Regularity in Sobolev Spaces for Doubly Nonlinear Parabolic Equations

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Abstract

The doubly nonlinear parabolic equation

$$u_t = \operatorname{div}\Big[|\nabla(|u|^{m-1}u)|^{p-2}\nabla(|u|^{m-1}u)\Big] \qquad (m>1,\ m(p-1)>1)$$

is considered in several dimensions and regularity results in fractional order Sobolev spaces are obtained. The main tools in the proof are a difference quotient technique and the imbedding theorem of Nikolskii spaces into Sobolev spaces.

1. Introduction.

We will be concerned with the Cauchy problem

$$\begin{cases} u_t = \operatorname{div}\Big[|\nabla(|u|^{m-1}u)|^{p-2}\nabla(|u|^{m-1}u)\Big] & \text{in} \quad \mathbb{R}^n \times (0,T] \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ for some $n \geq 2$, $t \in [0,T]$ for some $T < \infty$, $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and m,p are fixed constants such that $m > 1$ and $m(p-1) > 1$.

Equation (1.1) has a double nonlinearity. For $p = 2$ it is the porous medium equation

$$u_t = \Delta u^m \qquad (m > 1),$$

and the case $m = 1$ corresponds to the (degenerate) parabolic p-Laplace equation

$$u_t = \operatorname{div}[|\nabla u|^{p-2}\nabla u] \qquad (p > 2).$$

These two limit cases are prototypes for the main features presented by the solutions of (1.1) and are extensively studied in the literature (see, e.g., [1, 22, 24] for the

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porous medium equation and [9, 20] for the p-Laplacian). Despite this fact, regularity results in Sobolev spaces are in general not available and only recently (see [11]) a first contribution was put forward for the porous medium equation. Results of this type have an interest in their own right but are also very useful for numerical purposes since they provide detailed informations about the singularities of the solutions and this knowledge may be used to develop efficient numerical schemes.

The aim of this paper is to obtain regularity results in fractional order Sobolev spaces for certain powers of the solution of the problem. The main ingredients in the proof are the difference quotient technique as developed in [10, 12] and the *smoothing property*

$$u_t \geq -\frac{u}{[(p-1)m-1]\,t} \ ,$$

from which follows the crucial estimate

$$\|u_t\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{[(p-1)m-1]\,t}\,\|u_0\|_{L^1(\mathbb{R}^n)} \,.$$

This smoothing property is in itself interesting since it generalizes previous similar results for the porous medium equation (see [2]) and the p-Laplacian equation (see [15]); it is proven in [13]. A proof in the one-dimensional case can be found in [14]. Other results concerning the regularity of solutions can be found in [16], where L^p–properties are obtained (see also [4] for the porous medium equation). Moreover, the Hölder continuity of the solution is proven in [18, 23, 25].

Another interesting feature of the problem is the appearance of a free boundary. Assuming that spt u_0 is bounded, this free boundary is the set $\partial[\text{spt}\ u(\cdot,t)]$ and for the porous medium equation it is well known that it has a finite speed of propagation (cf. [7]). In this more general case, this property for $m(p-1)>1$ and the regularity of the free boundary is the object of current research by the authors.

We conclude this introduction by presenting the definition of a weak solution and stating the main results of the paper. We assume that $0 \leq u_0 \in L^\infty(\mathbb{R}^n)$ and spt u_0 is bounded and let $m>1$ and $m(p-1)>1$. Introducing the functions

$$\begin{aligned} b(z) &= |z|^{m-1}z \qquad \text{for } z \in \mathbb{R}, \\ a_i(s) &= |s|^{p-2}s_i \qquad \text{for } s \in \mathbb{R}^n \ , \quad i=1,\ldots,n \ , \end{aligned}$$

we may rewrite equation (1.1) in the form

$$\frac{\partial}{\partial t}u(x,t) = \sum_{i=1}^{n} \partial_i a_i(\nabla b(u(x,t))) \qquad \text{in } \mathbb{R}^n \times (0,T],$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

Definition: We say that $u(x,t)$ is a weak solution of (1.1) if

$$u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \; ; \quad b(u) \in L^p(0,T;W^{1,p}(\mathbb{R}^n)) \; ; \tag{1.2}$$

$$-\int_0^T\int_{\mathbb{R}^n} u\,\phi_t \,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^n} a_i(\nabla b(u))\,\partial_i\phi \,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^n} u_0\,\phi(0)\,\mathrm{d}x \; , \tag{1.3}$$

for all $\phi \in L^p(0,T;W^{1,p}(\mathbb{R}^n)) \cap W^{1,1}(0,T;L^1(\mathbb{R}^n))$ such that $\phi(T) \equiv 0$.

