21},x_{2,2});\ (c_1,c_2);[x_{ij}]]$ and $\mathcal{F} = [(a_2+x_{21});(x_{22});(c_1);[x_{22}]]$ with deletion-insertion matrix $F=[a_2+x_{21}]$. By previous lemma, the second row of the deletion matrix Y of $\mathcal{T}$ is defined by $y_{21}=\theta_2^{(2)}=x_{21}$ and $y_{22}=a_2+x_{21}-\theta_2^{(2)}=a_2$. On the other hand, $[y_{11}]$ is the deletion matrix of $\mathcal{T}^{(1)}=[(a_1^{(1)});(x_{11}+x_{21});\ (c_2);[x_{11}^{(1)}]]\in\overline{LRD}_1$ where $a_1^{(1)}=a_1-y_{21}$ and $x_{11}^{(1)}=x_{11}+\theta_2^{(2)}=x_{11}+x_{21}$. Therefore, $y_{11}=a_1-x_{21}=a_1-\theta_2^{(2)}$, $y_{21}=z_{21}-\theta_2^{(1)}+\theta_2^{(2)}$ and $y_{22}=z_{22}-x_{21}=a_2$.

Let $r>2$ and suppose the truth of the theorem for $r-1$. By previous lemma the r-th row of the deletion matrix Y of $\mathcal{T}$ is defined by $y_{r,i}=z_{ri}-\theta_i^{(r)}+\theta_{i+1}^{(r)}$, $i=1,2,\ldots,r$, where $\theta_i^{(r)}:=\min\{x_{i1},z_{ri}\}$, $i=1,\ldots,r$, and $\theta_{r+1}^{(r)}:=0$.

Again by previous lemma, when we apply the r-full deletion operation to $\mathcal{T}$, we are applying also the $(r-1)$-full deletion operation to $\mathcal{F}$. We obtain, therefore, $\mathcal{T}^{(r-1)}=[(a_1^{(r-1)},\ldots,\ a_{r-1}^{(r-1)});\ (b_1,\ldots,\ b_{r-1});\ (c_2,\ldots,c_r);\ X^{(r-1)}]\in\overline{LRD}_{r-1}$ with deletion matrix $Y'=[y_{ij}]_{i,j=1}^{r-1}$ (the principal submatrix in the first $r-1$ rows of Y), and $\mathcal{F}'=[(a_i^{(r-1)}+x_{i1}^{(r-1)})_{i=2}^{r};(b_2,\ldots,\ b_{r-1}),\ (c_2,\ldots,c_{r-1}),\ X'^{(r-1)}]\in\overline{LRD}_{r-2}$, with deletion matrix $F'=[f_{ij}]_{i,j=2}^{r-1}$ (the principal submatrix in the first $r-2$ rows of F). Let $Z'=[z_{ij}]_{i,j=1}^{r-1}$ be the matrix obtained from Z by suppressing the last row and the last column.

By induction hypothesis and taking into account (18), (19), the matrix $Y'=[y_{ij}]_{i,j=1}^{r-1}$ is defined as follows

For $k=r-1,\ldots,2$, define
$\theta_i^{(k)}:=\min\{x_{i1}^{(k)},z_{ki}\}$, $i=1,\ldots,k$, and $\theta_{k+1}^{(k)}:=0$,
$x_{i,1}^{(k-1)}:=x_{i1}^{(k)}-\theta_i^{(k)}+\theta_{i+1}^{(k)}$, $i=1,\ldots,k-1$.
$y_{k,i}:=z_{ki}-\theta_i^{(k)}+\theta_{i+1}^{(k)}$, $i=1,\ldots,k$, and $y_{11}:=a_1-\sum_{j=2}^{r}\theta_2^{(j)}$.
The theorem is proved. ■

Note that $\theta_r^{(r)}=x_{r1}^{(r)}$, $y_{r,r}=z_{r,r}-\theta_r^{(r)}=a_r$ and $\theta_1^{(k)}=0$, $y_{k1}=\theta_2^{(k)}$, for $k=2,\ldots,r$. Futhermore, by previous lemma, $z_{k,k}=a_k^{(k)}+x_{k1}^{(k)}$, for $k=2,\ldots,r$. So, $\theta_k^{(k)}=x_{k1}^{(k)}$ and $y_{kk}=z_{kk}-\theta_k^{(k)}=a_k^{(k)}$, for $k=2,\ldots,r$.

According to previous theorem, the matrix Y, defined in (16), equals to

$$\begin{bmatrix} a_1-\sum_{j=2}^{r}\theta_2^{(j)} & 0 & 0 & 0 & 0 & 0\\ \theta_2^{(2)} & z_{22}-\theta_2^{(2)} & 0 & 0 & 0 & 0\\ \theta_2^{(3)} & z_{32}-\theta_2^{(3)}+\theta_3^{(3)} & z_{33}-\theta_3^{(3)} & 0 & 0 & 0\\ \vdots & \vdots & \cdots & \cdots & 0 & 0\\ \theta_2^{(r-1)} & z_{r-1,2}-\theta_2^{(r-1)}+\theta_3^{(r-1)} & z_{r-1,3}-\theta_3^{(r-1)}+\theta_4^{(r-1)} & \cdots & z_{r-1,r-1}-\theta_{r-1}^{(r-1)} & 0\\ \theta_2^{(r)} & z_{r2}-\theta_2^{(r)}+\theta_3^{(r)} & z_{r3}-\theta_3^{(r)}+\theta_4^{(r)} & \cdots & z_{r,r-1}-\theta_{r-1}^{(r)}+\theta_r^{(r)} & a_r \end{bmatrix}.$$

In what follows we need

Remark 5 *(a)* $\theta_i^{(u)}=z_{ui}\Rightarrow y_{ui}=\theta_{i+1}^{(u)}$.
In particular, $z_{ui}=0\Rightarrow\theta_i^{(u)}=0\Rightarrow y_{ui}=\theta_{i+1}^{(u)}$, *and*
$z_{u-1,i}=z_{ui}=0\Rightarrow y_{u,i-1}=0$.
(b) $\theta_i^{(u)}=x_{i1}^{(u)}\Rightarrow x_{i1}^{(u-1)}=\theta_{i+1}^{(u)}$.
In particular, $x_{i1}^{(u)}=0\Rightarrow x_{i1}^{(u-1)}=\theta_{i+1}^{(u)}$.

