On limit stability of special classes of continuous maps

Maria Manuel Clementino and Dirk Hofmann*

Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal
email: mmc@mat.uc.pt, dirk@mat.uc.pt

Abstract

Biquotient, effective descent, triquotient, open and proper maps were described in [4] by their lifting properties of chains of convergent ultrafilters. In this paper we use these characterizations to prove their stability under special limits. By their similar behaviour on lifting chains of convergent ultrafilters, on one hand we obtain several results on limit stability of open and perfect maps, and, on the other hand we give unified proofs of the pullback stability and of the product stability of biquotient, effective descent and triquotient maps.

1 Introduction

Quotient maps in **Top** are neither stable under pullback nor under products, although there are important classes of quotient maps that have these properties. In this paper we show that the class of effective descent maps are among these classes.

It is well-known that open surjections and proper and perfect maps are pullback and product stable. These two classes are included in the class of triquotient maps, introduced by Michael in [11], with the goal of defining a notion that includes both open and proper surjections, and that still behaves nicely with respect to completeness. They are also closed under pullbacks and products, as it was shown recently by Richter [15] and Uspenskij [16]. Moreover, they are in particular biquotient maps (also called limit lifting maps [6], [7]), which are exactly the pullback stable quotient maps (see [10] and [5]).

Triquotient maps have been shown to be relevant in the realm of descent theory (see [13]): every triquotient map $f : X \to Y$ is an effective descent map (that is, the pullback change of base functor $f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$ is monadic), which in turn are descent maps (i.e, f^* is premonadic) and these maps coincide with biquotient maps (see [8] for details).

All these classes can be described by their lifting properties of chains of convergent ultrafilters (see also [4]). In this paper (Sections 2-5) we show that these descriptions can be used to prove several limit stability properties of open and perfect maps in a similar way, due to their

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"dual" characterizations (see Definition 3.1). In Section 6, using the common characterization of biquotient, effective descent and triquotient maps given in [4], we obtain the following:

> **Theorem.** *Biquotient, effective descent and triquotient maps are pullback and product stable.*

This includes results proved separately by Michael [10], by Reiterman, Sobral and Tholen [14], by Uspenskij [16], by Plewe [13] and by Richter [15], as well as the new fact that effective descent maps are product-stable.

Our techniques, introduced in [4], interact very nicely with the formation of limits. In fact, as it is well-known, although there are many equivalent ways of describing the topology of a space, namely by open or closed sets, its closure operator, and – filter or net – convergence, they have a different behaviour while describing initial structures.

This description is very handy if we use ultrafilters: for a given structured source $(f_i : X \to |X_i|)_{i\in I}$, the initial structure on X is defined as

$$\mathfrak{a} \to x :\iff (\forall i \in I)\ f_i(\mathfrak{a}) \to f_i(x),$$

for each ultrafilter $\mathfrak{a}$ and each $x \in X$. A relevant example of ultrafilters suitability is the proof of Tychonoff's Theorem. Indeed, it becomes trivial if one uses the description of the product topology above, together with the fact that a topological space X is compact if and only if each ultrafilter on X converges.

2 Basic results and notations

The main result of [4] is a characterization ot triquotient maps by a lifting property on chains of convergent ultrafilters. In order to define these chains, the category **URS** of ultrarelational spaces together with a well copointed endofunctor (Ult, p) was introduced. In this section we recall these definitions and results.

Definition 2.1. An *ultrarelation* on a set X is a subset $r \subseteq \mathfrak{U}(X) \times X$. An *ultrarelational space* is a set X equipped with an ultrarelation r on X. Given ultrarelational spaces (X, r) and (Y, s), a map $f : X \to Y$ is *continuous* if $(f(\mathfrak{a}), f(x)) \in s$ whenever $(\mathfrak{a}, x) \in r$.

Denoting by **URS** the category of ultrarelational spaces and continuous maps, we may define an endofunctor $\mathrm{Ult} : \mathbf{URS} \to \mathbf{URS}$ as follows:

- $\mathrm{Ult}(X, r) := (r, R_{(X,r)})$ with

  $$R_{(X,r)} := \{(\mathfrak{A}, (\mathfrak{a}, x)) \in \mathfrak{U}(r) \times r \,|\, p_{(X,r)}(\mathfrak{A}) = \mathfrak{a}\},$$

  where $p_{(X,r)}$ is the projection map $p_{(X,r)} : r \to X$, $(\mathfrak{a}, x) \mapsto x$;
- if $f : (X, r) \to (Y, s)$ belongs to **URS**, $\mathrm{Ult}(f) : \mathrm{Ult}(X, r) \to \mathrm{Ult}(Y, s)$ is given by $(\mathfrak{a}, x) \mapsto (f(\mathfrak{a}), f(x))$.

Note that $p_{(X,r)} : \mathrm{Ult}(X, r) \to (X, r)$ is a continuous map; in fact, it is the (X, r)-component of a natural transformation

$$p = (p_{(X,r)})_{(X,r)\in \mathsf{Ob}\,\mathbf{URS}} : \mathrm{Ult} \to \mathrm{Id}_{\mathbf{URS}}.$$

Moreover, $\mathrm{Ult}(p_{(X,r)}) = p_{\mathrm{Ult}(X,r)}$ holds true for each ultrarelational space (X,r), hence (Ult, p) is a *well copointed endofunctor* (see [9]).

We may then define endofunctors Ult^α and natural transformations p^α_β for ordinal numbers α, β with $\beta \leq \alpha$, by:

$\mathrm{Ult}^0 = \mathrm{Id}_{\mathbf{URS}}$, $p^0_0 = 1_{\mathrm{Ult}^0}$;

$\mathrm{Ult}^{\alpha+1} = \mathrm{Ult}(\mathrm{Ult}^\alpha)$, $p^{\alpha+1}_\beta = p^\alpha_\beta \cdot p_{\mathrm{Ult}^\alpha}$ and $p^{\alpha+1}_{\alpha+1} = 1_{\mathrm{Ult}^{\alpha+1}}$, for $\beta \leq \alpha$;

$\mathrm{Ult}^\lambda = \lim\limits_{\beta \leq \alpha < \lambda} p^\alpha_\beta$, $p^\lambda_\beta =$ the limit projection and $p^\lambda_\lambda = 1_{\mathrm{Ult}^\lambda}$, for every limit ordinal λ and every $\beta < \lambda$.

From now on, since we usually work with only one ultrarelation on a set X, for an ultrarelational space we relax our notation and write X instead of (X,r) and $\mathfrak{a} \to x$ instead of $(\mathfrak{a}, x) \in r$. Also, we will denote $\mathrm{Ult}^\alpha(X)$ by X_α and $\mathrm{Ult}^\alpha(f)$ by f_α, for every continuous map $f : X \to Y$ between ultrarelational spaces.