Remark: i) The existence of a solution for the problem in the sense of the previous definition is a well known fact (cf., e.g., [16]). For existence results concerning related equations see [5, 6, 19]. It is also known that the solution is Hölder continuous and (see [16])

$$\left(u^{\frac{m+1}{2}}\right)_t \in L^2(0,T;L^2(\mathbb{R}^n)) \; . \tag{1.4}$$

ii) Applying a comparison theorem (see, e.g., [6, 17]) we find that $0 \le u \le \|u_0\|_{L^\infty}$. Moreover, spt $u(x,T)$ is bounded. In fact, let us consider the selfsimilar Barenblatt solutions (cf. [3])

$$u^*(x,t;\alpha,\tau) = (t+\tau)^{-\frac{1}{\mu}} \left[\alpha - k\left(|x|(t+\tau)^{-\frac{1}{n\mu}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{m(p-1)-1}} ,$$

where $\mu = m(p-1) - 1 + \frac{p}{n}$, $k = \frac{m(p-1)-1}{mp}(n\mu)^{-\frac{1}{p-1}}$, and $\alpha, \tau > 0$. Clearly, spt $u^*(x,T;\alpha,\tau)$ is bounded for each $T > 0$. Let $0 \le u_0 \le u_0^*$ and $T > 0$. From the comparison theorem it follows that $u \le u^*$ on $\mathbb{R}^n \times [0,T]$. Thus, spt $u(x,t) \subset$ spt $u^*(x,T;\alpha,\tau)$ for all $t \in [0,T]$.

The main results of this paper deal with the question of the regularity in fractional order Sobolev spaces. The first theorem concerns the degenerate case $p > 2$.

Theorem 1.1: Let $p \ge 2$ and u be a weak solution of (1.1) in the sense of the previous definition. Then, for all $\varepsilon > 0$ and $q < \frac{p^2}{p+1}$, we have

$$b(u) \in L^q(0,T;W^{1+\frac{1}{p}-\varepsilon,q}(\mathbb{R}^n)) \; . \tag{1.5}$$

Remark: i) Recall $\nabla b(u) \in L^p(0,T;L^p(\mathbb{R}^n))$; using (1.5) and the Sobolev imbedding theorem we obtain a better space integrability for $p > n$; in fact,

$$\nabla b(u) \in L^q(0,T;L^s(\mathbb{R}^n)) \qquad \text{for all } \; s < \frac{np^2}{np+n-p} \;\text{ and }\; q < \frac{p^2}{p+1}\ .$$

ii) Assertion (1.5) is proven in [11] in the case $p = 2$, i.e., for the porous medium equation.

Now let us consider the singular case $1 < p < 2$.

Theorem 1.2: Let $1 < p < 2$ and u be a weak solution of (1.1) in the sense of the previous definition. Then, for all $\varepsilon > 0$ and $q < \frac{2}{3}p$, we have

$$b(u) \in L^q(0,T;W^{\frac{3}{2}-\varepsilon,q}(\mathbb{R}^n)). \tag{1.6}$$

Our last result is valid for all $p > 1$.

Theorem 1.3: Let $p > 1$ and $\frac{p}{2} \le \alpha < p - \frac{1}{2}$. Let u be a weak solution of (1.1) in the sense of the previous definition. Then, for all $1 \le q < \frac{2p}{2\alpha+1}$ and $\varepsilon > 0$, we have

$$|\nabla b(u)|^\alpha \in L^q(0,T;W^{\frac{1}{2}-\varepsilon,q}(\mathbb{R}^n))\ . \tag{1.7}$$

Remark: Let us note that (1.7) is a regularity result for $|\nabla b(u)|^{\frac{p}{2}}$, if $p > 1$, and for $|\nabla b(u)|^{p-1}$, if $p \ge 2$.

In the sequel let us write $\sum_{i,j,k}$ instead of $\sum_{i,j,k=1}^n$. Moreover, let c be a generic constant which may vary from line to line.

2. The basic estimates.

We denote the usual Sobolev spaces by $W^{s,p}(\mathbb{R}^n)$ and will consider also the Nikolskii spaces $\mathcal{N}^{s,p}(\mathbb{R}^n)$, defined as follows (cf. [21]). Let k be an integer, $0 < \sigma < 1$, $s = k + \sigma$, $z \in \mathbb{R}^n$, and $1 \le p < \infty$. The space $\mathcal{N}^{s,p}(\mathbb{R}^n)$ consist of all functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the norm

$$\|f\|_{\mathcal{N}^{s,p}(\mathbb{R}^n)} = \left(\|f\|^p_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|=k} \sup_{\substack{\delta>0 \\ 0<|z|<\delta}} \int_{\mathbb{R}^n} \frac{|\partial^\alpha f(x+z) - \partial^\alpha f(x)|^p}{|z|^{\sigma p}}\, dx \right)^{\frac{1}{p}}$$

is finite. The Nikolskii spaces are very close to the Sobolev spaces $W^{s,p}(\mathbb{R}^n)$. Since s is not an integer, there hold the following imbeddings (cf. [21]):

$$W^{s,p}(\mathbb{R}^n) \to \mathcal{N}^{s,p}(\mathbb{R}^n)$$

and

$$\mathcal{N}^{s,p}(\mathbb{R}^n) \to W^{s-\varepsilon,p}(\mathbb{R}^n) \qquad \text{for all } \varepsilon > 0.$$