(c) $\theta_{i+1}^{(u)} \neq 0 \Rightarrow x_{i1}^{(u-1)} > 0$ *and* $y_{ui} > 0$.

Let $r \geq 1$ and $\mathcal{T} \in \overline{LRD}_r$ of type $[a, b, c]$, with deletion-insertion matrix $Y \in \mathbb{Z}^{r,r}$, and b-decomposition defined by $b^{(s)}$, $s = 0, \ldots, r$. Let $\mathcal{T}^{(r-1)} \in \overline{LRD}_{r-1}$ of type $[a^{(r-1)}; (b_1, \ldots, b_{r-1}); (c_{r-s+1}, \ldots, c_r)]$, the LR tableau obtained from $\mathcal{T}$ by r-full deletion, and Y' the principal submatrix in the first $r-1$ rows of Y, the deletion matrix of $\mathcal{T}^{(r-1)}$. Since Y records the a-decomposition of $\mathcal{T}$. This means that if we delete the last row of $\phi(\mathcal{T}) = [b, a, c, Y]$, we obtain $\phi(\mathcal{T}^{(r-1)}) = [(b_1, \ldots, b_{r-1}); a^{(r-1)}; (c_{r-s+1}, \ldots, c_r); Y']$.

Now, we address the symmetric question: Let $\widehat{\mathcal{T}}$ be the LR tableau obtained from $\mathcal{T}$ by suppressing the r-th row. Let $\hat{Y}$ be the deletion-insertion matrix of $\widehat{\mathcal{T}}$. May we conclude that $[\phi(\mathcal{T})]^{(r)} = \phi(\widehat{\mathcal{T}})$ and $(b_1 - b_1^{(r-1)}, \ldots, b_{r-1} - b_{r-1}^{(r-1)}, b_{r+1})$ is the $(r+1)$-full deletion sequence of $\phi(\mathcal{T})$? This would mean that Y is obtained from $\widehat{Y}$ by inserting $(b_1 - b_1^{(r-1)}, \ldots, b_{r-1} - b_{r-1}^{(r-1)}, b_{r+1})$, and that the b-decomposition of $\phi(\mathcal{T})$ of type $[b, a, c]$, defines the b-decomposition of $\mathcal{T}$. In other words, if $\mathcal{T} = [a, b, c, X]$, the deletion-insertion matrix of $\phi(\mathcal{T}) = [b, a, c, Y]$ is X. We shall prove that this is the case.

Lemma 8 *Let* $r \geq 0$, $k \in \{0, 1, \ldots, r\}$ *and* $e_k = (0^{k-1}, 1, 0^{r-k})$. *Let* $\mathcal{H} = [(a_1, \ldots, a_r, a_{r+1});$ $(b_1, \ldots, b_k + 1, \ldots, b_r, b_{r+1});$ $(c_1, c_2, \ldots, c_{r+1});$ $H] \in \overline{LRD}_{r+1}$ *where* $H = \begin{bmatrix} X & 0 \\ e_k & b_{r+1} \end{bmatrix}$, *with deletion-insertion matrix* Y. *Let* $\mathcal{T} \in \overline{LRD}_r$ *obtained from* $\mathcal{H}$ *by suppressing the* $(r+1)$*-th row, with deletion-insertion matrix* $\widehat{Y}$. *Then,* Y *is obtained from* $\widehat{Y}$ *by inserting* $(0^{k-1}, 1, 0^{r-k}, a_{r+1})$. *(We let* $e_0 := (0^r)$.*)*

Proof: The proof will be handled by induction on k. Let $k = 0$, then $Y = \begin{bmatrix} \widehat{Y} & 0 \\ 0 & a_{r+1} \end{bmatrix}$ which is obtained from $\widehat{Y}$ by inserting $(0^r, a_{r+1})$.

Let $k \geq 1$ and suppose the result is true for $k-1$. Let $\mathcal{F} = [(a_i + x_{i1})_{i=2}^r; (b_i)_{i=2}^r; (c_i)_{i=2}^r; X'] \in \overline{LRD}_{r-1}$ where X' is obtained from X by deleting the first row and column of X, and $\mathcal{G} = [(a_2 + x_{21}, \ldots, a_r + x_{r1}, a_{r+1} + 1); (b_2, \ldots, b_k + 1, \ldots, b_r, b_{r+1}); (c_1, \ldots, c_r); H'] \in \overline{LRD}_r$, where $H' = \begin{bmatrix} X' & 0 \\ e_{k-1} & b_{r+1} \end{bmatrix}$. Let $\widehat{Z} = [\hat{z}_{ij}] \in \mathbb{Z}^{r+1,r+1}$ with $\hat{z}_{i1} = \hat{z}_{1i} = \hat{z}_{r+1,i} = 0$, for $i = 1, \ldots, r+1$, where $[\hat{z}_{ij}]_{i,j=2}^r$ is the deletion-insertion matrix of $\mathcal{F}$. Let $Z = [z_{ij}] \in \mathbb{Z}^{r+1,r+1}$ with $z_{i1} = z_{1i} = 0$, for $i = 1, \ldots, r+1$, where $[z_{ij}]_{i,j=2}^{r+1}$ is the deletion-insertion matrix of $\mathcal{G}$. By induction hypothesis, $[z_{ij}]_{i,j=2}^{r+1}$ is obtained from $[\hat{z}_{ij}]_{i,j=2}^r$ by inserting $(0^{k-2}, 1, 0^{r-k+1}, a_{r+1})$. Equivalently, Z is obtained from $[\hat{z}_{ij}]_{i,j=1}^r$ by inserting $(0^{k-2}, 1, 0^{r-k+1}, a_{r+1})$.

Without loss of generality, it is enough to consider $\widehat{Z}$: $\hat{z}_{k+j,j+2} > 0$, for $j = 0, 1, \ldots, r-k$.

