FINITE PREORDERS AND TOPOLOGICAL DESCENT II: ÉTALE DESCENT

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Abstract

It is known that every effective (global-) descent morphism of topological spaces is an effective étale-descent morphism. On the other hand in the predecessor of this paper we gave examples of:

- a descent morphism that is not an effective étale-descent morphism;
- an effective étale-descent morphism that is not a descent morphism.

Both of the examples in fact involved only finite topological spaces, i.e. just finite preorders, and now we characterize the effective étale-descent morphisms of preorders/finite topological spaces completely.

0 Introduction

Our main purpose in [1] was to show that essentially all the results of topological descent theory can be motivated by their finite instances, which become very simple and natural as soon as they are expressed in the language of finite preorders. Now we are making a next step in this direction by solving the finite version of the very hard (unsolved) problem of characterizing the effective étale-descent morphisms. It is not yet clear to us how to extend our result to the infinite spaces, but at least it shows where is the difficulty: roughly speaking, instead of working with the "two-step convergence" as J. Reiterman and W. Tholen [4] one would have to deal with certain equivalence relation for all "finite chains of convergences". We can also say that now we really see how far are the effective étale-descent morphisms from the descent morphisms and from the effective descent morphisms.

We will freely use the terminology and notation from [1] (see also [2] and [3]), and in particular the standard diagram

$$\begin{tikzcd} \mathbf{E}(B) \arrow[rr, "p^*"] \arrow[dr, "K^p"'] & & \mathbf{E}(E) \\ & Des_{\mathbf{E}}(p) \arrow[ur, "U^p"'] & \end{tikzcd} \tag{0.1}$$

in which:

- $p : E \to B$ is a continuous map of finite topological spaces, or equivalently a morphism of finite preorders;

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- **E** the class of étale maps (=local homeomorphisms) of finite topological spaces, and accordingly $\mathbf{E}(B)$ and $\mathbf{E}(E)$ are the categories of finite étale bundles over B and E respectively;

- $Des_E(p)$ the category of étale-descent data for p, and p^*, U^p, and K^p the pullback functor along p, the forgetful, and the comparison functor respectively.

Let us however recall from [1]:

- The morphism $p : E \to B$ being a morphism of preorders can also be considered as a functor from the category E to the category B - and then the functor $p^* : \mathbf{E}(B) \to \mathbf{E}(E)$ can be identified (up to an equivalence) with the functor $\mathbf{Sets}^{p^{op}} : \mathbf{Sets}^{B^{op}} \to \mathbf{Sets}^{E^{op}}$, which sends a functor $B^{op} \to \mathbf{Sets}$ to its composite with the functor $p^{op} : E^{op} \to B^{op}$ induced by p.

- Accordingly the category $Des_E(p)$ is to be replaced by an equivalent one, namely by the category $\mathbf{X}$ of pairs (X, ξ), where X is a functor from E^{op} to $\mathbf{Sets}$, and $\xi = (\xi_{e,e'})$ a family of maps $\xi_{e,e'} : X(e) \to X(e')$, defined whenever $p(e) = p(e')$ (for e and e' in E), with

$$\xi_{e',e''}\xi_{e,e'} = \xi_{e,e''}, \xi_{e,e} = 1_{X(e)}, \xi_{e,e'}X(e,\bar{e}) = X(e',\bar{e}')\xi_{\bar{e},\bar{e}'} \tag{0.2}$$

whenever $p(e) = p(e') = p(e''), p(\bar{e}) = p(\bar{e}'), e \to \bar{e}$, and $e' \to \bar{e}$ in E.

- Finally, the diagram (0.1) transforms into

$$\begin{tikzcd} \mathbf{Sets}^{B^{op}} \arrow[rr, "\mathbf{Sets}^{p^{op}}"] \arrow[dr, "k^p"'] & & \mathbf{Sets}^{E^{op}} \\ & \mathbf{X} \arrow[ur, "u^p"'] & \end{tikzcd} \tag{0.3}$$

where u^p is again the appropriate forgetful functor, and k^p is defined by

$$k^p(A) = (Ap^{op}, 1) \tag{0.4}$$

denoting by 1 the family of identity morphisms $1_{e,e'}$ of $A(p(e)) = A(p(e'))$ for all e, e' in E with $p(e) = p(e')$.