In what follows, let $0 < h < 1$ and $\zeta \in \mathbb{R}^n$ be a unit vector, i.e., $|\zeta| = 1$. We set

$$E_\zeta^h f(x) = f(x + h\zeta), \qquad E_\zeta^{-h} f(x) = f(x - h\zeta),$$

and define the differences

$$\Delta_\zeta^h f(x) = f(x + h\zeta) - f(x), \qquad \Delta_\zeta^{-h} f(x) = f(x - h\zeta) - f(x).$$

Next, we introduce the function ψ. Let $\alpha > 0$ be small and set

$$\psi(s) = \begin{cases} 1 - (1+s)^{-\alpha} & \text{if} \quad s \geq 0, \\ -1 + (1-s)^{-\alpha} & \text{if} \quad s < 0. \end{cases}$$

It's clear that $|\psi| \leq 1$ and $\psi'(s) = \alpha(1 + |s|)^{-1-\alpha}$, thus, $0 < \psi'(s) < \alpha$, for all $s \in \mathbb{R}$. For simplicity we will use the notations

$$\psi_{\pm h} = \psi(h^{-1}\Delta_\zeta^{\pm h} b(u)) \qquad \text{and} \qquad \psi'_{\pm h} = \psi'(h^{-1}\Delta_\zeta^{\pm h} b(u)) \ . \tag{2.1}$$

Finally, for each $\epsilon > 0$, we define in $[0, T]$ the nonnegative smooth function

$$\mu_\varepsilon(t) := t^\varepsilon \, (T - t)^\varepsilon \ . \tag{2.2}$$

A crucial ingredient in the sequel is the following *smoothing property*, that expresses a regularizing effect, and its corollary. We state this property without a proof; for details, see [13].

Lemma 2.1: *Let u be a non negative weak solution of (1.1). Then, in the distribution sense,*

$$u_t \geq -\frac{u}{[(p-1)m - 1]\, t} \ . \tag{2.3}$$

Corollary 2.2: *Under the assumptions of the previous lemma the following estimate holds*

$$\|u_t\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{[(p-1)m - 1]\, t} \, \|u_0\|_{L^1(\mathbb{R}^n)} \, . \tag{2.4}$$

Proof: We just sketch the proof for smooth u; more details can be found in [13]. Let $u_t = (u_t)^+ - (u_t)^-$. Thus, $|u_t| = (u_t)^+ + (u_t)^-$. We can show that

$$\|u(x,t)\|_{L^1(\mathbb{R}^n)} \le \|u_0\|_{L^1(\mathbb{R}^n)} \ , \qquad \text{a.e. } t>0 \ ,$$

from which follows that

$$\partial_t \int_{\mathbb{R}^n} |u| = \partial_t \int_{\mathbb{R}^n} u = \int_{\mathbb{R}^n} u_t \le 0.$$

Thus, we have

$$\int_{R^n} (u_t)^+ \le \int_{R^n} (u_t)^-$$

and

$$\int_{R^n} |u_t| = \int_{R^n} (|(u_t)^+| + |(u_t)^-|) \le 2\int_{R^n} |(u_t)^-|.$$

Due to (2.3), we get the assertion. □

The aim of this section is to prove the following lemma. It provides a weighted estimate of $|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u(x,t))|^2$, if $p \ge 2$. Further, we will treat the case $p<2$, as well; see Lemma 2.4 below.

Lemma 2.3: *Let u be a weak solution of (1.1) and $p \ge 2$. For each $\varepsilon > 0$ and each unit vector $\zeta \in \mathbb{R}^n$, there is a constant c, depending only on ε and the data, such that*

$$\sup_{0<h<1} \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon(t)\, \omega(x,t)\, \psi_h'(x,t) \left| h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u(x,t)) \right|^2 dxdt \le c, \tag{2.5}$$

where

$$\omega(x,t) = \int_0^1 |\tau \nabla b(u(x+h\zeta,t)) + (1-\tau)\nabla b(u(x,t))|^{p-2}\, d\tau$$

and $\psi_h'(x,t)$ and $\mu_\varepsilon(t)$ are as in (2.1) and (2.2).

Proof: Let $s \in \mathbb{R}^n$, $A(s) = \frac{1}{p}|s|^p$, $A_i(s) = \frac{\partial}{\partial s_i} A(s)$, and $A_{i,k}(s) = \frac{\partial}{\partial s_k} A_i(s)$. Let us note that $A_i(s) = |s|^{p-2}s_i$ and $A_{i,k}(s) = (p-2)|s|^{p-4}s_i s_k + |s|^{p-2}\delta_{ik}$. Thus, utilizing the Taylor expansion of $A(s)$ we find

$$\begin{aligned} A(s') - A(s) &= \sum_i (s'-s)_i A_i(s) \\ &\quad + \sum_{i,k} (s'-s)_i (s'-s)_k \int_0^1 (1-\tau) A_{i,k}(\tau s' + (1-\tau)s)\, d\tau \\ &\ge \sum_i (s'-s)_i A_i(s) \\ &\quad + |s'-s|^2 \int_0^1 (1-\tau)\, |\tau s' + (1-\tau)s|^{p-2}\, d\tau. \end{aligned} \tag{2.6}$$