The space X_α may be described by

$$X_\alpha = \{((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in \prod_{\beta\in\alpha} \mathfrak{U}(X_\beta) \times X \mid \mathfrak{a}_0 \to x \text{ and } (\forall \gamma \leq \beta < \alpha)\ (p^\beta_\gamma)_X(\mathfrak{a}_\beta) = \mathfrak{a}_\gamma\},$$

(see [4] for details), and each element of X_α can be seen as a chain of convergent ultrafilters. In fact, $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$ if and only if, for each $\gamma \in \alpha$, the ultrafilter $\mathfrak{a}_\gamma$ converges to $((\mathfrak{a}_\beta)_{\beta\in\gamma}, x)$ in X_γ, hence we write

$$\cdots \to \mathfrak{a}_{\beta+1} \to \mathfrak{a}_\beta \to \cdots \to \mathfrak{a}_1 \to \mathfrak{a}_0 \to x$$

whenever $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$.

Definition 2.2. A continuous map $f : X \to Y$ between ultrarelational spaces X and Y is called *α-surjective* (for an ordinal α or $\alpha = \Omega$ the class of ordinal numbers) if $f_\beta : X_\beta \to Y_\beta$ is surjective for every $\beta \in \alpha$.

$$\begin{tikzcd}
X \arrow[d, "f"'] & \cdots \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_{\beta+1} \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_\beta \arrow[r, dashed] \arrow[d, mapsto] & \cdots \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_0 \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
Y & \cdots \arrow[r] & \mathfrak{b}_{\beta+1} \arrow[r] & \mathfrak{b}_\beta \arrow[r] & \cdots \arrow[r] & \mathfrak{b}_0 \arrow[r] & y
\end{tikzcd}$$

For special instances of α, these maps are well-known:

Theorem 2.3 ([4]). *A continuous map $f : X \to Y$ between topological spaces X and Y is*

1. *biquotient if and only if it is 2-surjective,*
2. *effective descent if and only if it is 3-surjective,*
3. *triquotient if and only if it is Ω-surjective.*

So far we have not considered any axioms on an ultrarelational space. The following conditions are essential to recognize (pseudo)topological spaces inside **URS**.

Definition 2.4. An ultrarelational space (X,r) is called

1. *reflexive* if, for each $x \in X$, $(\dot{x}, x) \in r$, and

2. *transitive* if the map

$$\begin{array}{rcl} \mu_{(X,r)} : R_{(X,r)} & \to & \mathcal{U}(X) \times X \\ (\mathfrak{A}, (\mathfrak{a}, x)) & \mapsto & (\bigcup_{\mathcal{A} \in \mathfrak{A}} \bigcap_{(\mathfrak{a}', x') \in \mathcal{A}} \mathfrak{a}', x) \end{array}$$

factors via the inclusion $r \hookrightarrow \mathcal{U}(X) \times X$.

The category **PsTop** is isomorphic to the full subcategory of **URS** consisting of all reflexive ultrarelational spaces and **Top** – in perfect analogy to the finite case: finite topological spaces are exactly the reflexive and transitive relations – is isomorphic to the full subcategory of **URS** consisting of all reflexive and transitive ultrarelational spaces.

We will introduce two other axioms, which are motivated by the following observation. It is well-known that the set $\mathcal{U}(X)$ of all ultrafilters of a given set X, equipped with the Zariski-closure, is compact and Hausdorff. While dealing with $\mathcal{U}(X)$ we can make then use of results about compact Hausdorff spaces, in particular of the following Theorem (see [2]).

Theorem 2.5. *The codirected limit of non-empty compact Hausdorff spaces is again non-empty.*

Remark 2.6. This enables us, for a given codirected diagram $D : I \to \mathbf{Set}$, to choose ultrafilters $\mathfrak{a}_i \in \mathcal{M}_i \subseteq \mathcal{U}(D(i))$ on $D(i)$ $(i \in I)$ compatible with the connecting maps, provided that the subsets $\mathcal{M}_i$ of $\mathcal{U}(D(i))$ are Zariski-closed and non-empty, and $D(k)(\mathfrak{a}) \in \mathcal{M}_j$ holds for each $\mathfrak{a} \in \mathcal{M}_i$ and each $\varphi : i \to j$ in I.

Very often we will choose $\mathcal{M}_i = \{\mathfrak{a} \in \mathcal{U}(X) \,|\, \mathfrak{a} \to x_i\}$ for a given $x_i \in D(i)$, hence we have to ensure that this set is non-empty and Zariski-closed. This suggests the following definition.

Definition 2.7. An ultrarelational space X is called

1. *weak reflexive* if, for each $x \in X$, there exists $\mathfrak{a} \in \mathcal{U}(X)$ such that $\mathfrak{a} \to x$, and

2. *fibre-closed* if, for each $x \in X$, $\{\mathfrak{a} \in \mathcal{U}(X) \,|\, \mathfrak{a} \to x\}$ is closed in $\mathcal{U}(X)$ with respect to the Zariski topology.

Weak reflexivity just means that the projection $p_{(X,r)} : \mathrm{Ult}(X, r) \to (X, r)$ is surjective, while the second axiom is easily interpreted: the category **PrTop** is isomorphic to the full subcategory of **URS** of all reflexive and fibre-closed ultrarelational spaces.

While the functor Ult does not preserve neither reflexivity nor transitivity, nor even the combination of these two properties (apply Ult, for instance, to the three points chain $\{0 \to 1 \to 2\}$), it is easily seen that it preserves weak reflexivity and fibre-closedness.

Finally, while dealing with limits of codirected diagrams, we will use a particular instance of a well-known property of diagram schemes connected by an initial functor (see [12] for details): given a codirected diagram $D : I \to \mathbf{X}$, we may replace it by its restriction to an initial section $I_0 = \{i \in I | i \leq i_0\}$, for $i_0 \in I$, since a limit cone $(L, (p_i)_{i \in I_0})$ induces a limit cone $(L, (p_i)_{i \in I})$ for I and vice-versa. Thus we may assume, without loss of generality, that the codirected diagram we are working with has a largest element.

3 Perfect and open maps

The ultrafilter-characterizations of proper/perfect and open maps obtained in [4] justify the introduction of the following definitions.

Definition 3.1. A continuous map $f : X \to Y$ between ultrarelational spaces is called

1. *proper (perfect)* if, for each ultrafilter $\mathfrak{a}$ on X with $f(\mathfrak{a}) \to y$ in Y, there exists (a unique) $x \in f^{-1}(y)$ such that $\mathfrak{a} \to x$.

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ f(\mathfrak{a}) \arrow[r] & y \end{tikzcd}$$

2. *open (a weak local homeomorphism)* if, for each $x \in X$ and each ultrafilter $\mathfrak{b}$ on Y with $\mathfrak{b} \to f(x)$ in Y, there exists an (unique) ultrafilter $\mathfrak{a}$ such that $\mathfrak{a} \to x$ in X and $f(\mathfrak{a}) = \mathfrak{b}$.