In particular we have

Proposition 0.1 *The morphism p is an effective étale-descent morphism if and only if the functor k^p above is a category equivalence.*

That is, our problem of characterizing the effective étale-descent morphisms of finite topological spaces becomes a purely categorical one, and we simplify it further using double categories, and then give a simple solution in Section 1. The additional remarks made in Section 2 should help to clarify the relationship between étale and global descent.

1 Characterization of effective étale-descent morphisms

For an arbitrary category C, let $S(C)$ be the double category of commutative squares in C. Recall that for such a square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

the horizontal domain and codomain are g and h respectively, and the vertical ones are f and k. In particular the sets $S(C)_{10}$ and $S(C)_{01}$ of vertical and horizontal arrows in $S(C)$ are the same.

The functor $S : \mathbf{Cat} \to \mathbf{DoubleCat}$ has a left adjoint Z, which can be described as follows: For a double category D we take

- D_0 to be the discrete category with objects as in D;
- D_h and D_v to be the categories with the same objects and the morphisms to be respectively the horizontal and vertical arrows of D;
- D_+ the pushout in $\mathbf{Cat}$ of the embeddings $D_0 \to D_h$ and $D_0 \to D_v$;

after that, for every square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "s"] & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

in D, the pairs (h, f) and (k, g) become morphisms in D_+ from a to d, and we construct $Z(D)$ as the quotient category $D + / \sim$ under the smallest congruence $\sim$ for which $(h, f) \sim (k, g)$ for all such pairs (h, f) and (k, g).

Note that every morphism in $Z(D)$ can be presented as the equivalence class of a morphism in D_+, say, of the form

$$\begin{tikzcd} a_0 \arrow[r, "f_1"] & b_1 \arrow[d, "g_1"'] & & & \\ & a_1 \arrow[r, "f_2"] & b_2 \arrow[d, "g_2"'] & & \\ & & \cdots \arrow[d, "g_n"'] & & \\ & & a_n \arrow[r, "f_{n+1}"] & b_{n+1} \end{tikzcd} \tag{1.1}$$

where $f_1, \ldots, f_{n+1}$ and $g_1, \ldots, g_n$ are $n+1$ horizontal and n vertical arrows in D respectively. This suggests to call $Z(D)$ the category of zigzags in D.

Let $\mathbf{C}$ be a category with pullbacks and D an internal category in $\mathbf{C}$. For a (pseudo-) functor $F : \mathbf{C}^{op} \to \mathbf{Cat}$, one defines a category F^D of internal actions of D in F, and the category $Des_E(p)$ above is a special case of such an F^D with $D = Eq(p)$, the equivalence relation on E determined by p - see [3] for details. For $\mathbf{C} = \mathbf{Cat}$ it is easy to see that:

- an internal category in $\mathbf{C}$ is nothing but a double category (a well-known fact!);
- for a functor of the form $C^? : \mathbf{Cat}^{op} \to \mathbf{Cat}$ (where C is an arbitrary category) an internal action of D in $C^?$ is nothing but a morphism of double categories from D to $S(C)$, and therefore the same as a functor from $Z(D)$ to C.

Moreover, it is then easy to conclude that the diagram (0.3) can be identified with

$$\begin{tikzcd}
\mathbf{Sets}^{B^{op}} \arrow[rr, "\mathbf{Sets}^{p^{op}}"] \arrow[dr, "\mathbf{Sets}^{\varphi^{op}}"'] & & \mathbf{Sets}^{E^{op}} \\
& \mathbf{Sets}^{Z(Eq(p))^{op}} \arrow[ur, "\mathbf{Sets}^{\psi^{op}}"'] &
\end{tikzcd} \tag{1.2}$$

where $p = \varphi\psi$ is what one might call a canonical factorization of p. Explicitly, $\varphi : Z(Eq(p)) \to B$ is the unique functor that coincides with p on objects, and $\psi : E \to Z(Eq(p))$ is the unique functor with $\psi(e) = e$ on objects.