We set $s = \nabla b(u)$, $s' = E_\zeta^h \nabla b(u)$, and define

$$\omega_h = \int_0^1 (1-\tau)\left|\tau E_\zeta^h \nabla b(u) + (1-\tau)\nabla b(u)\right|^{p-2} d\tau.$$

Multiplying inequality (2.6) by $\mu_\varepsilon h^{-1}\psi'(h^{-1}\Delta_\zeta^h b(u)) \equiv \mu_\varepsilon h^{-1}\psi'_h$ and integrating over $\mathbb{R}^n \times [0,T]$ we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \omega_h \psi'_h h^{-1}\left|\Delta_\zeta^h \nabla b(u)\right|^2 dxdt \\
&\quad \leq \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1} \Delta_\zeta^h A(\nabla b(u)) dxdt \\
&\qquad - \sum_i \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1}\Delta_\zeta^h \partial_i b(u) A_i(\nabla b(u)) dxdt. \qquad (2.7)
\end{aligned}$$

Next, we take as test function in equation (1.3) $\phi = \mu_\varepsilon \psi(h^{-1}\Delta_\zeta^h b(u)) \equiv \mu_\varepsilon \psi_h$. Due to (1.4), we have that

$$\left[(b(u))_t\right]^2 = \left(\frac{2m}{m+1}\right)^2 u^{m-1}\left[(u^{\frac{m+1}{2}})_t\right]^2$$

belongs to L^1 and so this is an admissible test function. We obtain

$$-\int_0^T\int_{\mathbb{R}^n} u\,(\mu_\varepsilon\psi_h)_t + \sum_i \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon a_i(\nabla b(u))\partial_i \psi_h = 0$$

and, integrating by parts with respect to t,

$$\int_0^T\int_{\mathbb{R}^n} u_t \mu_\varepsilon \psi_h = -\sum_i \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon a_i(\nabla b(u))\psi'_h h^{-1}\Delta_\zeta^h \partial_i b(u). \qquad (2.8)$$

Let us note that $a_i(s) = A_i(s)$. Thus, due to (2.7) and (2.8), we find

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \omega_h \psi'_h \left|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u)\right|^2 \\
&\quad \leq \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1}\Delta_\zeta^h A(\nabla b(u)) + \int_0^T\int_{\mathbb{R}^n} u_t \mu_\varepsilon \psi_h. \qquad (2.9)
\end{aligned}$$

Likewise, we may test the equation by $\phi = \mu_\varepsilon \psi(h^{-1}\Delta_\zeta^{-h} b(u))$ and choose $s = \nabla b(u)$ and $s' = E_\zeta^{-h}\nabla b(u)$ in the Taylor expansion (2.6). This yields

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \omega_{-h} \psi'_{-h} \left|h^{-\frac{1}{2}}\Delta_\zeta^{-h} \nabla b(u)\right|^2 \\
&\quad \leq \int_0^T\int_{\mathbb{R}^n} \mu_\varepsilon \psi'_{-h} h^{-1}\Delta_\zeta^{-h} A(\nabla b(u)) + \int_0^T\int_{\mathbb{R}^n} u_t \mu_\varepsilon \psi_{-h}, \qquad (2.10)
\end{aligned}$$

where

$$\omega_{-h} = \int_0^1 (1-\tau) \left|\tau E_\zeta^{-h} \nabla b(u) + (1-\tau)\nabla b(u)\right|^{p-2} d\tau.$$

Now, we add the inequalities (2.9) and (2.10). Below, we will show that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega_h \psi_h' \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2 + \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega_{-h} \psi_{-h}' \left|h^{-\frac{1}{2}} \Delta_\zeta^{-h} \nabla b(u)\right|^2 \\ &= \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega \psi_h' \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2, \end{aligned} \tag{2.11}$$

where $\omega = \int_0^1 |\tau E_\zeta^h \nabla b(u) + (1-\tau)\nabla b(u)|^{p-2} d\tau$, and

$$\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi_h' h^{-1} \Delta_\zeta^h A(\nabla b(u)) + \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi_{-h}' h^{-1} \Delta_\zeta^{-h} A(\nabla b(u)) = 0. \tag{2.12}$$

Thus, adding (2.9) and (2.10) provides

$$\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega \psi_h' \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2 \le \int_0^T \int_{\mathbb{R}^n} u_t \mu_\varepsilon (\psi_h + \psi_{-h}). \tag{2.13}$$

Let us note that $\|\psi\|_{L^\infty(\mathbb{R}^n)} \le 1$ and $\|u_t\|_{L^1(\mathbb{R}^n)} \le ct^{-1}$ according to (2.4). Hence, the right-hand side of (2.13) is bounded by a constant of the type $c\,\epsilon^{-1}\,T^\epsilon$. This yields the assertion.