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ \mathfrak{b} \arrow[r] & f(x) \end{tikzcd}$$

The reason for introducing *weak local homeomorphisms* is the fact that we do not know whether the necessary lifting property above is also sufficient for a topological map to be a local homeomorphism.

It is well-known that open and perfect maps are triquotient maps in **Top**. This fact remains true in this general setting; in fact, it follows immediately from the following characterizations of proper and open maps.

Lemma 3.2. *A continuous map $f : X \to Y$ in* **URS** *is:*

1. *proper if and only if, for each ordinal α and each ultrafilter $\mathfrak{a}_\alpha$ on X_α such that the ultrafilter $(p_0^\alpha)_Y(f_\alpha(\mathfrak{a}_\alpha)) \to y$ in Y, there exists $x \in f^{-1}(y)$ such that $\mathfrak{a}_0 = (p_0^\alpha)_X(\mathfrak{a}_\alpha) \to x$;*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{a}_\alpha \arrow[r, dashed] \arrow[d, mapsto] & \cdots \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_0 \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ f_\alpha(\mathfrak{a}_\alpha) \arrow[r] & \cdots \arrow[r] & f(\mathfrak{a}_0) \arrow[r] & y \end{tikzcd}$$

2. *open if and only if, for each ordinal α, $x \in X$ and $((\mathfrak{b}_\beta)_{\beta\in\alpha}, f(x)) \in Y_\alpha$, there exists $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$ such that $f_\alpha((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) = ((\mathfrak{b}_\beta)_{\beta\in\alpha}, f(x))$, provided that X is fibre-closed.*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \cdots \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_{\beta+1} \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_\beta \arrow[r, dashed] \arrow[d, mapsto] & \cdots \arrow[r, dashed] \arrow[d, mapsto] & \mathfrak{a}_0 \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ \cdots \arrow[r] & \mathfrak{b}_{\beta+1} \arrow[r] & \mathfrak{b}_\beta \arrow[r] & \cdots \arrow[r] & \mathfrak{b}_0 \arrow[r] & f(x) \end{tikzcd}$$

Proof. 1. is obvious. To prove 2., since the case $\alpha = 0$ is trivial and the case α is a successor is straightforward (see Lemma 3.10 below), we assume that α is a limit ordinal and that the assertion is true for all $\beta \in \alpha$. Then, for $((\mathfrak{b}_\beta)_{\beta\in\alpha}, f(x)) \in Y_\alpha$ and each $\beta \in \alpha$, there exists $\mathfrak{a}_\beta$ such that $f_\beta(\mathfrak{a}_\beta) = \mathfrak{b}_\beta$ and $(p_0^\beta)_X(\mathfrak{a}_\beta) \to x$. Since X is fibre-closed, we can choose the $\mathfrak{a}'_\beta$s compactible with respect to $(p_{\beta'}^\beta)_X$. □

It is easily seen that:

Lemma 3.3 (Composition-cancellation rules). *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between ultrarelational spaces.*

1. *$g \circ f$ is proper (perfect) provided that g and f are proper (perfect).*
2. *g is proper (perfect) provided that f and $g \circ f$ are proper (perfect) and f is surjective.*
3. *f is proper (perfect) provided that g is separated and $g \circ f$ is proper (perfect).*

The results above remain true if we replace proper by open and perfect by weak local homeomorphism:

Lemma 3.4 (Composition-cancellation rules). *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between ultrarelational spaces.*

1. *$g \circ f$ is open (a weak local homeomorphism) provided that g and f are open (weak local homeomorphisms).*
2. *g is open (a weak local homeomorphism) provided that f and $g \circ f$ are open (weak local homeomorphisms) and f is surjective.*
3. *f is open (a weak local homeomorphism) provided that g is a weak local homeomorphism and $g \circ f$ is open (a weak local homeomorphism).*

We summarize below invariance and inverse invariance of weak-reflexivity and fibre-closedness. For that we first remark that an ultrarelational space X is weak reflexive if and only if $X \to 1$ is open.

Lemma 3.5. *Let $f : X \to Y$ be a continuous map betweem ultrarelational spaces.*

1. *If X is weak-reflexive, then $f(X)$ is weak reflexive.*
2. *If f is open, then:*
   (a) *X is weak-reflexive whenever Y is;*
   (b) *$f(X)$ is fibre-closed whenever X is.*

Proof. 1. Since $p_Y \circ f_1 = f \circ p_X$, with f and p_X, also p_Y is surjective.

2.(a) is obvious since open maps are closed under composition.

2.(b) Whenever $\{\mathfrak{a} \in \mathcal{U}(X) \,|\, \mathfrak{a} \to x\}$ is closed, its image under the closed map $\mathcal{U}(f)$, that coincides with $\{\mathfrak{b} \in \mathcal{U}(Y) \,|\, \mathfrak{b} \to f(x)\}$ whenever f is open, is closed. □

Next we are going to show that, in our setting, many "canonical" maps are perfect.

Lemma 3.6. *The map* $p_X : \mathrm{Ult}(X) \to X$ *is perfect for each ultrarelational space* X.

Proof. Let $\mathfrak{A}$ be an ultrafilter on $\mathrm{Ult}(X)$ such that $p_X(\mathfrak{A}) \to x$ for some $x \in X$. We put $\mathfrak{a} = p_X(\mathfrak{A})$. Then we have

$$\mathfrak{A} \to (\mathfrak{a}, x) \quad \text{and} \quad p_X(\mathfrak{a}, x) = x,$$

and $(\mathfrak{a}, x)$ is the only element of $\mathrm{Ult}(X)$ having these properties. □

Lemma 3.7. *For each continuous map* $f : X \to Y$ *between ultrarelational space* X *and* Y, *the canonical map* $\kappa : \mathrm{Ult}(X) \to X \times_Y \mathrm{Ult}(Y)$ *making the diagram*

$$\begin{tikzcd}
\mathrm{Ult}(X) \arrow[dr, "\kappa"] \arrow[drr, bend left, "\mathrm{Ult}(f)"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Y \mathrm{Ult}(Y) \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathrm{Ult}(Y) \arrow[d, "p_Y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute is perfect.