Therefore, Z is such that $z_{k+j,j+2} = \hat{z}_{k+j,j+2} - 1$ and $z_{k+j+1,j+2} = \hat{z}_{k+j+1,j+2} + 1$, for $j = 0, 1, \ldots, r-k$, $z_{r+1,r+1} = a_{r+1}$ and $z_{ij} = \hat{z}_{ij}$, otherwise.

Let $\hat{Z}$ as defined above. By previous theorem, given $[\hat{z}_{ij}]_{i,j=2}^r$ the deletion-insertion matrix of $\mathcal{F}$, the deletion-insertion matrix $\hat{Y}$ of $\mathcal{T}$ is determined recursively as follows

$x_{i1}^{(r+1)} := x_{i1}$, $i = 1, \ldots, r$,

$x_{r+1,1}^{(r+1)} := 0$.

For $u = r+1, r, \ldots, 2$,

$\theta_i^{(u)} := \min\{x_{i1}^{(u)}, \hat{z}_{ui}\}$, $i = 1, \ldots, u$, and $\theta_{u+1}^{(u)} := 0$,
$x_{i,1}^{(u-1)} := x_{i1}^{(u)} - \theta_i^{(u)} + \theta_{i+1}^{(u)}$, $i = 1, \ldots, u-1$,
$\hat{y}_{u,i} = \hat{z}_{ui} - \theta_i^{(u)} + \theta_{i+1}^{(u)}$, $i = 1, \ldots, u$, and $\hat{y}_{11} = a_1 - \sum_{j=2}^{r} \theta_2^{(j)}$.
For $i = 1, \ldots, r+1$, we have $\theta_i^{(r+1)} = 0$, $x_{i1}^{(r)} = x_{i1}^{(r+1)}$ and $\hat{y}_{r+1,i} = 0$.

Again, by previous theorem, given Z the deletion-insertion matrix of $\mathcal{G}$, the deletion-insertion matrix Y of $\mathcal{H}$, is determined by setting
$h_{i1}^{(r+1)} := x_{i1}$, $i = 1, \ldots, r$,
$h_{r+1,1}^{(r+1)} := 0$.
For $u = r+1, \ldots, 2$,
$\lambda_i^{(u)} := \min\{h_{i1}^{(u)}, z_{u,i}\}$, $i = 1, \ldots, u$, and $\lambda_{u+1}^{(u)} = 0$,
$h_{i1}^{(u-1)} := h_{i1}^{(u)} - \lambda_i^{(u)} + \lambda_{i+1}^{(u)}$, $i = 1, \ldots, u-1$,
$y_{ui} = z_{ui} - \lambda_i^{(u)} + \lambda_{i+1}^{(u)}$, $i = 1, \ldots, u$, and $y_{11} := a_1 - \sum_{j=2}^{r+1} \lambda_2^{(j)}$.

We want to relate the deletion-insertion matrix $Y \in \mathbb{Z}^{r+1,r+1}$ of $\mathcal{H}$ with the deletion-insertion matrix $\hat{Y} \in \mathbb{Z}^{r,r}$ of $\mathcal{T}$. We shall relate $\hat{Y} \oplus [0]$ with Y by comparing each row.

For $u = r+1$, $z_{r+1,r-k+2} = \hat{z}_{r+1,r-k+2} + 1$, $z_{r+1,i} = \hat{z}_{r+1,i}$, for $i \neq r-k+2, r+1$.
We have $\lambda_i^{(r+1)} = \theta_i^{(r+1)}$, for $i \neq r-k+2$, and

$$\lambda_{r-k+2}^{(r+1)} = \min\{x_{r-k+2,1}^{(r+1)}, \hat{z}_{r+1,r-k+2} + 1\}.$$

Here, we have to distinguish two cases:

$$(I_{r+1}) \quad \lambda_{r-k+2}^{(r+1)} = \theta_{r-k+2}^{(r+1)} + 1$$

and

$$(II_{r+1}) \quad \lambda_{r-k+2}^{(r+1)} = \theta_{r-k+2}^{(r+1)} = x_{r-k+2,1}.$$

Case $(\mathbf{I_{r+1}})$: $\lambda_{r-k+2}^{(r+1)} = \theta_{r-k+2}^{(r+1)} + 1$.
Clearly, $\theta_{r-k+2}^{(r+1)} = \hat{z}_{r+1,r-k+2}$ and $x_{r-k+2,1}^{(r+1)} > \theta_{r-k+2}^{(r+1)}$. This implies $x_{r-k+2,1}^{(r)} > 0$.
Attending to remark 5, it follows

$$\theta_{r-k+2}^{(r+1)} = \hat{z}_{r+1,r-k+2} \Rightarrow \hat{y}_{r+1,r-k+2} = \theta_{r-k+3}^{(r+1)} = 0.$$

Now notice that we have the following implications
$x_{r-k+2,1}^{(r)} > 0$ and $\hat{z}_{r,r-k+2} > 0 \Rightarrow \theta_{r-k+2}^{(r)} > 0$,
$\theta_{r-k+2}^{(r)} > 0 \Rightarrow x_{r-k+1,1}^{(r-1)} > 0$ and $\hat{y}_{r,r-k+1} > 0$,
$x_{r-k+1,1}^{(r-1)} > 0$ and $\hat{z}_{r-1,r-k+1} > 0 \Rightarrow \theta_{r-k+1}^{(r-1)} > 0$,
...
$x_{3,1}^{(k+1)} > 0$ and $\hat{z}_{k+1,3} > 0 \Rightarrow \theta_3^{(k+1)} > 0$,
$\theta_3^{(k+1)} > 0 \Rightarrow x_{2,1}^{(k)} > 0$ and $\hat{y}_{k+1,2} > 0$,
$x_{2,1}^{(k)} > 0$ and $\hat{z}_{k+1,3} > 0 \Rightarrow \theta_2^{(k)} > 0$.
This means that

$$\hat{y}_{k,1}, \hat{y}_{k+1,2}, \ldots, \hat{y}_{r-1,r-k}, \hat{y}_{r,r-k+1} > 0. \tag{20}$$

Then,
$h_{i1}^{(r)} = x_{i1},\ i \neq r-k+1, r-k+2,$
$h_{r-k+1,1}^{(r)} = x_{r-k+1,1} + 1,$
$h_{r-k+2,1}^{(r)} = x_{r-k+2,1} - 1.$
The $(r+1)$-th row of Y is, therefore,

$$y_{r+1,i} = \hat{y}_{r+1,i} = 0, \text{ for } i \neq r-k+1, r-k+2, r+1,$$

$$y_{r+1,r-k+1} = 1, y_{r+1,r-k+2} = 0 \text{ and } y_{r+1,r+1} = a_{r+1}.$$

Proceed to the r-th row of Y.