In order to prove (2.11) we consider

$$J_1 = \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega_h \psi_h' \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2 + \int_0^T \int_{\mathbb{R}^n} E_\zeta^h \left(\mu_\varepsilon \omega_{-h} \psi_{-h}' \left|h^{-\frac{1}{2}} \Delta_\zeta^{-h} \nabla b(u)\right|^2 \right).$$

Clearly, J_1 is equal to the left-hand side of (2.11). Now, let us remark that

$$\begin{aligned} E_\zeta^h \omega_{-h} &= \int_0^1 (1-\tau) \left|\tau \nabla b(u) + (1-\tau) E_\zeta^h \nabla b(u)\right|^{p-2} d\tau \\ &= \int_0^1 s \left|(1-s)\nabla b(u) + s E_\zeta^h \nabla b(u)\right|^{p-2} ds. \end{aligned}$$

Here, we have substituted τ by $1-s$. Thus, it follows that

$$E_\zeta^h \omega_{-h} + \omega_h = \int_0^1 (\tau + 1 - \tau) \left|\tau E_\zeta^h \nabla b(u) + (1-\tau)\nabla b(u)\right|^{p-2} d\tau = \omega.$$

Moreover, it holds that

$$E_\zeta^h \left|\Delta_\zeta^{-h} \nabla b(u)\right| = \left|\nabla b(u) - E_\zeta^h \nabla b(u)\right| = \left|\Delta_\zeta^h \nabla b(u)\right|$$

and

$$E^h_\zeta \psi'_{-h} = E^h_\zeta \psi'\left(\left|h^{-1}\Delta^{-h}_\zeta b(u)\right|\right) = \psi'\left(\left|h^{-1}\Delta^h_\zeta b(u)\right|\right) = \psi'_h.$$

Thus, we may conclude that

$$J_1 = \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega \psi'_h \left|h^{\frac{1}{2}}\Delta^h_\zeta \nabla b(u)\right|^2$$

and (2.11) is proven. Finally, due to

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} E^h_\zeta \left(\mu_\varepsilon \psi'_{-h} h^{-1}\Delta^{-h}_\zeta A(\nabla b(u))\right) \\
&\quad = \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1}\left[A(\nabla b(u)) - E^h_\zeta A(\nabla b(u))\right] \\
&\quad = -\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1}\Delta^h_\zeta A(\nabla b(u)),
\end{aligned}$$

we obtain

$$\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi'_h h^{-1}\Delta^h_\zeta A(\nabla b(u)) + \int_0^T \int_{\mathbb{R}^n} E^h_\zeta \left(\mu_\varepsilon \psi'_{-h} h^{-1}\Delta^{-h}_\zeta A(\nabla b(u))\right) = 0$$

which implies equality (2.12). □

Next, we consider the case that $p < 2$. Let us define

$$\omega = \begin{cases} 1 & \text{if } \; E^h_\zeta \nabla b(u) = \nabla b(u), \\ \displaystyle\int_0^1 \left|\tau E^h_\zeta \nabla b(u) + (1-\tau)\nabla b(u)\right|^{p-2} d\tau & \text{if } \; \text{not}. \end{cases} \tag{2.14}$$

Now we may proceed as in the proof of Lemma 2.3. Modifying also the definitions of the weights ω_h and ω_{-h}, we obtain the following analogue of Lemma 2.3.

Lemma 2.4: *Let u be a weak solution of (1.1) and $1 < p < 2$. For each $\varepsilon > 0$ and each unit vector $\zeta \in \mathbb{R}^n$ there is a constant c, depending only on ε and the data, such that*

$$\sup_{0<h<1} \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon(t)\, \omega(x,t)\, \psi'_h(x,t) \left|h^{-\frac{1}{2}}\Delta^h_\zeta \nabla b(u(x,t))\right|^2 dxdt \le c, \tag{2.15}$$

where $\omega(x,t)$ is as in (2.14).

3. Proofs of the main results.

Proof of Theorem 1.1: Applying Taylor's expansion we find

$$\begin{aligned} J_1 &:= \sum_i \big(A_i(E_\zeta^h \nabla b(u)) - A_i(\nabla b(u))\big)\big(E_\zeta^h \partial_i b(u) - \partial_i b(u)\big) \\ &= \sum_{i,k} \Delta_\zeta^h \partial_i b(u) \int_0^1 A_{i,k}\big(\tau E_\zeta^h \nabla b(u) + (1-\tau)\nabla b(u)\big)\, \Delta_\zeta^h \partial_k b(u)\, d\tau \\ &\leq c\,\omega \left|\Delta_\zeta^h \nabla b(u)\right|^2, \end{aligned} \tag{3.1}$$

where $\omega = \int_0^1 |\tau E_\zeta^h \nabla b(u) + (1-\tau)\nabla b(u)|^{p-2} d\tau$, since $|A_{i,k}(s)| \leq C(n,p)|s|^{p-2}$.