Proof. Let $\mathfrak{A}$ be an ultrafilter on $\mathrm{Ult}(X)$ and $(x, \mathfrak{b}, f(x)) \in X \times_Y \mathrm{Ult}(Y)$ such that $\kappa(\mathfrak{A}) \to (x, \mathfrak{b}, f(x))$, that is, $p_X(\mathfrak{A}) \to x$ and $p_Y(\mathrm{Ult}(f)(\mathfrak{A})) = \mathfrak{b}$. We have to find an element of $\mathrm{Ult}(X)$ mapped by κ into $(x, \mathfrak{b}, f(x))$ and such that $\mathfrak{A}$ converges to it. The only candidate, $(p_X(\mathfrak{A}), x)$, satisfies obviously $\mathfrak{A} \to (p_X(\mathfrak{A}), x)$, and, moreover,

$$\kappa(p_X(\mathfrak{A}), x) = (x, f(p_X(\mathfrak{A})), f(x)) = (x, p_Y(\mathrm{Ult}(f)(\mathfrak{A})), f(x)) = (x, \mathfrak{b}, f(x)).$$

□

Lemma 3.8. *For each diagram* $D : I \to \mathbf{URS}$, *the canonical map* $\kappa : \mathrm{Ult}(\mathsf{lim}_{i\in I} D) \to \mathsf{lim}_{i\in I} \mathrm{Ult} \circ D$ *making the diagram*

$$\begin{tikzcd}
\mathrm{Ult}(\mathsf{lim}_{i\in I} D) \arrow[r, "\kappa"] \arrow[d, "\mathrm{Ult}(\pi_i)"'] & \mathsf{lim}_{i\in I} \mathrm{Ult} \circ D \arrow[dl, "\rho_i"] \\
\mathrm{Ult}(D(i)) &
\end{tikzcd}$$

commute, for each $i \in I$, *is perfect, where* $(\pi_i : \mathsf{lim}_{i\in I} D \to D(i))_{i\in I}$ *and* $(\rho_i : \mathsf{lim}_{i\in I} \mathrm{Ult} \circ D \to \mathrm{Ult}(D(i)))_{i\in I}$ *denote the respective limit cones.*

Proof. Let $\mathfrak{A}$ be an ultrafilter on $\mathrm{Ult}(\mathsf{lim}_{i\in I} D)$ such that $\kappa(\mathfrak{A}) \to (\mathfrak{a}_i, x_i)_{i\in I}$ for some $(\mathfrak{a}_i, x_i)_{i\in I} \in \mathsf{lim}_{i\in I} \mathrm{Ult} \circ D$. We have to find an (unique) element $(\mathfrak{a}, x) \in \mathrm{Ult}(\mathsf{lim}_{i\in I} D)$ such that $\mathfrak{A} \to (\mathfrak{a}, x)$ and $\kappa(\mathfrak{a}, x) = (\mathfrak{a}_i, x_i)_{i\in I}$. But the only choice we have is to put $\mathfrak{a} = p_{\mathsf{lim}_{i\in I} D}(\mathfrak{A})$ and $x = (x_i)_{i\in I}$. Since $\kappa(\mathfrak{A}) \to (\mathfrak{a}_i, x_i)_{i\in I}$ we have

$$\mathrm{Ult}(\pi_i)(\mathfrak{A}) = \rho_i(\kappa(\mathfrak{A})) \to (\mathfrak{a}_i, x_i)$$

and therefore

$$\pi_i \circ p_{\mathsf{lim}_{i\in I} D}(\mathfrak{A}) = p_{D(i)} \circ \mathrm{Ult}(\pi_i)(\mathfrak{A}) = \mathfrak{a}_i \to x_i$$

for each $i \in I$, hence $p_{\mathsf{lim}_{i\in I} D}(\mathfrak{A}) \to (x_i)_{i\in I}$ and $\kappa(p_{\mathsf{lim}_{i\in I} D}(\mathfrak{A}), (x_i)_{i\in I}) = (\mathfrak{a}_i, x_i)_{i\in I}$. □

The following observation will be very useful in the sequel. Given a map $f : X \to Y$ and filters $\mathfrak{f}$ on X and $\mathfrak{g}$ on Y, $\mathfrak{f} \cup f^{-1}(\mathfrak{g})$ is a filter base provided that $f(\mathfrak{f}) \subseteq \mathfrak{g}$. As a particular instance, consider a commutative square

$$\begin{tikzcd} C \arrow[r, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \tag{1}$$

and ultrafilters $\mathfrak{a}$ on X and $\mathfrak{b}$ on Y such that $f(\mathfrak{a}) = g(\mathfrak{b})$. Then $\pi^{-1}(\mathfrak{a}) \cup \rho^{-1}(\mathfrak{b})$ is a filter base provided that (1) satisfies the *Beck-Chevalley condition*, that is:

(BC) $\qquad \forall A \subseteq X : g^{-1}(f(A)) = \rho(\pi^{-1}(A)).$

Note that the commutativity of (1) implies already the inclusion $\rho(\pi^{-1}(A)) \subseteq g^{-1}(f(A))$. We also remark that (1) satisfies (BC) if and only if the canonical map $\kappa : C \to X \times_Z Y$, $c \mapsto (\pi(c), \rho(c))$ is surjective.

We have the following obvious link to open maps.

Lemma 3.9. *A continuous map $f : X \to Y$ between ultrarelational spaces X and Y is open (a weak local homeomorphism) if and only if*

$$\begin{tikzcd} \mathrm{Ult}(X) \arrow[r, "\mathrm{Ult}(f)"] \arrow[d, "p_X"'] & \mathrm{Ult}(Y) \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

satisfies the Beck-Chevalley condition (is a pullback).

Lemma 3.10. *Let $f : X \to Y$ be a continuous map between ultrarelational spaces X and Y. Then* $\mathrm{Ult}(f)$ *is open (proper, perfect) provided that f is open (proper, perfect).*

Proof. In case f is open this is a consequence of Lemma 3.9. If f is proper or perfect, then the assertion follows from Lemma 3.3 and Lemma 3.6. □

Lemma 3.11. *Assume that the diagram (1) satisfies (BC) with π proper and g separated. Then also the diagram*

$$\begin{tikzcd} \mathrm{Ult}(C) \arrow[r, "\mathrm{Ult}(\rho)"] \arrow[d, "\mathrm{Ult}(\pi)"'] & \mathrm{Ult}(Y) \arrow[d, "\mathrm{Ult}(g)"] \\ \mathrm{Ult}(X) \arrow[r, "\mathrm{Ult}(f)"'] & \mathrm{Ult}(Z) \end{tikzcd}$$

satisfies (BC).

Proof. Let $(\mathfrak{a}, x) \in \mathrm{Ult}(X)$ and $(\mathfrak{b}, y) \in \mathrm{Ult}(Y)$ be given such that $f(\mathfrak{a}) = g(\mathfrak{b})$ and $f(x) = g(y)$. Hence there exists an ultrafilter $\mathfrak{c}$ on C with $\pi(\mathfrak{c}) = \mathfrak{a}$ and $\rho(\mathfrak{c}) = \mathfrak{b}$. Since π is proper, there exists $c \in C$ such that $\pi(c) = x$ and $\mathfrak{c} \to c$. Then $\mathfrak{b} = \rho(\mathfrak{c}) \to \rho(c)$ and $g\rho(c) = f\pi(c) = f(x) = g(y)$, hence $\rho(c) = y$ by the separatedness of g. □

4 Limit stability of perfect maps

Topological perfect maps are known to be the "right" map-generalization of compact Hausdorff spaces, since any perfect map $f : X \to Y$ can be viewed as a compact Hausdorff object in the slice category $\mathbf{Top}/Y$ (see [3]). Many results about compact Hausdorff spaces can be extended to perfect maps in **Top**, and some of them in such a way that neither reflexivity nor transitivity are needed in their proofs, hence they hold in **URS** as well. However, combinations of this kind of properties with surjectivity conditions may fail to be true, basically because the required properties may fail at the level of sets. For instance, in contrast to Theorem 2.5, the codirected limit of non-empty sets might be empty even if all connecting maps are surjective [17]. Fibre-closedness and weak reflexivity seem to be exactly the properties needed, as we are going to show in this section.

