On the eigenvalues of normal matrices

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Abstract

Characterizations of eigenvalues of normal matrices using the lexicographical order in $\mathbb{C}$ are presented, with some applications.

1 Introduction

The classes of Hermitian and unitary matrices have a rich structure and much is known about the eigenvalues of these types of matrices. The more general class of normal (*i.e.* unitarily diagonalizable) complex matrices is less well understood. And not much is known about spectral problems involving normal matrices, even with their eigenvalues being described in terms of those of their Hermitian and skew-Hermitian parts.

The difference between Hermitian and general normal matrices is that the latter can have as eigenvalues arbitrary complex numbers. $\mathbb{C}$, of course, is not an ordered field. But it turns out that the simple fact that $\mathbb{C}$ can be totally ordered as a vector space over the reals is enough to obtain useful information on spectra of normal matrices using Hermitian matrices as an inspiration. This is the object of the present note.

2 Total orders in $\mathbb{C}$

A total order in $\mathbb{C}$ compatible with addition of complex numbers and multiplication by positive reals is the lexicographic order. It is characterized by its positive cone $H = \{a + ib : a > 0 \text{ or, if } a = 0,\ b > 0\}$.

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Compatibility with addition means $H + H \subseteq H$, and compatibility with multiplication by positive reals means $\lambda H \subseteq H$ for $\lambda > 0$. The order being total means $H \cup -H = \mathbb{C} \setminus \{0\}$.

The lexicographic order is not Archimedian and, apart from rotations of the positive cone, is the only total order in $\mathbb{C}$ compatible with the above mentioned operations. We shall use the notation $\leq^{\text{lex}}$ for it, and, for real θ, we use $\leq^{\text{lex}}_{\theta}$ for the total order with positive cone $e^{i\theta}H$.

3 Eigenvalues of normal matrices

Let A be an $n \times n$ complex normal matrix. Let $\alpha_1, \ldots, \alpha_n$ be its eigenvalues, ordered so that $\alpha_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \alpha_n$, and let $v_1, \ldots, v_n$ be corresponding orthonormal eigenvectors of A. For $j = 1, \ldots, n$ denote by E_j and E'_j the subspaces of $\mathbb{C}^n$ spanned by $v_1, \ldots, v_j$ and $v_j, \ldots, v_n$, respectively.

Applying the argument used to obtain the corresponding result for Hermitian matrices, we get:

Theorem 1. *For $j = 1, \ldots, n$ we have*

$$\alpha_j = \min_{x \in E_j,\, \|x\|=1} x^*Ax = \max_{x \in E'_j,\, \|x\|=1} x^*Ax \,.$$

In addition, we have

$$\alpha_j = \max_{\dim\, H=j} \ \min_{x \in H,\, \|x\|=1} x^*Ax = \min_{\dim\, H=n-j+1} \ \max_{x \in H,\, \|x\|=1} x^*Ax \,.$$

(Here max *and* min *are used in the lexicographic sense.)*

Analogous characterizations hold for any order of the type $\leq^{\text{lex}}_{\theta}$, either using the same proof or applying the theorem to the normal matrix $e^{-i\theta}A$.

Note how these results make immediately visible the fact that the numerical range $W(A) = \{x^*Ax : \|x\| = 1\}$ of a normal matrix A is the convex hull of its eigenvalues: any straight line moving in the plane parallel to itself must touch $W(A)$ first at an eigenvalue of A.

From the above theorem we immediately obtain, again repeating the Hermitian argument, a result concerning principal normal submatrices of normal matrices:

Theorem 2. *Let A be an $n \times n$ normal matrix with eigenvalues $\alpha_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \alpha_n$. If B is a principal $k \times k$ normal submatrix of A with eigenvalues $\beta_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \beta_k$, we have*

$$\alpha_j \geq^{\text{lex}} \beta_j \geq^{\text{lex}} \alpha_{j+n-k} \ , \quad j = 1, \ldots, k \, .$$

An analogous result holds for any order of the type $\leq_\theta^{\text{lex}}$.

For other interlacing results in this setting see [2], [1].

The result in [2] shows that for a $n \times n$ normal matrix to have a principal $(n-1) \times (n-1)$ normal principal submatrix is a highly restrictive condition, essentially forcing the matrix, apart from a rotation and a translation, to be Hermitian. It seems plausible that one can obtain this from Theorem 2 above (this is easy to see for small values of n).

In [1] an interlacing result is presented for the arguments of eigenvalues of a normal matrix and a normal principal submatrix: a relation with Theorem 2 above is unclear.

And then there is the general interlacing theorem for singular values [11], which for normal matrix and submatrix yields a statement whose relation with the above result is again unclear.

Note also that Theorem 2 does not follow directly from the interlacing theorem for Hermitian matrices applied to the Hermitian and skew-Hermitian parts of A and B.

4 Sums of normal matrices

A generalization of the first part of Theorem 1 can be obtained by mimicking the corresponding result for Hermitian matrices [5]. To present it we need some notation.

Take a sequence $V = (V_1, ..., V_n)$ of subspaces of $\mathbb{C}^n$ with $V_1 \subset \ldots \subset V_n$ and $\dim(V_i) = i$, for $i = 1, ..., n$. Given a sequence $I = (i_1, ..., i_r)$, with $1 \leq i_1 < \ldots < i_r \leq n$, the *Schubert variety* associated to V and I is

$$\Omega_I(V) = \{L \text{ subspace of } \mathbb{C}^n : \dim(L) = r,\ \dim(L \cap V_{i_d}) \geq d,\ d = 1, ..., r\}\,.$$

Keep the notation of the previous section and write $E = (E_1, ..., E_n)$, $E' = (E'_n, ..., E'_1)$. Put also $I' = (n - i_r + 1, ..., n - i_1 + 1)$.