From the well-known inequality, valid for $p > 2$ (cf. [8]),

$$\exists c > 0 \ : \ \big(|x|^{p-2}x - |y|^{p-2}y\big)(x-y) \geq c|x-y|^p \ , \quad \forall x, y \in \mathbb{R}^n$$

it follows that

$$J_1 \geq c \left|E_\zeta^h \nabla b(u) - \nabla b(u)\right|^p \equiv c \left|\Delta_\zeta^h \nabla b(u)\right|^p. \tag{3.2}$$

Using (3.1) and (3.2) we get

$$\left|h^{-\frac{1}{p}} \Delta_\zeta^h \nabla b(u)\right|^p \leq c\,\omega \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2.$$

Thus, in view of (2.5) we have

$$\begin{aligned} &\sup_{0<h<1} \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi_h' \left|h^{-\frac{1}{p}} \Delta_\zeta^h \nabla b(u)\right|^p \\ &\quad \leq \ c \sup_{0<h<1} \int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \omega \psi_h' \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^2 \\ &\quad \leq \ c. \end{aligned} \tag{3.3}$$

Let $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. The Hölder inequality yields

$$\begin{aligned} J_2 &:= \int_0^T \int_{\mathbb{R}^n} \left|h^{-\frac{1}{p}} \Delta_\zeta^h \nabla b(u)\right|^{\frac{p}{q_3}} \\ &= \int_0^T \int_B (\mu_\varepsilon \psi_h')^{-\frac{1}{q_3}} \left(\mu_\varepsilon \psi_h' \left|h^{-\frac{1}{p}} \Delta_\zeta^h \nabla b(u)\right|^p\right)^{\frac{1}{q_3}} \\ &\leq \left[\int_0^T \int_B (tT - t^2)^{-\frac{\varepsilon q_1}{q_3}}\right]^{\frac{1}{q_1}} \left[\int_0^T \int_B {\psi_h'}^{-\frac{q_2}{q_3}}\right]^{\frac{1}{q_2}} \left[\int_0^T \int_B \mu_\varepsilon \psi_h' \left|h^{-\frac{1}{p}} \Delta_\zeta^h \nabla b(u)\right|^p\right]^{\frac{1}{q_3}}, \end{aligned} \tag{3.4}$$

where $B \subset \mathbb{R}^n$ is a ball such that $\left(\operatorname{spt} u(x,T) \cup \operatorname{spt} E_\zeta^h u(x,T)\right) \subset B$. Now, we estimate the integrals on the right-hand side of (3.4). We set $q_1 = \frac{p+1+\delta}{2\varepsilon p}$, $q_2 = \frac{p+1+\delta}{1+\varepsilon}$, $q_3 = \frac{p+1+\delta}{p}$, and $\delta = (2p+1)\varepsilon$. Notice that $q_1 = \frac{q_3}{2\varepsilon}$ and so the first integral on the right-hand side of (3.4) is bounded. Next, let us note that

$$\psi_h' = \alpha \left(1 + \left|h^{-1}\Delta_\zeta^h b(u)\right|\right)^{-1-\alpha}.$$

Let us put $\alpha = \varepsilon$. Noting that $q_2 = \frac{pq_3}{1+\varepsilon}$ we get

$$\int_0^T\int_B {\psi_h'}^{-\frac{q_2}{q_3}} = \varepsilon^{-\frac{p}{1+\varepsilon}} \int_0^T\int_B \left(1 + \left|h^{-1}\Delta_\zeta^h b(u)\right|\right)^p \le c_\varepsilon.$$

Further, utilizing (3.3) it follows that the third integral on the right-hand side of (3.4) is bounded. Altogether, we have shown that J_2 is bounded. We may conclude that

$$\sup_{\substack{\zeta\in\mathbb{R}^n\\ |\zeta|=1}} \sup_{0<h<1} \int_0^T\int_{\mathbb{R}^n} \left|h^{-\frac{1}{p}}\Delta_\zeta^h \nabla b(u)\right|^{\frac{p^2}{p+1+\delta}} \le c.$$

This implies that

$$b(u) \in L^{\frac{p^2}{p+1+\delta}}(0,T;\mathcal{N}^{1+\frac{1}{p},\frac{p^2}{p+1+\delta}}(\mathbb{R}^n)).$$

The imbedding theorem of Nikolskii spaces into Sobolev spaces (cf. [21])

$$\mathcal{N}^{s,q}(\mathbb{R}^n) \to W^{s-\varepsilon,q}(\mathbb{R}^n) \qquad \text{for all } \varepsilon > 0$$

provides

$$b(u) \in L^q\left(0,T;W^{s,q}(\mathbb{R}^n)\right)$$

for all $s < 1 + \frac{1}{p}$ and $q < \frac{p^2}{p+1}$ This yields the assertion. □

Proof of Theorem 1.2: Let $\delta > 0$. Applying the Hölder inequality (with $q = \frac{3+\delta}{3-p+\delta}$ and $\hat{q} = \frac{3+\delta}{p}$) we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} \left|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u)\right|^{\frac{2p}{3+\delta}} = \int_0^T\int_B \left|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u)\right|^{\frac{2p}{3+\delta}}\\
&= \int_0^T\int_B (\mu_\varepsilon\psi_h'\omega)^{-\frac{p}{3+\delta}} \left[\mu_\varepsilon\psi_h'\omega \left|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u)\right|^2\right]^{\frac{p}{3+\delta}}\\
&\le \left[\int_0^T\int_B (\mu_\varepsilon\psi_h'\omega)^{-\frac{p}{3-p+\delta}}\right]^{\frac{3-p+\delta}{3+\delta}} \left[\int_0^T\int_B \mu_\varepsilon\psi_h'\omega \left|h^{-\frac{1}{2}}\Delta_\zeta^h \nabla b(u)\right|^2\right]^{\frac{p}{3+\delta}},
\end{aligned}$$