The following result is well-known in **Top** and has a straightforward proof.

Proposition 4.1. *Let $D : I \to \mathbf{URS}$ be a codirected diagram such that, for each $\varphi : i \to j$ in I, $D(\varphi)$ is perfect. Then all limit projections $\pi_i : \mathsf{lim}_{i\in I}D \to D(i)$, $i \in I$, are perfect.*

Theorem 4.2. *Let $D : I \to \mathbf{PsTop}$ be a codirected diagram such that $D(\varphi)$ is perfect for each $\varphi : i \to j$ in I and let $(\pi_i : L \to D(i))_{i\in I}$ be a compatible cone for D. Then the assertions below are equivalent.*

1. *$(\pi_i : L \to D(i))_{i\in I}$ is a limit of D.*
2. *The following conditions are fulfilled:*
   - *(a) $(\pi_i : L \to D(i))_{i\in I}$ is point separating and initial;*
   - *(b) For each $i \in I$, π_i is perfect;*
   - *(c) For each $i \in I$,*

$$\mathsf{im}\pi_i = \bigcap_{j \xrightarrow{\varphi} i} \mathsf{im}D(\varphi).$$

Proof. We assume first that $(\pi_i : L \to D(i))_{i\in I}$ is a limit of D. Then, as for every concrete limit source, it is point separating and initial, moreover, Proposition 4.1 implies 2.(b). To prove 2.(c) we remark first that the inlusion

$$\mathsf{im}\pi_i \subseteq \bigcap_{j \xrightarrow{\varphi} i} \mathsf{im}D(\varphi)$$

follows from the fact that $(\pi_i : L \to D(i))_{i\in I}$ is compatible for D. Now let $i \in I$ and $x_i \in \bigcap_{j \xrightarrow{\varphi} i} \mathsf{im}D(\varphi)$ be given. Without loss of generality we may assume that i is the largest element of I, that is $I = \{j \in I \,|\, j \le i\}$. For each $\varphi_j : j \to i$ in I, there exists an ultrafilter $\mathfrak{a}_j$ on $D(j)$ such that $D(\varphi_j)(\mathfrak{a}_j) = \dot{x}_i$. Hence the set

$$\mathcal{M}_j = \{\mathfrak{a} \in \mathcal{U}(D(j)) \,|\, D(\varphi_j)(\mathfrak{a}) = \dot{x}_i\}$$

is, for each $j \in I$, non-empty and Zariski closed; according to Remark 2.6 we can find a family $(\mathfrak{a}_j)_{j\in I}$ of ultrafilters $\mathfrak{a}_j \in \mathcal{M}_j$ such that $D(\varphi)(\mathfrak{a}_j) = \mathfrak{a}_{j'}$ for each $\varphi : j \to j'$ in I. Each $D(\varphi_j) : D(j) \to D(i))$ $(j \in I)$ is perfect, hence there exists a family $(x_j)_{j\in I}$ of elements $x_j \in D(j)$ such that $\mathfrak{a}_j \to x_j$ and $D(\varphi_j)(x_j) = x_i$. But we have, by the perfectness of $D(\varphi)$ for each $\varphi : j \to j'$, also $D(\varphi)(x_j) = x_{j'}$. Hence $(x_j)_{j\in I} \in L$.

Now we assume that 2.(a), 2.(b) and 2.(c) hold. Let $(f_i : X \to D(i))_{i\in I}$ be a compatible cone for D. We have to prove the existence of a map $f : X \to L$ such that $\pi_i \circ f = f_i$ for each $i \in I$. Let $x \in X$. For each $i \in I$ we have

$$f_i(x) \subseteq \bigcap_{j \xrightarrow{\varphi} i} \mathsf{im} D(\varphi) = \mathsf{im}\pi_i,$$

therefore (and by the codirectedness of I) $\{\pi_i^{-1}(f_i(x)) \,|\, i \in I\}$ is a filter base on L which can be refined to an ultrafilter $\mathfrak{a}$. It holds $\pi_i(\mathfrak{a}) = \dot{f_i}(x)$ for each $i \in I$. Each π_i ($i \in I$) is perfect, there exists an element $y^i \in L$ with $\mathfrak{a} \to y^i$ and $\pi_i(y^i) = f_i(x)$. But the perfectness of each $D(\varphi)$ implies $y = y^i = y^{i'}$ for all $i, i' \in I$, we can define $f(x) = y$. $\square$

In the proof above we used the (trivial) fact that, if $x \in \mathsf{im}D(\varphi)$, then $\dot{x}$ is a ultrafilter on $\mathsf{im}D(\varphi)$. If we replace reflexivity by weak reflexivity and hence $\dot{x}$ by an arbitrary ultrafilter $\mathfrak{a} \to x$, this argument cannot be used. However, in case all $D(\varphi)$ are surjective a sligh modification of the proof above goes through.

Proposition 4.3. *Let $D : I \to$ **URS** be a codirected diagram such that $D(i)$ is weak reflexive and fibre-closed and $D(\varphi)$ is a perfect surjection for each $\varphi : i \to j$ in I and let $(\pi_i : L \to D(i))_{i\in I}$ be compatible cone for D. Then the assertions below are equivalent.*

1. *$(\pi_i : L \to D(i))_{i\in I}$ is a limit of D.*
2. *The following conditions are satisfied:*
   - *(a) $(\pi_i : L \to D(i))_{i\in I}$ is point separating and initial;*
   - *(b) L is weak reflexive and fibre-closed;*
   - *(c) For each $i \in I$, π_i is a perfect surjection.*

Corollary 4.4. *If X is weak reflexive and fibre-closed, then, for each ordinal α, X_α is weak reflexive and fibre-closed.*

Proposition 4.5. *Let $D, D' : I \to$ **URS** be diagrams and $(f_i)_{i\in I} : D \to D'$ be a natural transformation such that, for each $i \in I$, f_i is perfect. Then $f = \mathsf{lim}_{i\in I} f_i$ is perfect.*

Also the following Proposition is a generalization of a well-known result about compact Hausdorff spaces: the codirected limit of surjections between compact Hausdorff spaces is surjective. This stands in sharp contrast to the situation in **Set**, where even the limit of a sequence of surjections need not be surjective.