Theorem 1.1 *The following conditions are equivalent:*

(a) the morphism $p : E \to B$ is étale conservative in the sense of [2], i.e. the comparison functor $K^p : \mathbf{E}(B) \to Des_E(p)$ is conservative;

(b) K^p is faithful;

(c) $p : E \to B$ is essentially surjective on objects (i.e. for every element b in B there exists an element e in E with $p(e) \to b$ and $b \to p(e)$).

Proof. According to the previous remarks we can replace $K^p : \mathbf{E}(B) \to Des_E(p)$ by k^p, i.e. by $\mathbf{Sets}^{\varphi^{op}} : \mathbf{Sets}^{B^{op}} \to \mathbf{Sets}^{Z(Eq(p))^{op}}$, which makes $(a) \Leftrightarrow (b) \Leftarrow (c)$ obvious, and gives $(b) \Rightarrow (c)$ since B is a preorder. Let us only sketch the proof of $(b) \Rightarrow (c)$:

We observe that any B-indexed family $S = (S_b, s_b)_{b \in B}$ of pointed sets with $S_b = S_{b'}$ whenever $b \to b'$ and $b' \to b$ in B, yields a functor $\check{S} : B^{op} \to \mathbf{Sets}$ with $\check{S}(b) = S_b$ and

$$\check{S}(b \to b') = \begin{cases} \text{the identity map of } S_b \text{ if } b' \to b \text{ in } B; \\ \\ \text{the constant map } S_{b'} \to S_b \text{ with the image } \{s_b\}, \text{ if not.} \end{cases} \tag{1.3}$$

Moreover, every map $S \to T$ of such families determines a natural transformation $\check{S} \to \check{T}$, and if the condition (c) does not hold it is easy to find two such natural trasformations σ and σ' with $\sigma \neq \sigma'$ and $\sigma\varphi^{op} = \sigma'\varphi^{op}$ (say, whenever each S_b and each T_b has at least two elements) - in contradiction with (b).

The same result can be quickly deduced from the observations made in [2] - which is not at all the case for

Theorem 1.2 *The morphism $p : E \to B$ is an effective étale-descent morphism if and only if the functor $\varphi : Z(Eq(p)) \to B$ is a category equivalence.*

Proof. Replacing again $K^p : \mathbf{E}(B) \to Des_E(p)$ by k^p (or, say, using Proposition 0.1) we see that the morphism p is an effective étale-descent morphism if and only if the functor k^p, i.e. the functor $\mathbf{Sets}^{\varphi^{op}} : \mathbf{Sets}^{B^{op}} \to \mathbf{Sets}^{Z(Eq(p))^{op}}$, is a category equivalence. After that we can use standard arguments:

- if $\mathbf{Sets}^{\varphi^{op}} : \mathbf{Sets}^{B^{op}} \to \mathbf{Sets}^{Z(Eq(p))^{op}}$ is an equivalence, then so is its left adjoint $L : \mathbf{Sets}^{Z(Eq(p))^{op}} \to \mathbf{Sets}^{B^{op}}$;

- since L composed with the Yoneda embedding of $Z(Eq(p))$ is isomorphic to the Yoneda embedding of B composed with $\varphi : Z(Eq(p)) \to B$, we then conclude that φ is full and faithful;

- together with Theorem 1.1 this tells us that if $\mathbf{Sets}^{\varphi^{\mathbf{op}}}$ is an equivalence, then so is φ – and

- the converse is trivial.