where $B \subset \mathbb{R}^n$ is a ball such that $\big(\text{spt } u(x,T) \cup \text{spt } E_\zeta^h u(x,T)\big) \subset B$. Due to (2.15), the second integral on the right-hand side is bounded. Now let us consider the first integral. Let $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. The Hölder inequality entails

$$\begin{aligned} J_1 &:= \int_0^T \int_B (\mu_\varepsilon \psi_h' \omega)^{-\frac{p}{3-p+\delta}} \\ &\leq \left[\int_0^T \int_B (tT - t^2)^{-\frac{\varepsilon p}{3-p+\delta} q_1}\right]^{\frac{1}{q_1}} \left[\int_0^T \int_B |\psi_h'|^{-\frac{p}{3-p+\delta} q_2}\right]^{\frac{1}{q_2}} \left[\int_0^T \int_B |\omega|^{-\frac{p}{3-p+\delta} q_3}\right]^{\frac{1}{q_3}} . \end{aligned}$$

We put $q_1 = \frac{3-p+\delta}{2\varepsilon p}$, $q_2 = \frac{3-p+\delta}{1+\varepsilon}$, $q_3 = \frac{3-p+\delta}{2-p}$, and $\delta = (1+2p)\varepsilon$. Clearly, the first integral on the right-hand side is bounded. Moreover, let us take $\alpha = \varepsilon$ in the definition of ψ_h', i.e., $\psi_h' = \varepsilon \left(1 + \left|h^{-1}\Delta_\zeta^h b(u)\right|\right)^{-1-\varepsilon}$. Then we have

$$\int_0^T \int_B |\psi_h'|^{-\frac{p}{1+\varepsilon}} = \varepsilon^{-\frac{p}{1+\varepsilon}} \int_0^T \int_B \left(1 + \left|h^{-1}\Delta_\zeta^h b(u)\right|\right)^p \leq c_\varepsilon.$$

Next, let us note that $p < 2$, thus,

$$\omega \geq \left(|E_\zeta^h \nabla b(u)| + |\nabla b(u)|\right)^{p-2}.$$

This yields

$$\int_0^T \int_B |\omega|^{-\frac{p}{2-p}} \leq \int_0^T \int_B \left(|E_\zeta^h \nabla b(u)| + |\nabla b(u)|\right)^p \leq c.$$

Collecting results it follows that J_1 is bounded. We may conclude that

$$\sup_{\substack{\zeta \in \mathbb{R}^n \\ |\zeta|=1}} \sup_{0<h<1} \int_0^T \int_{\mathbb{R}^n} \left|h^{-\frac{1}{2}} \Delta_\zeta^h \nabla b(u)\right|^{\frac{2p}{3+\delta}} \leq c,$$

thus,

$$b(u) \in L^{\frac{2p}{3+\delta}}(0,T; \mathcal{N}^{\frac{3}{2}, \frac{2p}{3+\delta}}(\mathbb{R}^n)).$$

Applying the imbedding theorem of Nikolskii spaces into Sobolev spaces we get the assertion. □

Proof of Theorem 1.3: Let $\frac{p}{2} \leq \alpha < p - \frac{1}{2}$. For $s \in \mathbb{R}^n$ we define $F(s) = |s|^\alpha$ and $F_i(s) = \frac{\partial}{\partial s_i} F(s) = \alpha |s|^{\alpha-2} s_i$. The Taylor expansion provides

$$\begin{aligned} |F(s') - F(s)| &= \left|\sum_i (s' - s)_i \int_0^1 F_i(\tau s' + (1-\tau)s)\, d\tau\right| \\ &\leq c|s' - s| \int_0^1 |\tau s' + (1-\tau)s|^{\alpha-1} d\tau. \end{aligned}$$

Let $E^h_\zeta \nabla b(u) \neq \nabla b(u)$. Putting $s = \nabla b(u)$ and $s' = E^h_\zeta \nabla b(u)$ we get

$$\left| h^{-\frac{1}{2}} \Delta^h_\zeta |\nabla b(u)|^\alpha \right|^{\frac{p}{\alpha}} \leq c \left| h^{-\frac{1}{2}} \Delta^h_\zeta \nabla b(u) \right|^{\frac{p}{\alpha}} \left[\int_0^1 \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{\alpha - 1} d\tau \right]^{\frac{p}{\alpha}}.$$

Recalling the definition of w, the Hölder inequality entails

$$\begin{aligned}
&\left[\int_0^1 \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{\alpha - 1} \right]^{\frac{p}{\alpha}} \\
&= \left[\int_0^1 \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{\frac{p-2}{2}} \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{\frac{2\alpha - p}{2}} \right]^{\frac{p}{\alpha}} \\
&\leq \left[\int_0^1 \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{p-2} \right]^{\frac{p}{2\alpha}} \times \\
&\qquad \times \left[\int_0^1 \left| \tau E^h_\zeta \nabla b(u) + (1-\tau) \nabla b(u) \right|^{2\alpha - p} \right]^{\frac{p}{2\alpha}} \\
&\leq \omega^{\frac{p}{2\alpha}} \left[|\nabla b(u)| + |E^h_\zeta \nabla b(u)| \right]^{\frac{p}{2\alpha}(2\alpha - p)}.
\end{aligned}$$