Theorem 4.6. *Let $D, D' : I \to$ **PsTop** be codirected diagrams and $(f_i)_{i\in I} : D \to D'$ be a natural transformation such that, for each $i \in I$, f_i is a perfect surjection. Then $f = \mathsf{lim}_{i\in I} f_i$ is a perfect surjection.*

Proof. By the Proposition above, f is perfect. To prove the surjectivity of f, let $(\pi_i : L \to D(i))_{i\in I}$ and $(\pi'_i : L' \to D'(i))_{i\in I}$ be limit cones for D and D' respectively, and let $x' \in L'$. For each $i \in I$, let $\mathfrak{a}'_i$ be the principal ultrafilter induced by $\pi'_i(x')$. Since f_i is surjective, there exists an ultrafilter $\mathfrak{a}_i$ on $D(i)$ such that $f(\mathfrak{a}_i) = \mathfrak{a}'_i$. Again, according to Remark 2.6, we can choose these ultrafilters compatible with the connecting maps $D(\varphi)$. The perfectness of each f_i ($i \in I$) guarantees the existence of a family $(x_i)_{i\in I}$ of elements of $D(i)$ such that $\mathfrak{a}_i \to x_i$ and $f_i(x_i) = x'_i$. Hence we have also $D(\varphi)(x_i) = x_j$ for each $\varphi : i \to j$ in I and therefore $x = (x_i)_{i\in I} \in L$. $\square$

Again, the reflexivity argument used in the proof can be substituted by weak reflexivity together with fibre-closedness: weak reflexivity guarantees the existence of $\mathfrak{a}'_i$ converging to $\pi'_i(x')$ and fibre-closedness allows us to choose them compatibly with the connecting maps.

Proposition 4.7. *Let $D, D' : I \to \mathbf{URS}$ be codirected diagrams and $(f_i)_{i\in I} : D \to D'$ be a natural transformation such that, for each $i \in I$, f_i is a perfect surjection and $D'(i)$ is weak reflexive and fibre-closed. Then $f = \mathsf{lim}_{i\in I} f_i$ is a perfect surjection.*

Finally, in the special case of building the functors Ult^{λ} for a limit ordinal λ, the existence and compatibility of the ultrafilters $\mathfrak{a}'_i$ is guaranteed by the construction. Hence, in this case the result of the proposition above does not require extra conditions.

Proposition 4.8. *If $f : X \to Y$ in **URS** is a perfect surjection, then, for each ordinal α, $f_\alpha : X_\alpha \to Y_\alpha$ is a perfect surjection.*

5 Limit stability of open maps

We are now going to prove stability of open maps under special codirected limits. We start with a result about cartesian natural transformations, that is, those natural transformations $(f_i)_i \in I : D \to D'$ such that, for each $\varphi : i \to j$ in I, the diagram

$$\begin{tikzcd} D(i) \arrow[r, "f_i"] \arrow[d, "D(\varphi)"'] & D'(i) \arrow[d, "D'(\varphi)"] \\ D(j) \arrow[r, "f_j"'] & D'(j) \end{tikzcd} \tag{2}$$

is a pullback.

Lemma 5.1. *Let $D, D' : I \to \mathbf{C}$ codirected diagrams in a category $\mathbf{C}$ and let $(f_i)_{i\in I} : D \to D'$ be a cartesian natural transformation. Then, for each $i \in I$, the square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_i"'] & Y \arrow[d, "\rho_i"] \\ D(i) \arrow[r, "f_i"'] & D'(i) \end{tikzcd}$$

is a pullback, where $(\pi_i : X \to D(i))_{i\in I}$ and $(\rho_i : Y \to D'(i))_{i\in I}$ are the respective limit cones.

Proof. Let $j \in I$ be any element of I, we may assume that j is the largest element of I. Let $h_j : Z \to D(j)$ and $k : Z \to Y$ be $\mathbf{C}$-morphism such that $f_j \circ h_j = \rho_j \circ k$. Since $(f_i)_{i\in I} : D \to D'$ is cartesian, h_j is the j-component of a compatible cone $(h_i : Z \to D(i))_{i\in I}$ for D such that $f_i \circ h_i = \rho_i \circ k$ holds for each $i \in I$. By the universal property of the limit cone $(\pi_i : X \to D(i))_{i\in I}$, there exists a $\mathbf{C}$-morphism $h : Z \to X$ such that $\pi_i \circ h = h_i$ holds for each $i \in I$, hence in particular $\pi_j \circ h = h_j$. Since $(\rho_i : Y \to D'(i))_{i\in I}$ is a limit cone it holds also $f \circ h = k$.

Assume now that we have $\mathbf{C}$-morphisms $h', h'' : Z \to X$ fulfilling the equations $f \circ h' = f \circ h'' = k$ and $\pi_j \circ h' = \pi_j \circ h'' = h_j$. For each $i \in I$, the universal property of the pullback diagram 2 implies $\pi_i \circ h' = \pi_i \circ h''$, hence $h' = h''$. $\square$

Proposition 5.2. *Let $D, D' : I \to \mathbf{URS}$ be codirected diagrams in $\mathbf{URS}$ and let $(f_i)_{i\in I} : D \to D'$ be a natural transformation such that, for each $\varphi : i \to j$ in I, $D(\varphi)$ and $D'(\varphi)$ are perfect and the diagram (2) satisfies (BC). Moreover, we assume that, for each $i \in I$, $D(i)$ is weak reflexive and fibre-closed and $D'(i)$ is fibre-closed. Then, for each $i \in I$, the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_i"'] & Y \arrow[d, "\rho_i"] \\ D(i) \arrow[r, "f_i"'] & D'(i) \end{tikzcd}$$

satisfies (BC), where $(\pi_i : X \to D(i))_{i\in I}$ and $(\rho_i : Y \to D'(i))_{i\in I}$ are the respective limit cones.