Since the r-th row of Z is defined by $z_{ri} = \hat{z}_{ri}$, for $i \neq r-k+1, r-k+2$, $z_{r,r-k+1} = \hat{z}_{r,r-k+1} + 1$ and $z_{r,r-k+2} = \hat{z}_{r,r-k+2} - 1$, we have
$\lambda_i^{(r)} = \theta_i^{(r)}$, for $i \neq r-k+1, r-k+2,$
$\lambda_{r-k+1}^{(r)} = \theta_{r-k+1}^{(r)} + 1,$
$\lambda_{(r-k+2)}^{(r)} = \theta_{r-k+2}^{(r)} - 1.$
Hence,
$h_{i1}^{(r-1)} = x_{i1}^{(r-1)}$, for $i \neq r-k, r-k+1,$
$h_{r-k,1}^{(r-1)} = x_{r-k,1}^{(r-1)} + 1$, and
$h_{r-k,1}^{(r-1)} = x_{r-k+1,1}^{(r-1)} - 1.$
Finally, we get
$y_{r,i} = \hat{y}_{r,i}$, for $i \neq r-k, r-k+1,$
$y_{r,r-k} = \hat{y}_{r,r-k} + 1$, and $y_{r,r-k+1} = \hat{y}_{r,r-k+1} - 1$.

Proceeding to the $(r-1)$-th row of Y, we verify that we are reduced to the previous situation. So, by an inductive argument, we conclude that Y is defined by

$$y_{k,1} = \hat{y}_{k,1} - 1,$$

$$y_{k+1,1} = \hat{y}_{k+1,1} + 1,\ y_{k+1,2} = \hat{y}_{k+1,2} - 1,$$

$$\dots$$

$$y_{r-1,r-k-1} = \hat{y}_{r-1,r-k-1} + 1,\ y_{r-1,r-k} = \hat{y}_{r-1,r-k} - 1,$$

$$y_{r,r-k} = \hat{y}_{r,r-k} + 1,\ y_{r,r-k+1} = \hat{y}_{r,r-k+1} - 1,$$

and

$$y_{r+1,r-k+1} = 1,\ y_{r+1,r+1} = a_{r+1},$$

and

$$y_{i,j} = \hat{y}_{i,j},\ \text{otherwise}.$$

Considering (20), Y is obtained from $\hat{Y}$ by inserting $(0^{k-1}, 1, 0^{r-k}, a_{r+1})$.

Case($\mathbf{II_{r+1}}$): $\lambda_{r-k+2}^{(r+1)} = \theta_{r-k+2}^{(r+1)} = x_{r-k+2,1}^{(r+1)}$.

First notice that $\theta_{r-k+2}^{(r+1)} = x_{r-k+2,1}^{(r+1)}$ and

$$\theta_{r-k+2}^{(r+1)} = x_{r-k+2,1}^{(r+1)} \Rightarrow x_{r-k+2,1}^{(r)} = \theta_{r-k+3,1}^{(r+1)} = 0,$$

$$x^{(r)}_{r-k+2,1} = 0 \Rightarrow \theta^{(r)}_{r-k+2} = 0, \tag{21}$$

$$\theta^{(r)}_{r-k+2} = 0 \text{ and } \hat{z}_{r,r-k+2} > 0 \Rightarrow \hat{y}_{r,r-k+2} > 0.$$

In this case the $(r+1)$-th row of Y is defined by

$$y_{r+1,r-k+2} = 1,\ y_{r+1,r+1} = a_{r+1},\ \text{and } y_{r+1,i} = \hat{y}_{r+1,i},\ \text{otherwise.}$$

Proceeding to the r-th row of Y, we start by checking

$$\lambda^{(r)}_i = \theta^{(r)}_i,\ \text{for } i \neq r-k+1, r-k+2.$$

Considering $x^{(r)}_{r-k+2,1} = 0 = \theta^{(r)}_{r-k+2}$, we have

$$\lambda^{(r-k+2)}_r = \theta^{(r)}_{r-k+2} = 0.$$

Now,

$$\lambda^{(r-k+1)}_r = \min\{x^{(r)}_{r-k+1,1}, \hat{z}_{r,r-k+1} + 1\}.$$

So, here, we have again to distinguish two cases

$$(I_r)\ \lambda^{(r)}_{r-k+1} = \theta^{(r)}_{r-k+1} + 1,$$

and

$$(II_r)\ \lambda^{(r)}_{r-k+1} = \theta^{(r)}_{r-k+1} = x^{(r)}_{r-k+1,1}.$$

Subcase ($\mathbf{I_r}$): $\lambda^{(r)}_{r-k+1} = \theta^{(r)}_{r-k+1} + 1$.
Clearly, $\theta^{(r)}_{r-k+1} = \hat{z}_{r+1,r-k+1}$ and $x^{(r)}_{r-k+1,1} > \theta^{(r)}_{r-k+1}$. This implies $x^{(r-1)}_{r-k+1,1} > 0$.
Considering remark 5 and (21), $x^{(r)}_{r-k+2,1} = \theta^{(r)}_{r-k+2} = 0$, it follows

$$\theta^{(r)}_{r-k+1} = \hat{z}_{r,r-k+1} \Rightarrow \hat{y}_{r,r-k+1} = \theta^{(r)}_{r-k+2} = 0.$$