Corollary 1.3 *The morphism $p : E \to B$ is an effective étale-descent morphism if and only if the following conditions hold:*

(a) the map $p : E \to p(E)$ induced by p is a quotient map;

(b) $Z(Eq(p))$ is a preorder;

(c) $p : E \to B$ is essentially surjective on objects.

Proof. Just observe that the conditions (a) and (b) hold if and only if the functor $\varphi : Z(Eq(p)) \to B$ is full and faithful respectively.

2 Additional remarks

Let us consider again the zigzag (1.1), and call it now an n-zigzag $a_0 \to b_{n+1}$ in $Eq(p)$ if $D = Eq(p)$. In this case the morphisms $f_1, \ldots, f_{n+1}$ and $g_1, \ldots, g_n$ are uniquely determined by their domains and codomains, and it is convenient to write

$$f_i = [a_{i-1}, b_i], \quad g_i = (b_i, a_i) \quad \text{(for each appropriate } i). \tag{2.1}$$

With this notation the category $Z(Eq(p))$ is to be described as the quotient category $Eq(p)+/\sim$ under the smallest congruence $\sim$ for which

$$(e', \bar{e}')[e, e'] \sim [\bar{e}, \bar{e}'](e, \bar{e}) \tag{2.2}$$

whenever $p(e) = p(e'), p(\bar{e}) = p(\bar{e}'), e \to \bar{e}$, and $e' \to \bar{e}'$ in E.

Proposition 2.1 *For every two elements e and e' in E every two 1-zigzags $e \to e'$ in $Eq(p)$ are equivalent (under $\sim$).*

Proof. Let $(e \to x \to y \to e') = [y, e'](x, y)[e, x]$ and be $(e \to u \to v \to e') = [v, e'](u, v)[e, u]$ be 1-zigzags. Applying (2.1) twice we obtain
$[y, e'](x, y)[e, x] = [v, e'][y, v](x, y)[e, x] \sim [v, e'](u, v)[x, u][e, x] \sim [v, e'](u, v)[e, u]$.

Since any n-zigzag can obviously be presented as a composite of n 1-zigzags, we conclude

Corollary 2.2 *If every* 2*-zigzag in* $Eq(p)$ *is equivalent to a* 1*-zigzag, then* $Z(Eq(p))$ *is a preorder.*

Let us now recall from [1, Proposition 3.4] that p is an effective descent morphism if and only if it is surjective on composable pairs, i.e. for every $b_0 \to b_1 \to b_2$ in B there exists $e_0 \to e_1 \to e_2$ in E with $p(e_i) = b_i (i = 0, 1, 2)$. We have

Proposition 2.3 *If* p *is surjective on composable pairs, then every* 2*-zigzag in* $Eq(p)$ *is equivalent to a* 1*-zigzag.*

Proof. For a 2-zigzag $(e \to x \to y \to z \to t \to e') = [t, e'](z, t)[y, z](x, y)[e, x]$ in $Eq(p)$ we take b_0 to be $p(e) = p(x), b_1 = p(y) = p(z)$, and $b_2 = p(t) = p(e')$; then $b_0 \to b_1 \to b_2$ in B, and we choose $e_0 \to e_1 \to e_2$ in E as above. After that we have $[t, e'](z, t)[y, z](x, y)[e, x] = [t, e'](z, t)[e_1, z][y, e_1](x, y)[e, x] \sim [t, e'][e_2, t]\ (e_1, e_2)(e_0, e_1)[x, e_0][e, x] \sim [e_2, e'](e_0, e_2)[e, e_0]$ as desired.

As shown in [2], every effective descent morphism is an effective étale-descent morphism, and it is natural to ask if this fact (restricted to finite spaces of course!) can easily be deduced from our characterizations of those classes of morphisms, namely from [1, Proposition 3.4] and Corollary 1.3. The affirmative answer clearly follows from Proposition 2.3 and Corollary 2.2. On the other hand using Corollary 1.3 one could simplify our arguments in [1] on counter-examples for "descent implies effective étale-descent" and for the converse, and easily construct many other counter-examples.

References

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