Altogether, we may conclude that

$$\begin{aligned}
J_0 &:= \int_0^T \int_{\mathbb{R}^n} (\mu_\varepsilon \psi_h')^{\frac{p}{2\alpha}} \left| h^{-\frac{1}{2}} \Delta^h_\zeta |\nabla b(u)|^\alpha \right|^{\frac{p}{\alpha}} \\
&\leq c \int_0^T \int_{\mathbb{R}^n} \left[\mu_\varepsilon \psi_h' \omega \left| h^{-\frac{1}{2}} \Delta^h_\zeta \nabla b(u) \right|^2 \right]^{\frac{p}{2\alpha}} \left[|\nabla b(u)| + |E^h_\zeta \nabla b(u)| \right]^{\frac{p}{2\alpha}(2\alpha - p)}.
\end{aligned}$$

Applying the Hölder inequality (with $q_1 = \frac{2\alpha}{p}$ and $q_2 = \frac{2\alpha}{2\alpha - p}$) we obtain

$$J_0 \leq c \left[\int_0^T \int_{\mathbb{R}^n} \mu_\varepsilon \psi_h' \omega \left| h^{-\frac{1}{2}} \Delta^h_\zeta \nabla b(u) \right|^2 \right]^{\frac{p}{2\alpha}} \left[\int_0^T \int_{\mathbb{R}^n} \left[|\nabla b(u)| + |E^h_\zeta \nabla b(u)| \right]^p \right]^{\frac{2\alpha - p}{2\alpha}}.$$

Due to (2.5), (2.15), and the fact that $\nabla b(u) \in L^p(\mathbb{R}^n \times (0,T))$ the integrals on the right-hand side are bounded. Thus, we have shown

$$J_0 \equiv \int_0^T \int_{\mathbb{R}^n} (\mu_\varepsilon \psi_h')^{\frac{p}{2\alpha}} \left| h^{-\frac{1}{2}} \Delta^h_\zeta |\nabla b(u)|^\alpha \right|^{\frac{p}{\alpha}} \leq c. \tag{3.5}$$

Now, we proceed as above, cf. (3.4). Let $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. Utilizing again the Hölder inequality it follows that

$$J_1 := \int_0^T \int_B (\mu_\varepsilon \psi_h')^{-\frac{p}{2\alpha q_3}} (\mu_\varepsilon \psi_h')^{\frac{p}{2\alpha q_3}} \left| h^{-\frac{1}{2}} \Delta^h_\zeta |\nabla b(u)|^\alpha \right|^{\frac{p}{\alpha q_3}}$$

$$\leq \left[\int_0^T\int_B (tT-t^2)^{-\varepsilon\frac{pq_1}{2\alpha q_3}}\right]^{\frac{1}{q_1}} \left[\int_0^T\int_B \psi_h'^{\,-\frac{pq_2}{2\alpha q_3}}\right]^{\frac{1}{q_2}} \times$$
$$\times \left[\int_0^T\int_B (\mu_\varepsilon\psi_h')^{\frac{p}{2\alpha}} \left|h^{-\frac{1}{2}}\Delta_\zeta^h|\nabla b(u)|^\alpha\right|^{\frac{p}{\alpha}}\right]^{\frac{1}{q_3}},$$

where $B\subset\mathbb{R}^n$ is a ball such that $\big(\text{spt } u(x,T)\cup \text{spt } E_\zeta^h u(x,T)\big)\subset B$. Let us choose $q_1=\frac{2\alpha+1+\delta}{2\varepsilon p}$, $q_2=\frac{2\alpha+1+\delta}{1+\varepsilon}$, $q_3=\frac{2\alpha+1+\delta}{2\alpha}$, and $\delta=(2p+1)\varepsilon$. In view of (3.5) and the definition of ψ_h' the integral J_1 is bounded. This implies that

$$\sup_{\substack{\zeta\in\mathbb{R}^n\\|\zeta|=1}}\sup_{0<h<1}\int_0^T\int_{\mathbb{R}^n}\left|h^{-\frac{1}{2}}\Delta_\zeta^h|\nabla b(u)|^\alpha\right|^{\frac{2p}{2\alpha+1+\delta}}\leq c.$$

We may conclude that

$$|\nabla b(u)|^\alpha\in L^q(0,T;\mathcal{N}^{\frac{1}{2},q}(\mathbb{R}^n))\qquad\text{for all } q<\frac{2p}{2\alpha+1}.$$

The imbedding theorem of Nikolskii spaces into Sobolev spaces entails

$$|\nabla b(u)|^\alpha\in L^q(0,T;W^{\frac{1}{2}-\varepsilon,q}(\mathbb{R}^n))$$

for all $\varepsilon>0$ and $q<\frac{2p}{2\alpha+1}$. This yields the assertion. □

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