Proof. Let $i_0 \in I$ be any element of I. We may assume that i_0 is the largest element of I. We can factorize $(f_i)_{i\in I} : D \to D'$ by a natural transformation $(f_i^{\#})_{i\in I} : D \to D^{\#}$ consisting of perfect surjections followed by a cartesian natural transformation $(f_i^*)_{i\in I} : D^{\#} \to D'$. Moreover, $D^{\#}(i)$ is weak reflexive and fibre-closed for each $i \in I$. Let $(\pi_i : X \to D(i))_{i\in I}$, $(\xi_i : Z \to D^{\#}(i))_{i\in I}$ and $(\rho_i : Y \to D'(i))_{i\in I}$ denote the respective limit cones and $f = \mathsf{lim}_{i\in I} f_i$, $f^{\#} = \mathsf{lim}_{i\in I} f_i^{\#}$ and $f^* = \mathsf{lim}_{i\in I} f_i^*$. We have the following diagram,

$$\begin{tikzcd} X \arrow[r, "f^{\#}"] \arrow[rr, bend left, "f"] \arrow[d, "\pi_{i_0}"'] & Z \arrow[r, "f^*"] \arrow[d, "\xi_{i_0}"'] & Y \arrow[d, "\rho_{i_0}"] \\ D(i_0) \arrow[r, "f_{i_0}^{\#}"'] \arrow[rr, bend right, "f_{i_0}"'] & D^{\#}(i_0) \arrow[r, "f_{i_0}^*"'] & D'(i_0) \end{tikzcd}$$

where the right hand square is a pullback (Lemma 5.1) and $f^{\#}$ is a (perfect) surjection (Theorem 4.6), which proves our assertion. $\square$

Corollary 5.3. *Under the conditions of the proposition above, assuming, in addition, that f_i is open (open and surjective) for each $i \in I$, then $f = \mathsf{lim}_{i\in I} f_i$ is open (open and surjective).*

For the special case of the limit step in the construction on Ult^{λ}, exactly as in the study of perfect maps, one may choose the ultrafilters compatibly.

Proposition 5.4. *If $f : X \to Y$ is an open map in $\mathbf{URS}$ and X is fibre-closed, then, for each pair of ordinal numbers α, β with $\beta \leq \alpha$, the diagram*

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "(p^\alpha_\beta)_X"'] & Y_\alpha \arrow[d, "(p^\alpha_\beta)_Y"] \\ X_\beta \arrow[r, "f_\beta"'] & Y_\beta \end{tikzcd}$$

satisfies (BC). In particular, f_β is open.

Proof. Combine Proposition 5.2 with Lemmas 3.2, 3.9 and 3.11. $\square$

6 Limit stability of α-surjective maps

In this section we consider α-surjective maps and prove their stability under various kinds of limits in **URS**, in particular their pullback and product stability. As special cases occure **Top**-results of Day and Kelly [5] and Michael [10] about pullback and product stability of biquotient maps, a result of Uspenskij [16] about the product stability of triquotient maps and a result of Richter [15] about pullback stability of triquotient maps. Moreover, we obtain the pullback and product stability of effective descent maps in **Top**. Their pullback stability is a consequence of a result of Reiterman, Sobral and Tholen [14] whereby their product stability seems to be new.

To prove pullback stability of biquotient maps in **Top** one might argue as follows. Given a pullback diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in **Top** with f biquotient and $\mathfrak{b} \to y$ in Y, then $g(\mathfrak{b}) \to g(y)$ in Z, hence there exists $\mathfrak{a} \to x$ in X such that $g(\mathfrak{b}) = f(\mathfrak{a})$ and $f(x) = g(y)$. The fact that each pullback diagram satisfies the Beck-Chevalley condition guarantees now that $\pi^{-1}(\mathfrak{a}) \cup \rho^{-1}(\mathfrak{b})$ generates a filter on $X \times_Z Y$ which can be refined to an ultrafilter $\mathfrak{c}$. We have $\rho(\mathfrak{c}) = \mathfrak{b}$ and $\mathfrak{c} \to (x, y)$, hence ρ is biquotient. In fact, we have proved that the commutative square

$$\begin{tikzcd} \mathrm{Ult}(X \times_Z Y) \arrow[r, "\mathrm{Ult}(\rho)"] \arrow[d, "\mathrm{Ult}(\pi)"'] & \mathrm{Ult}(Y) \arrow[d, "\mathrm{Ult}(g)"] \\ \mathrm{Ult}(X) \arrow[r, "\mathrm{Ult}(f)"'] & \mathrm{Ult}(Z) \end{tikzcd}$$

satisfies (BC). Iterating this argument we obtain

Lemma 6.1. *Let*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a pullback diagram in **URS**. *Then, for each ordinal α, the diagram*

$$\begin{tikzcd} (X \times_Z Y)_\alpha \arrow[r, "\rho_\alpha"] \arrow[d, "\pi_\alpha"'] & Y_\alpha \arrow[d, "g_\alpha"] \\ X_\alpha \arrow[r, "f_\alpha"'] & Z_\alpha \end{tikzcd}$$

satisfies (BC).

Proof. It is well-known that each pullback diagram satisfies (BC), hence the assertion is true for $\alpha = 0$.

Let $\alpha > 0$ and assume that $g_\beta^{-1}(f_\beta(\mathcal{A})) = \rho_\beta(\pi_\beta^{-1}(\mathcal{A}))$ holds for each $\beta \in \alpha$ and each $\mathcal{A} \subseteq X_\beta$. Let $\mathcal{A} \subseteq X_\alpha$. The inclusion $g_\alpha^{-1}(f_\alpha(\mathcal{A})) \supseteq \rho_\alpha(\pi_\alpha^{-1}(\mathcal{A}))$ holds obviously, so we only have to show that $g_\alpha^{-1}(f_\alpha(\mathcal{A})) \subseteq \rho_\alpha(\pi_\alpha^{-1}(\mathcal{A}))$. To this end, let $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in Y_\alpha$ be such that

$$(f_\beta((\mathfrak{a}_\beta)_{\beta\in\alpha}), f(x)) = (g_\beta((\mathfrak{b}_\beta)_{\beta\in\alpha}), g(y))$$

for some $((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \in \mathcal{A}$. By induction hypothesis,

$$\pi_\beta^{-1}(\mathfrak{a}_\beta) \cup \rho_\beta^{-1}(\mathfrak{b}_\beta)$$

induces, for each $\beta \in \alpha$, a filter on $(X \times_Z Y)_\beta$ which can be refined to an ultrafilter $\mathfrak{c}_\beta$. Moreover, according to Remark 2.6, we can choose $\mathfrak{c}_\beta$ such that $(p_\beta^{\beta'})_X(\mathfrak{c}_{\beta'}) = \mathfrak{c}_\beta$ for all $\beta \leq \beta' \in \alpha$. We have

$$\pi(\mathfrak{c}_0) = \mathfrak{a}_0 \to x \quad \text{and} \quad \rho(\mathfrak{c}_0) = \mathfrak{b}_0 \to y$$

and therefore $\mathfrak{c}_0 \to (x, y)$. Hence $((\mathfrak{c}_\beta)_{\beta\in\alpha}, (x, y)) \in (X \times_Z Y)_\alpha$ and

$$(\pi_\beta((\mathfrak{c}_\beta)_{\beta\in\alpha}), x) = ((\mathfrak{a}_\beta)_{\beta\in\alpha}, x) \quad \text{and} \quad (\rho_\beta((\mathfrak{c}_\beta)_{\beta\in\alpha}), y) = ((\mathfrak{b}_\beta)_{\beta\in\alpha}, y)$$

hold. We conclude that $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in \rho_\alpha(\pi_\alpha^{-1}(\mathcal{A}))$. ☐

Theorem 6.2. *For each ordinal α and for $\alpha = \Omega$, the class of α-surjective maps is pullback stable in* **URS**.

