The quantale of Galois connections

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Abstract

Galois connections were originally expressed in a contravariant form with transformations that reverse (rather than preserve) order. Nowadays its covariant form (as residuated maps) is more often used since it is more convenient, namely compositions of residuated maps are handled more easily. In this paper we show that this is not a serious disadvantage of the contravariant form (at least in the natural context for uniform structures, where we need it), by introducing an operation of composition in the complete lattice $Gal(L, L)$ of all (contravariant) Galois connections in a complete lattice L, that allows us to work with Galois connections in the same way as one usually works with residuated maps. This operation endows $Gal(L, L)$ with a structure of quantale whenever L is a locale. As an application, it is shown that this structure plays an important role on the entourage-like approaches to uniform structures.

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1. Introduction and motivation

Originally Galois connections were expressed in a symmetric but contravariant form with transformations that reverse order (early references to this form are [1], [4] and [14]). A *Galois connection* [1] between the partially ordered sets A and B is a pair (f, g) of order-reversing (antitone) maps $f : A \to B$ and $g : B \to A$ such that

$$a \leq gf(a) \text{ for all } a \in A \text{ and } b \leq fg(b) \text{ for all } b \in B. \tag{1.1}$$

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$Gal(A,B)$, also denoted $A \otimes B$ [18], is the set of all antitone maps $f : A \to B$ for which there is $f^+ : B \to A$ such that (f, f^+) is a Galois connection, with the pointwise partial order.

Galois connections appear in many areas of mathematics (cf. the survey [3]).

Much of the theory of order structures can be reformulated in purely categorical terms by observing that preordered classes are in bijective correspondence with *thin categories*, that is, categories for which each Hom-set has at most one element. Under this bijection, posets correspond to small thin categories for which no two distinct elements are isomorphic. Explicitly, given a pair $(P, \leq)$, the objects of the corresponding category are the elements of P and

$$\begin{cases} |P(a,b)| = 1 & \text{if } a \leq b \\ \\ |P(a,b)| = 0 & \text{otherwise.} \end{cases}$$

In particular, products are exactly meets whereas coproducts are exactly joins. Further, a functor between two posets is exactly an order-preserving map whereas a contravariant functor is an antitone map. In this context the concept of Galois connection is just the categorical concept of dual adjunction; maps $A \to B$ that are the first half of a Galois connection are precisely the left adjoint functors from A to B^{op}, the dual of B.

Nowadays many authors prefer to work with Galois connections in the covariant form because of its convenience (the survey [3] contains a list of references to this form). In order to avoid confusion we refer to this dualized form as a residuated pair: a *residuated pair* between the partially ordered sets A and B is a pair (f, g) of order-preserving maps $f : A \to B$ and $g : B \to A$ such that

$$gf(a) \leq a \text{ for all } a \in A \text{ and } b \leq fg(b) \text{ for all } b \in B. \tag{1.2}$$

The map $f : A \to B$ in a residuated pair (f, g) is called *residuated* and the map $g : B \to A$ is called *residual.* Thus f is residuated (resp. residual) if and only if $f \in Gal(A, B^{op})$ (resp. $f \in Gal(A^{op}, B)$).

As Blyth and Janowitz (in their monograph on Residuation Theory [2], p. 19) and Erné, Koslowski, Melton and Strecker (in [3]) remarked, the reason for preferring residuated maps rather than Galois connections can be found in the fact that compositions of residuated pairs are handled more easily: two residuated maps may be composed to yield a new residuated map and this is not the case with antitone maps. Indeed, if $f : A \to B$ and $g : B \to C$ are residuated maps, then $g \circ f$ is clearly residuated with $(g \circ f)^+ = f^+ \circ g^+$ [2]. This composition endows $Gal(A, A^{op})$ with a quantalic (complete residuated semigroup) structure. We call it the *quantale of residuated maps.*

But there remains the question: how to compose Galois connections directly, in order to get a quantalic structure for $Gal(A, A)$? This is the problem which motivated us to write this paper. Its relevance comes from the fact that this quantale reveals to be a nice tool in the entourage-like approaches to uniform structures [5].

A more detailed description of this paper is as follows. In Section 2 we recall terminology, notation and facts about frames, locales, quantales and Galois connections. We also describe tensor products of complete join-semilattices. In the third section we introduce a composition $\circ$ for Galois connections and in Section 4 we prove, in detail, that $\circ$ is an associative operation when we specialize to frames. In Section 5 we show that $\circ$ endows $Gal(L, L)$ with a structure of quantale. In the last section we discuss the importance of this quantale, by applying it to the description of uniform structures. We then conclude that the theory of uniform structures (both classical and pointfree) may be completely described in terms of Galois connections. It is also observed that the equivalence between the somewhat different approaches to uniform locales of [6] and [16] is a special case of a very general phenomenon on Galois connections.