If L is a subspace of dimension r and $x_1, ..., x_r$ is an orthonormal basis of L, the *Rayleigh trace* of A with respect to L is

$$tr(A_{|L}) = \sum_{d=1}^{r} x_d^* A x_d\,.$$

(This does not depend on the basis.)

Theorem 3. *If the eigenvalues of a normal matrix A are $\alpha_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \alpha_n$, one has*

$$\alpha_{i_1} + \ldots + \alpha_{i_r} = \min_{L \in \Omega_I(E)} tr(A_{|L}) \;=\; \max_{L \in \Omega_{I'}(E')} tr(A_{|L})$$

where again max *and* min *are used in the lexicographic sense.*

This characterization (of course also valid for any order of the type $\leq_\theta^{\text{lex}}$) can be applied to obtaining inequalities for the eigenvalues of a sum of two normal matrices if this sum is itself normal.

Let A and B be $n \times n$ normal matrices with eigenvalues $\alpha_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \alpha_n$ and $\beta_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \beta_n$, respectively. Suppose that $A + B$ is normal, with eigenvalues $\gamma_1 \geq^{\text{lex}} \ldots \geq^{\text{lex}} \gamma_n$. Let E, E', F, F' and G, G' be sequences of subspaces built from the eigenvectors of A, B and $A + B$, as before. Let I, J and K be sequences of r indices:

$$\begin{aligned} I &= (i_1, \ldots, i_r)\ ,\ \ 1 \leq i_1 < \ldots < i_r \leq n\,, \\ J &= (j_1, \ldots, j_r)\ ,\ \ 1 \leq j_1 < \ldots < j_r \leq n\,, \\ K &= (k_1, \ldots, k_r)\ ,\ \ 1 \leq k_1 < \ldots < k_r \leq n\,. \end{aligned}$$

Then, using the characterizations of Theorem 3, it is easy to see that:

Theorem 4. *If*

$$\Omega_K(G) \cap \Omega_{I'}(E') \cap \Omega_{J'}(F') \neq \emptyset ,$$

then

$$\gamma_{k_1} + \ldots + \gamma_{k_r} \leq^{\text{lex}} \alpha_{i_1} + \ldots + \alpha_{i_r} + \beta_{j_1} + \ldots \beta_{j_r} .$$

For the Hermitian case this appears in [7], [4].

So a geometric condition (nonempty intersection of the three Schubert varieties) implies a linear inequality between the eigenvalues of the three normal matrices A, B and $A+B$. We abreviate this inequality to

$$\Sigma\, \gamma_K \leq^{\text{lex}} \Sigma\, \alpha_I + \Sigma\, \beta_J .$$

For the *Hermitian* case, a recent paper by Klyachko [8] has shown that the inequalities arising from all such geometric conditions actually yield a complete list of restrictions for the eigenvalues of a sum of two Hermitian matrices in terms of the eigenvalues of the summands. For recent surveys on this see [3], [9].

Klyachko's results, coupled with the combinatorial work of Knutson and Tao [10], imply the classical Horn conjecture [6] on eigenvalues of Hermitian matrices, which we now recall.

For two real ordered spectra α and β, denote by $E(\alpha, \beta)$ the set of all possible ordered spectra of sums of two Hermitian matrices with spectra α and β. For each r-tuple $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \ldots < i_r \leq n$ define

$$\rho(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1).$$

Then Horn's conjecture, now proved, can be presented as the following recursive description of the set E:

$$E(\alpha,\beta) = \{\gamma :\, \Sigma\,\gamma \,=\, \Sigma\,\alpha \,+\, \Sigma\,\beta \;\text{ and}$$

$$\Sigma\,\gamma_K \,\leq\, \Sigma\,\alpha_I \,+\, \Sigma\,\beta_J \;\text{ whenever }\; \rho(K) \,\in\, E[\rho(I),\rho(J)]\,,\; 1 \leq r < n\}\,.$$

By the Schubert calculus (see for example [4]), the geometric condition $\Omega_K(G) \,\cap\, \Omega_{I'}(E') \,\cap\, \Omega_{J'}(F') \,\neq\, \emptyset$ is equivalent to $\rho(K) \,\in\, LR[\rho(I),\rho(J)]$, meaning that the r-tuple $\rho(K)$ can be obtained from $\rho(I)$ and $\rho(J)$ using the combinatorial Littlewood-Richardson rule. From the results in [8] and [10] it turns out that it is also equivalent to $\rho(K) \,\in\, E[\rho(I),\rho(J)]$.

Return now to normal matrices A with spectrum α, B with spectrum β and $A+B$ with spectrum γ, with notations as above. As we have seen, the condition $\Omega_K(G) \,\cap\, \Omega_{I'}(E') \,\cap\, \Omega_{J'}(F') \neq \emptyset$ implies $\Sigma\,\gamma_K \,\leq^{\text{lex}}\, \Sigma\,\alpha_I \,+\, \Sigma\,\beta_J$. Therefore, bearing in mind the results quoted, we can now state:

Theorem 5. *For $1 \leq r < n$, whenever one has $\rho(K) \,\in\, E[\rho(I),\rho(J)]$, the inequality*

$$\Sigma\,\gamma_K \,\leq^{\text{lex}}\, \Sigma\,\alpha_I \,+\, \Sigma\,\beta_J$$

holds for the eigenvalues of the normal matrices A, B and $A+B$.

And the same, of course, for any order of the type $\leq^{\text{lex}}_{\theta}$.

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