Corollary 6.3. *Triquotient maps, effective descent maps and biquotient maps are pullback stable in* **Top**.

Corollary 6.4. *Let $D, D' : I \to$ **URS** be codirected diagrams in **URS** and let $(f_i)_{i\in I} : D \to D'$ be a cartesian natural transformation. Then $f = \mathsf{lim}_{i\in I} f_i : X \to Y$ is α-surjective (α an ordinal or $\alpha = \Omega$) provided that all f_i ($i \in I$) are, where $(\pi_i : X \to D(i))_{i\in I}$ and $(\rho_i : Y \to D'(i))_{i\in I}$ are the respective limit cones.*

Proof. It is an immediate consequence of Lemma 5.1. ☐

Proposition 6.5. *Let $D, D' : I \to$ **URS** be codirected diagrams in **URS** and let $(f_i)_{i\in I} : D \to D'$ be a natural transformation such that, for each $\varphi : i \to j$ in I, $D(\varphi)$ and $D'(\varphi)$ are perfect, $D(i)$ is weak reflexive and fibre-closed, $D'(i)$ is fibre-closed and the diagram*

$$\begin{tikzcd} D(i) \arrow[r, "f_i"] \arrow[d, "D(\varphi)"'] & D'(i) \arrow[d, "D'(\varphi)"] \\ D(j) \arrow[r, "f_j"'] & D'(j) \end{tikzcd}$$

satisfies (BC). Then $f = \mathsf{lim}_{i\in I} f_i : X \to Y$ is an α-surjective map (α an ordinal or $\alpha = \Omega$) provided that all f_i ($i \in I$) are, where $(\pi_i : X \to D(i))_{i\in I}$ and $(\rho_i : Y \to D'(i))_{i\in I}$ are the respective limit cones.

Proof. We can factorize $(f_i)_{i\in I} : D \to D'$ by a natural transformation $(f_i^\#)_{i\in I} : D \to D^\#$ consisting only of perfect surjections followed by a cartesian natural transformation $(f_i^*)_{i\in I} : D^\# \to D'$. ☐

As for pullbacks, the ideas which work for biquotients can be also used to prove product stability of α-surjective maps.

Lemma 6.6. *Let $(\pi_i : X \to X_i)_{i\in I}$ and $(\rho_i : Y \to Y_i)_{i\in I}$ be products in* **URS**, *let $(f_i : X_i \to Y_i)_{i\in I}$ be a family of continuous maps and let $f = \prod_{i\in I} f_i$. For each ordinal α, if f_i is α-surjective for every $i \in I$, then the generalized Beck-Chevalley condition*

$$f_\alpha(\bigcap_{i\in I} \pi_{i\alpha}^{-1}(\mathcal{A}_i)) = \bigcap_{i\in I} \rho_{i\alpha}^{-1}(f_{i\alpha}(\mathcal{A}_i))$$

holds true for every family $(\mathcal{A}_i)_{i\in I}$ of subsets $\mathcal{A}_i$ of $X_{i\alpha}$.

Proof. It is well-known that

$$f(\bigcap_{i\in I} \pi_i^{-1}(A_i)) = \bigcap_{i\in I} \rho_i^{-1}(f_i(A_i))$$

holds for each family $(A_i)_{i\in I}$ of subsets A_i of X_i, hence the assertion is true for $\alpha = 0$.

Let now $\alpha > 0$. Assume that f_i is α-surjective for each $i \in I$ and that the assertion is true for each $\beta \in \alpha$. Let $(\mathcal{A}_i)_{i\in I}$ be a family of subsets $\mathcal{A}_i$ of $X_{i\alpha}(:= \mathrm{Ult}^\alpha(X_i))$. The inclusion

$$f_\alpha(\bigcap_{i\in I} \pi_{i\alpha}^{-1}(\mathcal{A}_i)) \subseteq \bigcap_{i\in I} \rho_{i\alpha}^{-1} f_{i\alpha}(\mathcal{A}_i)$$

holds obviously, so we only have to show

$$f_\alpha(\bigcap_{i\in I} \pi_{i\alpha}^{-1}(\mathcal{A}_i)) \supseteq \bigcap_{i\in I} \rho_{i\alpha}^{-1} f_{i\alpha}(\mathcal{A}_i).$$

To this end, let $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in Y_\alpha$ be such that, for each $i \in I$, there exists $((\mathfrak{a}_{i\beta})_{\beta\in\alpha}, x_i) \in \mathcal{A}_i$ with

$$((f_{i\beta}(\mathfrak{a}_{i\beta}))_{\beta\in\alpha}, f_i(x_i)) = ((\rho_{i\beta}(\mathfrak{b}_\beta))_{\beta\in\alpha}, \rho_i(y)).$$

By the induction hypothesis, for each $\beta \in \alpha$,

$$f_\beta^{-1}(\mathfrak{b}_\beta) \cup \{\bigcap_{i\in F} \pi_{i\beta}^{-1}(A_i) \mid F \subseteq I \text{ finite, } A_i \in \mathfrak{a}_{i\beta}\}$$

generates a filter on X_β which can be refined to an ultrafilter $\mathfrak{a}_\beta$. Moreover, according to Remark 2.6 we can choose $\mathfrak{a}_\beta$ such that $(p_\beta^{\beta'})_X(\mathfrak{a}_{\beta'}) = \mathfrak{a}_\beta$ for all $\beta \leq \beta' \in \alpha$. We then have $\pi_i(\mathfrak{a}_0) = \mathfrak{a}_{i0} \to x_i$ for each $i \in I$ and therefore $\mathfrak{a}_0 \to (x_i)_{i\in I}$. That is $((\mathfrak{a}_\beta)_{\beta\in\alpha}, (x_i)_{i\in I}) \in X_\alpha$,

$$\pi_{i\alpha}((\mathfrak{a}_\beta)_{\beta\in\alpha}, (x_i)_{i\in I}) = ((\mathfrak{a}_{i\beta})_{\beta\in\alpha}, x_i)$$

for each $i \in I$ and

$$f_\alpha((\mathfrak{a}_\beta)_{\beta\in\alpha}, (x_i)_{i\in I}) = ((\mathfrak{b}_\beta)_{\beta\in\alpha}, y),$$

hence $((\mathfrak{b}_\beta)_{\beta\in\alpha}, y) \in f_\alpha(\bigcap_{i\in F} \pi_{i\alpha}^{-1}(\mathcal{A}_i))$. ☐

Theorem 6.7. *For each ordinal α and for $\alpha = \Omega$, the class of α-surjective maps is product stable in* **URS**.

Corollary 6.8. *Triquotient maps, effective descent maps and biquotient maps are product stable in* **Top**.

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