2. Background

In this section we introduce terminology and notation along with an overview of the construction of tensor products of complete join-semilattices, presented in the way which is appropriate for its application in the later sections.

2.1. Frames and locales. A *frame* is a complete lattice in which the infinite distributive law

$$a \wedge \bigvee S = \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. Frames are exactly complete Heyting algebras with the Heyting implication given by

$$a \to b = \bigvee \{s \in L \mid s \wedge a \leq b\}.$$

Frame homomorphisms are defined as maps between frames which preserve finitary meets (including the top element $1 = \bigwedge \emptyset$) and arbitrary joins (including the bottom element $0 = \bigvee \emptyset$). The resulting category is called the category of frames. The study of frames was motivated by the following observation. Let X be a topological space and let $\mathcal{O}(X)$ be the complete lattice of open subsets of X. Since finite meets and arbitrary joins correspond to the set-theoretic operations of intersection and union, $\mathcal{O}(X)$ is a frame. Moreover, if $f : X \to Y$ is a continuous map of spaces, the induced map $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a frame homomorphism. The functor $X \mapsto \mathcal{O}(X)$, $f \mapsto f^{-1}$ is contravariant and therefore the category of "generalized spaces" [10] should be the opposite of the category of frames. The objects of this category are called *locales* and morphisms *locale maps.* Thus locales form an order-theoretic counterpart of topological spaces. Locales were introduced by J. Isbell [10] to provide a better categorical context for topology.

For more information on frames and locales we refer the reader to [11].

2.2. Quantales. Replacing, in the definition of a frame, the commutative operation $\wedge$ with a general associative binary operation, we obtain a quantale (a "non-commutative generalized space"). Thus a quantale is a complete lattice Q equipped with an associative multiplication $\cdot$ which distributes over arbitrary joins:

$$a \cdot \bigvee S = \bigvee \{a \cdot s \mid s \in S\} \text{ and } (\bigvee S) \cdot a = \bigvee \{s \cdot a \mid s \in S\}$$

for all $a \in Q$ and $S \subseteq Q$. A quantale homomorphism is a map between quantales which preserves arbitrary joins and the multiplication.

The term *quantale* was suggested by C. J. Mulvey [13] as a "quantization" of the term locale. Its definition provides a non-commutative extension of the concept of locale. The intention was to develop the concept of non-commutative topology, introduced by R. Giles and H. Kummer [7], while providing a constructive, and non-commutative, context for quantum mechanics and, more generally, non-commutative logic. Further details about quantales can be found in [15] and [17].

2.3. Galois connections. Let A and B be complete lattices. As it is well-known (and very easy to prove), condition (1.1) is equivalent to

(1) $a \leq g(b) \Leftrightarrow b \leq f(a)$ for every $a \in A$ and $b \in B$,

and implies the following ones:

(2) $f(\bigvee S) = \bigwedge f(S)$ for every $S \subseteq A$ and $g(\bigwedge S) = \bigvee g(S)$ for every $S \subseteq B$,

(3) $gfg = g$ and $fgf = f$,

(4) $Im\, f = fg(B) = \{b \in B \mid fg(b) = b\}$ and $Im\, g = gf(A) = \{a \in A \mid gf(a) = a\}$,

(5) f and g restrict to order dual-isomorphisms $\overline{f} : Im\, g :\to Im\, f$ and $\overline{g} : Im\, f \to Im\, g$, inverse to each other.

Maps f and g uniquely determine each other; indeed

$$f(a) = \bigvee(\{b \in B \mid g(b) \geq a\})$$

and

$$g(b) = \bigvee(\{a \in A \mid f(a) \geq b\}).$$

Furthermore $f \in Gal(A, B)$ if and only if $f(\bigvee S) = \bigwedge f(S)$ for every $S \subseteq A$ (in particular, for $S = \emptyset$, $f(0) = 1$), that is, if and only if f is a complete join-morphism $A \to B^{op}$. The set $Gal(A, B)$ is also a complete lattice (meets are just defined pointwisely). On the other hand, $f : A \to B$ is residuated if and only if it is a join-homomorphism.

For a more categorical approach to Galois connections we refer to [9].