Now, notice that we have the following implications
$x^{(r-1)}_{r-k+1,1} > 0$ and $\hat{z}_{r-1,r-k+1} > 0 \Rightarrow \theta^{(r-1)}_{r-k+1} > 0$,
$\theta^{(r-1)}_{r-k+1,1} > 0 \Rightarrow x^{(r-2)}_{r-k+1} > 0$ and $\hat{y}_{r-1,r-k} > 0$,
...
$x^{(k+1)}_{3,1} > 0$ and $\hat{z}_{k+1,3} > 0 \Rightarrow \theta^{(k+1)}_3 > 0$,
$\theta^{(k+1)}_3 > 0 \Rightarrow x^{(k)}_{2,1} > 0$ and $\hat{y}_{k+1,2} > 0$,
$x^{(k)}_{2,1} > 0$ and $\hat{z}_{k+1,3} > 0 \Rightarrow \theta^{(k)}_2 > 0$.
This means that

$$\hat{y}_{k,1}, \hat{y}_{k+1,2}, \ldots, \hat{y}_{r-1,r-k} > 0. \tag{22}$$

We have
$h^{(r-1)}_{i1} = x^{(r-1)}_{i1}$, for $i \neq r-k, r-k+1$,
$h^{(r-1)}_{r-k,1} = x^{(r-1)}_{r-k,1} + 1$ and
$h^{(r-1)}_{r-k,1} = x^{(r-1)}_{r-k+1,1} - 1$.

Finally, we get the r-th row of Y

$$y_{r,i} = \hat{y}_{r,i}, \text{ for } i \neq r-k, r-k+1, r-k+2,$$

$$y_{r,r-k} = \hat{y}_{r,r-k} + 1, \; y_{r,r-k+1} = \hat{y}_{r,r-k+1} = 0, \text{ and } y_{r,r-k+2} = \hat{y}_{r,r-k+2} - 1.$$

Proceeding to the $(r-1)$-th row of Y, we verify that this situation has been already studied in case (I_{r+1}). So, Y is defined by

$$y_{k,1} = \hat{y}_{k,1} - 1,$$

$$y_{k+1,1} = \hat{y}_{k+1,1} + 1, \; y_{k+1,2} = \hat{y}_{k+1,2} - 1,$$

$$\dots$$

$$y_{r-1,r-k-1} = \hat{y}_{r-1,r-k-1} + 1, \; y_{r-1,r-k} = \hat{y}_{r-1,r-k} - 1,$$

$$y_{r,r-k} = \hat{y}_{r,r-k} + 1, \; y_{r,r-k+1} = \hat{y}_{r,r-k+1} - 1,$$

and

$$y_{r,r-k+1} = 0, y_{r,r-k+2} = \hat{y}_{r,r-k+2} - 1,$$

$$y_{r+1,r-k+2} = 1, \text{ and } \hat{y}_{r+1,r+1} = a_{r+1},$$

and $y_{i,j} = \hat{y}_{i,j}$, otherwise.

Considering (22), Y is obtained from $\hat{Y}$ by inserting $(0^{k-1}, 1, 0^{r-k}, a_{r+1})$.

It remains to consider

Subcase($\mathbf{II_r}$): $\lambda^{(r)}_{r-k+1} = \theta^{(r)}_{r-k+1} = x^{(r)}_{r-k+1,1}$.

First notice that $\theta^{(r)}_{r-k+1} = x^{(r+1)}_{r-k+1,1}$ and

$$\theta^{(r)}_{r-k+1} = x^{(r+1)}_{r-k+1,1} \Rightarrow x^{(r-1)}_{r-k+1,1} = \theta^{(r)}_{r-k+2,1} = 0,$$

$$x^{(r-1)}_{r-k+1,1} = 0 \Rightarrow \theta^{(r-1)}_{r-k+1} = 0, \tag{23}$$

$$\theta^{(r-1)}_{r-k+1} = 0 \text{ and } \hat{z}_{r-1,r-k+1} > 0 \Rightarrow \hat{y}_{r-1,r-k+1} > 0.$$

In this case, the r-th row of Y is defined by

$$y_{r,r-k+1} = \hat{y}_{r,r-k+1} + 1, \; y_{r,r-k+2} = \hat{y}_{r,r-k+2} - 1,$$

and $y_{r,i} = \hat{y}_{r,i}$, otherwise.

Proceeding to the $(r-1)$-th row of Y, we start by checking that

$$\lambda^{(r-1)}_i = \theta^{(r-1)}_i, \text{ for } i \neq r-k, r-k+1.$$

Considering $x^{(r-1)}_{r-k+1,1} = 0 = \theta^{(r-1)}_{r-k+1}$, we have

$$\lambda^{(r-1)}_{(r-k+1)} = \theta^{(r-1)}_{r-k+1} = 0.$$

Now,

$$\lambda^{(r-k)}_{(r-1)} = \min\{x^{(r-1)}_{r-k,1}, \hat{z}_{r-1,r-k} + 1\}.$$

We verify that this situation has been already studied, and, therefore, we have two possibilities for Y: either it is defined by

$$y_{k,1} = \hat{y}_{k,1} - 1,$$

$$y_{k+1,1} = \hat{y}_{k+1,1} + 1,\ y_{k+1,2} = \hat{y}_{k+1,2} - 1,$$

$$\ldots$$

$$y_{k+u,u+1} = \hat{y}_{k+u,u+1} + 1,\ y_{k+u,u+2} = 0,\ y_{k+u,u+3} = \hat{y}_{k+u,u+3} - 1, \text{for some } 1 \leq u < r-k-1,$$

$$y_{k+u+1,u+3} = \hat{y}_{k+u+1,u+3} + 1,\ y_{k+u+1,u+3} = \hat{y}_{k+u+1,u+3} - 1$$

and

$$y_{r,r-k+1} = \hat{y}_{r,r-k+1} + 1, y_{r,r-k+2} = \hat{y}_{r,r-k+2} - 1, y_{r+1,r-k+2} = 1, \text{ and } y_{r+1,r+1} = a_{r+1},$$

and $y_{i,j} = \hat{y}_{i,j}$, otherwise; or by

$$y_{k+j,j+2} = \hat{y}_{k+j,j+2} - 1, y_{k+j+1,j+2} = \hat{y}_{k+j+1,j+2} + 1, \text{ for } j = 0, 1, \ldots, r-k.$$

$$y_{r+1,r-k+2} = 1, \text{ and } y_{ij} = \hat{y}_{ij}, \text{ otherwise.}$$

In both cases, we have Y obtained from $\widehat{Y}$ by inserting $(0^{k-1}, 1, 0^{r-k}, a_{r+1})$. ■