2.4. Tensor products. Let A and B be complete join-semilattices. As Shmuely proved in [18], $Gal(A, B)$ is the tensor product of A and B in the category of complete join-semilattices (whose objects are complete join-semilattices, that is, complete lattices, and morphisms are maps preserving arbitrary joins). This is the analogue for complete join-semilattices of the construction of tensor products of modules over a commutative ring (cf. [12]). There is an equivalent way of describing the tensor product of complete join-semilattices, which will be very useful in the next section. As Shmuely proved ([18], Theorem 1.3; see also [8]) the elements of $Gal(A, B)$ stand in a one-to-one correspondence with certain semi-ideals of $A \times B$. Let us say that $E \subseteq A \times B$ is a G-*ideal* of $A \times B$ when

(a) E is a *down-set* of $A \times B$, that is,

$$\downarrow E := \{(x, y) \mid (x, y) \leq (a, b) \text{ for some } (a, b) \in E\}$$

coincides with E;

(b) $\Big(\{x\} \times S \subseteq E \Rightarrow (x, \bigvee S) \in E\Big)$ and $\Big(S \times \{y\} \subseteq E \Rightarrow (\bigvee S, y) \in E\Big)$.

Note that the case $S = \emptyset$ in (b) implies that every G-ideal contains the set $\downarrow(1, 0) \cup \downarrow(0, 1)$. We denote by $A \otimes B$ the set of all G-ideals of $A \times B$, partially ordered by inclusion. This is clearly a complete lattice, because the intersection of G-ideals is always a G-ideal. In summary, $A \otimes B \cong Gal(A, B)$ is the tensor product in the category of complete join-semilattices.

For every $a \in A$ and $b \in B$,

$$\begin{aligned} p_{a,b} : A &\longrightarrow B \\ x &\longmapsto \begin{cases} 1 & \text{if } x = 0 \\ b & \text{if } 0 < x \leq a \\ 0 & \text{if } x \not\leq a. \end{cases} \end{aligned}$$

is a Galois map. These maps generate $Gal(A, B)$ by joins, that is, every $f \in Gal(A, B)$ is a join of some family $\{p_{a_i,b_i}\}_{i \in I} \subseteq Gal(A, B)$ [18]. Under the isomorphism $Gal(A, B) \cong A \otimes B$, the Galois map $p_{a,b}$ corresponds to the G-ideal

$$a \otimes b := \downarrow(a, b) \cup \downarrow(1, 0) \cup \downarrow(0, 1).$$

This is called a *pure tensor.* The tensor product construction above gives us the construction of products in the category of locales. Indeed, as Wigner proved in [20], $A \otimes B$ is a frame whenever A and B are frames and $(A \otimes B, u_1, u_2)$ is the coproduct of A and B in the category of frames, where

$$\begin{aligned} u_1 : A &\longrightarrow A \otimes B \\ a &\longmapsto a \otimes 1 \end{aligned}$$

and

$$\begin{aligned} u_2 : B &\longrightarrow A \otimes B \\ b &\longmapsto 1 \otimes b. \end{aligned}$$

3. The composition of Galois connections

From now on A, B and C will denote complete lattices and $Ant(A, B)$ will denote the set of all antitone maps from A to B.

For $f \in Ant(A, B)$ and $g \in Ant(B, C)$ let $E_{f,g}$ denote the G-ideal

$$\bigvee \{a \otimes c \in A \otimes C \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}.$$

The *Galois composition* $g \circ f : A \to C$ is defined by

$$g \circ f(x) = \bigvee \{y \in C \mid (x, y) \in E_{f,g}\}.$$

Since f is order-reversing, this also defines an order-reversing map. Furthermore

$$(g \circ f)(\bigvee_{i \in I} x_i) = \bigwedge_{i \in I} (g \circ f)(x_i).$$

Indeed, $g \circ f$ being order-reversing, $(g \circ f)(\bigvee_{i \in I} x_i) \leq (g \circ f)(x_i)$ for every $i \in I$; if $z \leq (g \circ f)(x_i)$ for all $i \in I$, then

$$\Big(x_i, \bigvee\{y \in C \mid (x_i, y) \in E_{f,g}\}\Big) \in E_{f,g}$$

so $(x_i, z) \in E_{f,g}$; therefore $(\bigvee_{i \in I} x_i, z) \in E_{f,g}$, which implies $z \leq (g \circ f)(\bigvee_{i \in I} x_i)$. Thus $g \circ f \in Gal(A, C)$.

For a categorical description of the Galois composition, that justifies the "correctness" of the chosen definition, see [5].

Sometimes it is more convenient to express the $\circ$ operation by Galois maps:

$$g \circ f(x) = \bigvee\{p_{a,c}(x) \mid (a, c) \in E_{f,g}\}.$$

This description follows immediately from a straightforward property of G-ideals:

(1) For every $x \in A$ and every $E \in A \otimes B$,

$$\bigvee\{y \in B \mid (x, y) \in E\} = \bigvee\{p_{a,b}(x) \mid (a, b) \in E\}.$$

It may be simplified by the following fact about $Ant(A, B)$ (where we denote the set of all down-sets of $A \times B$ by $\mathcal{D}(A \times B)$):

Lemma 3.1. *Let $f \in Ant(A, B)$ and $E \in \mathcal{D}(A \times B)$ satisfying*

$$(x, y) \in E \Rightarrow y \leq f(x).$$

Then

$$(x, y) \in < E > \Rightarrow y \leq f(x),$$

where $< E >$ denotes the G-ideal generated by E.