Proposition 5 *Let $r \geq 0$ and $k \in \{0, 1, \ldots, r\}$. Let $\mathcal{H} = [(a_1, \ldots, a_r, a_{r+1}); (b_1, \ldots, b_k + \alpha, \ldots, b_r, b_{r+1}); (c_1, c_2, \ldots, c_{r+1}); H] \in \overline{LRD}_{r+1}$, where $\alpha > 0$, $H = \begin{bmatrix} X & 0 \\ \alpha e_k & b_{r+1} \end{bmatrix}$, with deletion-insertion matrix Y. Let $\mathcal{T} \in \overline{LRD}_r$ obtained from $\mathcal{H}$ by suppressing the $(r+1)$-th row, with deletion-insertion matrix $\widehat{Y}$. Then, Y is obtained from $\widehat{Y}$ by inserting $(0^{k-1}, \alpha,\ 0^{r-k}, a_{r+1})$.*

Proof: When $k = 0$, this situation has been already studied in the previous lemma. Let $k \geq 1$. The proof will be handled by induction on α.

By previous lemma, the result is true for $\alpha = 1$.

Let $\alpha > 1$ and suppose the result is true for $\alpha - 1$. For $i = 1, \ldots, \alpha - 1$, let $\mathcal{H}_i = [(a_1, \ldots, a_r, 0); (b_1, \ldots, b_k + i, \ldots, b_r, b_{r+1});\ (c_1 + \alpha - i, c_2, \ldots, c_{r+1}); H_i]$, where $H_i = \begin{bmatrix} X & 0 \\ ie_k & b_{r+1} \end{bmatrix}$. Let Y_i be the deletion-insertion matrix of $\mathcal{H}_i$, for $i = 1, \ldots, \alpha - 1$.

Let Y_1' be the principal submatrix in the first r rows of Y_1. We shall show that $Y \in \mathbb{Z}^{r+1,r+1}$, is obtained as follows: (i) by inserting $(0^{k-1}, 1, 0^{r-k+1})$ to $\widehat{Y}$, we get $Y_1 \in \mathbb{Z}^{r+1,r+1}$; then (ii) by inserting $(0^{k-1}, \alpha-1, 0^{r-k}, a_{r+1})$ in Y_1', we get an $(r+1)\times(r+1)$ matrix, let us say, W; and eventually (iii) by adding the $(r+1)$-th row of Y_1 to the $(r+1)$-th row of W, we get Y. This means that Y is obtained from $\widehat{Y}$ by inserting $(0^{k-1}, \alpha, 0^{r-k}, a_{r+1})$.

By previous lemma, the deletion-insertion matrix Y_1 of $\mathcal{H}_1$, is obtained from $\widehat{Y}$ by inserting $(0^{k-1}, 1,\ 0^{r-k+1}, 0)$. Let $(\mathcal{H}_1)^{(r)}$ be the LR rectangular tableau obtained from $\mathcal{H}_1$ by $(r+1)$-full deletion. The deletion-insertion matrix of $(\mathcal{H}_1)^{(r)}$ is Y_1'.

Let us denote by $k_1, \ldots, k_{\alpha-1}, k_\alpha$, from left to right, the α symbols k in the $(r+1)$-th row of $\mathcal{H}$. Let $\mathcal{K}_1, \ldots, \mathcal{K}_{\alpha-1}, \mathcal{K}_\alpha$ be the deletion paths generated by $k_1, \ldots, k_{\alpha-1},\ k_\alpha$, respectively.

Now, let $\mathcal{H}^{\mathcal{K}_\alpha}$ be the LR rectangular tableau obtained from $\mathcal{H}$ by deletion generated by the symbol k_α, and $\widehat{\mathcal{H}^{\mathcal{K}_\alpha}}$ the one obtained from $\mathcal{H}^{\mathcal{K}_\alpha}$ by suppression of the $(r+1)$-th row. Therefore, $\widehat{\mathcal{H}^{\mathcal{K}_\alpha}} = (\mathcal{H}_1)^{(r)}$ and the deletion-insertion matrix of $\widehat{\mathcal{H}^{\mathcal{K}_\alpha}}$ is Y_1'.

Let W be the deletion-insertion matrix of $\mathcal{H}^{\mathcal{K}_\alpha}$. By inductive hypothesis, W is obtained from Y_1' by inserting $(0^{k-1}, \alpha-1,\ 0^{r-k}, a_{r+1})$.

We claim that $Y = W + \begin{bmatrix} 0_{r,r} & 0 \\ w & 0 \end{bmatrix}$, *where* $[w, 0]$, $1 \times (r+1)$, *is the* $(r+1)$*-th row of* Y_1.

Let W' and Y' be the principal submatrices in the first r rows of W and Y, respectively.

Let $\mathcal{H}^{(r)}$ and $(\mathcal{H}^{\mathcal{K}_\alpha})^{(r)}$ be the LR rectangular tableaux obtained, respectively, from $\mathcal{H}$ and $\mathcal{H}^{\mathcal{K}_\alpha}$ by $(r+1)$-full deletion. Then, $\mathcal{H}^{(r)} = (\mathcal{H}^{\mathcal{K}_\alpha})^{(r)}$ and the deletion-insertion matrix of $\mathcal{H}^{(r)}$ is $Y' = W'$.

Finally, note that the $(r+1)$-deletion sequence of $\mathcal{H}$ is the $(r+1)$-th row of W plus the $(r+1)$-th row of Y_1, wich, by definition of deletion matrix, is precisely the $(r+1)$-th row of Y. Hence, the claim is true. This means that Y, the deletion-insertion matrix of $\mathcal{H}$, is obtained from $\widehat{Y}$ by inserting $(0^{k-1}, \alpha,\ 0^{r-k}, a_{r+1})$.■

Proposition 6 *Let* $r \geq 0$ *and* $\mathcal{H} = [(a_1, \ldots, a_r, a_{r+1});\ (b_1 + \alpha_1, \ldots, b_r + \alpha_r, b_{r+1});\ (c_1, c_2, \ldots, c_{r+1}); H] \in \overline{LRD}_{r+1}$, *where* $H = \begin{bmatrix} & X & & 0 \\ \alpha_1 & \ldots & \alpha_r & b_{r+1} \end{bmatrix}$, $\alpha_i \geq 0$, $i = 1, \ldots, r$. *Let* Y *be the deletion-insertion matrix of* $\mathcal{H}$. *Let* $\mathcal{T} \in \overline{LRD}_r$ *obtained from* $\mathcal{H}$ *by deleting the* $(r+1)$*-th row, with deletion-insertion matrix* $\widehat{Y}$. *Then,* Y *is obtained from* $\widehat{Y}$ *by inserting* $(\alpha_1, \ldots, \alpha_r,\ a_{r+1})$.

Proof: Let $m := \#\{i \in \{1, \ldots, r\} : \alpha_i > 0\}$. The proof will be handled by induction on m. By previous proposition the result is true for $m = 0, 1$.

Let $m > 1$, and suppose the result is true for $m-1$. Without loss of generality, we may assume that $\alpha_1, \ldots, \alpha_m > 0$. Let $\mathcal{H}_{\alpha_m} = [(a_1, \ldots, a_r, 0); (b_1, \ldots, b_m + \alpha_m, \ldots, b_r, b_{r+1});\ (c_1 + \alpha_1 + \ldots + \alpha_{m-1}, c_2, \ldots, c_{r+1}); H_m] \in \overline{LRD}_{r+1}$ with deletion-insertion matrix Y_{α_m}. Let $(\mathcal{H}_{\alpha_m})^{(r)}$ be the LR rectangular tableau obtained from $\mathcal{H}_{\alpha_m}$ by $(r+1)$-full deletion. Then, the deletion-insertion matrix of $(\mathcal{H}_{\alpha_m})^{(r)}$ is the principal submatrix in the first r rows of Y_{α_m}, denoted by $(Y_{\alpha_m})'$.

Let $\mathbf{Z}$ be the deletion path chain generated by the α_m symbols m. Now, let $\mathcal{H}^{\mathbf{Z}}$ be the LR rectangular tableau obtained by deletion generated by the α_m symbols m in the $(r+1)$-th row of $\mathcal{H}$. Let W be the deletion-insertion matrix of $\mathcal{H}^{\mathbf{Z}}$. Since, $\widehat{\mathcal{H}^{\mathbf{Z}}} = (\mathcal{H}_{\mathbf{Z}})^{(r)}$ it follows by induction hypothesis that W is obtained from $(Y_{\alpha_m})'$ by inserting $(\alpha_1, \ldots, \alpha_{m-1}, 0^{r-m}, a_{r+1})$.

Let $(\mathcal{H}^{\mathbf{Z}})^{(r)}$ and $(\mathcal{H})^{(r)}$ be the LR rectangular tableaux obtained, respectively, from $\mathcal{H}^{\mathbf{Z}}$ and $\mathcal{H}$ by $(r+1)$-full deletion. Let W' and Y' be the principal submatrices in the first r rows of W and Y, respectively.

Then, $(\mathcal{H}^{\mathbf{Z}})^{(r)} = (\mathcal{H})^{(r)}$ and the deletion-insertion matrix of $(\mathcal{H})^{(r)}$ is $W' = Y'$. But the $(r+1)$-deletion sequence of $\mathcal{H}$ is precisely the $(r+1)$-th row of W plus the $(r+1)$-th row of Y_{α_m}. ■

Proof of theorem 11.

We claim that if $[a, b, c, X] \in \overline{LRD}_r$ has deletion- insertion matrix Y, then the deletion-insertion matrix of $[b, a, c, Y] \in \overline{LRD}_r$ is X.

We prove the claim by induction on r. For $r = 1$, we have $[(a_1), (b_1), (c_1), X = [b_1]] \in \overline{LRD}_1$ with deletion-insertion matrix $Y = [a_1]$. So, $[(b_1), (a_1), (c_1), Y = [a_1]] \in \overline{LRD}_1$ and the deletion-insertion matrix is clearly $X = [b_1]$.

Let $r > 1$ and suppose the claim is true for $r-1$. Let $\mathcal{T} = [(a_1, \ldots, a_r);\ (b_1, \ldots, b_r);\ (c_1, \ldots, c_r);\ X = [x_{ij}]] \in \overline{LRD}_r$ with deletion-insertion matrix Y. Let $\widehat{\mathcal{T}} = [(a_1, \ldots, a_{r-1});\ (b_1 - \alpha_1, \ldots, b_{r-1} - \alpha_{r-1});\ (c_2, \ldots, c_r);\ X'] \in \overline{LRD}_{r-1}$ obtained from $\mathcal{T}$ by suppressing the last row. Therefore, $X = \begin{bmatrix} & X' & & 0 \\ \alpha_1 & \ldots & \alpha_{r-1} & b_r \end{bmatrix}$, and, in particular, recall that $a_{r-1} + \sum_{j=1}^{k-1} x_{r-1,j} \geq a_r + \sum_{j=1}^{k} \alpha_j$, for $k = 1, \ldots, r-1$.

Let $\widehat{Y}$ be the deletion-insertion matrix of $\widehat{\mathcal{T}}$. According to previous proposition, Y is obtained from $\widehat{Y}$ by inserting $(\alpha_1, \ldots, \alpha_{r-1}, b_r)$.

Now, let $\mathcal{H} = [(b_1, \ldots, b_r);\ (a_1, \ldots, a_r);\ (c_1, \ldots, c_r);\ Y] \in \overline{LRD}_r$ and $\mathcal{H}' = [(b_1 - \alpha_1, \ldots, b_{r-1} - \alpha_{r-1});\ (a_1, \ldots, a_{r-1})\ (c_2, \ldots, c_r);\ \widehat{Y}] \in \overline{LRD}_{r-1}$. By induction hypothesis, the deletion-insertion matrix of $\mathcal{H}'$ is X' and, in particular, $(x_{r-1,1}, \ldots, x_{r-1,r-2})$ is the $(r-1)$-deletion sequence of $\mathcal{H}'$.

By theorem 6, $(\alpha_1, \ldots, \alpha_{r-1}, b_r)$ is a full insertion sequence $mod(a_r)$ of $\mathcal{H}'$. Therefore, by inserting $mod(a_r)$, $(\alpha_1, \ldots, \alpha_{r-1}, b_r)$, on $\mathcal{H}'$ we obtain an LR rectangular tableau. Considering the relationship between Y and $\widehat{Y}$, this LR tableau is precisely $\mathcal{H}$, and considering the definition of insertion matrix, the deletion-insertion matrix of $\mathcal{H}$ is X. The claim is proved. ■

5.3 An example

Here we give illustrations of the algorithm of insertion defining π and of the algorithm of deletion defining ϕ.

The following example is an illustration of π in $\overline{LRD}_4$. Let

$$\mathcal{T} = [(6,5,2,0); (5,4,1,0); (4,3,2,0); X = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}] \in \overline{LRD}_4,$$

graphically represented by

						1	1
					1		
		1	2	2			
1	2	2	3				

To calculate the image of this LR rectangular tableau under π, the algorithm of insertion runs as follows.

We consider the following rectangular numbered diagram, where the boxes of length x_{ij} can be thought as being x_{ij} unitary boxes labelled with 0,

<table>
<tr><td colspan="2">x_{11}</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>x_{21}</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td colspan="2">x_{32}</td><td>x_{31}</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

Now, the row insertion and column sliding operations are going as follows.

The first row of this numbered diagram defines an LR rectangular tableau of type $[x_{11}, a_1, 0]$.

Insert $x_{21} = 1$ symbol 0 in the first row by sliding down, to the second row, the left most $x_{21} = 1$ symbol 1

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td colspan="2">x_{32}</td><td>x_{31}</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

The two first rows of this numbered diagram define an LR rectangular tableau of type $[(x_{11} + x_{21}, 0); (a_1, a_2); (c_3, 0)]$.

Insert $x_{32} = 2$ symbols 0 in the second row by sliding down, to the third row, the left most $x_{32} = 2$ symbols which are strictly larger than 0 (one symbol 1 and one symbol 2); insert $x_{31} = 1$ symbol 0 in the first row by sliding down, to the second row, the left most $x_{31} = 1$ symbol 1; on each turn, this $x_{31} = 1$ slided symbol 1 is inserted in the second row by sliding down, to the third row, the left most $x_{31} = 1$ symbol which is strictly larger than 1 (one symbol 2),

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>x_{31}</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">x_{32}</td><td>1</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

The first three rows of this numbered diagram defines an LR rectangular tableau of type $[(\sum_{i=1}^{3} x_{i1}, x_{32}, 0); (a_1, a_2, a_3); (c_2, c_3, 0)]$.

Insert $x_{43} = 1$ symbol 0 in the third row by sliding down, to the 4th row, the left most $x_{43} = 1$ symbol which is strictly larger than 0 (one symbol 1); insert $x_{42} = 2$ symbols 0 in the second row by sliding down, to the third row, the left most $x_{42} = 2$ symbols (one symbol 1 and one symbol 2) which are strictly larger than 0; on each turn, these slided $x_{42} = 2$ symbols are inserted in the third row by sliding down, to the 4th row, the left most $x_{42} = 2$ symbols which are strictly larger respectively than 1 and 2 (one symbol 2 and one symbol 3),

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>x_{31}</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">x_{32}</td><td colspan="2">x_{42}</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td>x_{43}</td><td>1</td><td>2</td><td>2</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

Now, insert $x_{41} = 1$ symbols 0 in the first row by sliding down, to the second row, one symbol 1; this symbol 1 will be inserted in the second row by sliding down to the third row, one symbol 2; and this symbol 2 will be inserted in the third row by sliding down to the 4th row one symbol 3; finally, the symbol 3 is inserted in the 4th row,

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \multicolumn{2}{|c|}{x_{11}} & x_{21} & x_{31} & x_{41} & 1 & 1 & 1 \\ \hline \multicolumn{2}{|c|}{x_{32}} & \multicolumn{2}{c|}{x_{42}} & 1 & 2 & & \\ \hline x_{43} & 1 & 2 & 2 & 2 & & & \\ \hline 1 & 2 & 3 & 3 & & & & \\ \hline \end{array}$$

The output is $[b, a, c, Y = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}]$. Now applying π to

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \multicolumn{3}{|c|}{y_{11}} & 1 & 1 & 1 & 1 & 1 \\ \hline y_{22} & y_{21} & 2 & 2 & 2 & 2 & & \\ \hline \multicolumn{3}{|c|}{y_{32}} & y_{31} & 3 & & & \\ \hline \multicolumn{2}{|c|}{y_{43}} & y_{42} & y_{41} & & & & \\ \hline \end{array}$$

we obtain $[a, b, c, X]$.

Now, we illustrate the mapping ϕ. To calculate $\phi(\mathcal{T})$ we have to perform the following projections starting with $\mathcal{T}$.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 \\ \hline & & & & & 1 & & \\ \hline & & 1 & 2 & 2 & & & \\ \hline 1 & 2 & 2 & 3 & & & & \\ \hline \end{array}\ , \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 1 \\ \hline & & & & 1 & 2 & & \\ \hline 1 & 2 & 2 & 2 & 3 & & & \\ \hline \end{array}\ , \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & 1 & 1 & 1 & 1 \\ \hline & 1 & 2 & 2 & 2 & 2 & & \\ \hline \end{array}\ , \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

We obtain the following sequence of deletion sequences $(y_{41} = 1, y_{42} = 1,\ y_{43} = 2, y_{44} = 0)$; $(y_{31} = 1, y_{32} = 3, y_{33} = 0)$; $(y_{22} = 1, y_{21} = 1)$; $(y_{11} = 3)$.

So we have $\pi(\mathcal{T}) = \phi(\mathcal{T})$.

Acknowledgments : The author is grateful to A. Kovačec, Marques de Sá, J. F. Queiró, A. P. Santana and I. Zaballa, for interesting discussions. This research was partially undertaken while the author was on sabbatical leave visiting the University of Basque Country, Spain.

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