Proof. Consider the non-empty set

$$\mathcal{F} = \{F \in \mathcal{D}(A \times B) \mid E \subseteq F \subseteq < E >, (x, y) \in F \Rightarrow y \leq f(x)\}.$$

If $F \in \mathcal{F}$ then also

$$F_0 := \Big\{(a, \bigvee S) \mid \{a\} \times S \subseteq F\Big\} \cup \Big\{(\bigvee S, b) \mid S \times \{b\} \subseteq F\Big\}$$

is in $\mathcal{F}$:

Let $(x, y) \in F_0$. If $(x, y) = (x, \bigvee S)$ for some S with $\{x\} \times S \subseteq F$, then $s \leq f(x)$ for every $s \in S$. Thus $\bigvee S \leq f(x)$. On the other hand, if $(x, y) = (\bigvee S, y)$

for some S with $S \times \{y\} \subseteq F$, then $y \leq f(s)$ for every $s \in S$. This implies $y \leq \bigwedge f(S) = f(\bigvee S) = f(x)$.

Besides, for any non-empty $\mathcal{H} \subseteq \mathcal{F}$, $\bigcup_{H \in \mathcal{H}} H \in \mathcal{F}$. Therefore $T := \bigcup_{F \in \mathcal{F}} F$ belongs to $\mathcal{F}$, that is, $\mathcal{F}$ has a largest element T. But then, as we proved above, T_0 also belongs to $\mathcal{F}$ so $T = T_0$, that is, T is a G-ideal. Hence $< E >= T \in \mathcal{F}$ which means that $y \leq f(x)$ whenever $(x, y) \in < E >$. □

Proposition 3.2. *For every $f \in Ant(A, B)$, $g \in Ant(B, C)$ and $x \in A$,*

$$g \circ f(x) = \bigvee \{p_{a,c}(x) \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}.$$

Proof. Consider $p_{a,c}(x)$ with $f(a) \geq b \neq 0$ and $g(b) \geq c$. If $x = 0$ then $p_{a,c}(x) = 1$; but in this case $(x, 1) \in E_{f,g}$. On the other hand, if $0 < x \leq a$ then $p_{a,c}(x) = c$; in this case $(x, c) \in E_{f,g}$ since $f(x) \geq f(a) \geq b \neq 0$ and $g(b) \geq c$. Hence

$$\bigvee \{p_{a,c}(x) \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\} \leq \bigvee \{y \mid (x, y) \in E_{f,g}\}.$$

Now let us prove the reverse inequality. One can easily check that

$$y \leq \bigvee \{p_{a,c}(x) \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}$$

whenever (x, y) belongs to a pure tensor $a \otimes c$ in

$$\{a \otimes c \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}.$$

The conclusion follows immediately from Lemma 3.1. □

4. The associativity of the Galois composition

In order to prove the associativity of $\circ$ we need to introduce the following auxiliary operation on antitone maps. Let $f \in Ant(A, B)$ and $g \in Ant(B, C)$; the *antitone composition* $g \cdot f : A \to C$ is defined by

$$(g \cdot f)(x) = \bigvee \{g(b) \mid b \in B \setminus \{0\}, b \leq f(x)\}.$$

Note that $(g \cdot f)(x) \geq g(f(x))$ and $g \cdot f \in Ant(A, C)$. Further $(g \cdot f)(x)$ is equal to

$$\bigvee \{y \in C \mid \exists b \in B \setminus \{0\} : f(x) \geq b \text{ and } g(b) \geq y\}$$

or, which is the same, to

$$\bigvee\Big\{y \in C \mid (x,y) \in \bigcup\{a \otimes c \in A \otimes C \mid \exists b \in B \setminus \{0\} : f(a) \geq b \text{ and } g(b) \geq c\}\Big\}.$$

For any $f \in Ant(A,B)$ let us denote by $< f >$ the Galois map generated by f, that is, $< f >= \bigwedge\{g \in Gal(A,B) \mid f \leq g\}$. We prepare the proof of the associativity of $\circ$ with three lemmata. The first one shows that $g \circ f$ is precisely the Galois map generated by $g \cdot f$.

Lemma 4.1. *For every $f \in Ant(A,B)$ and $g \in Ant(B,C)$, $< g \cdot f >= g \circ f$.*

Proof. Let us begin by checking that $g \cdot f \leq g \circ f$. For each $x \in A$ consider all $y \in C$ for which there exists $b \in B \setminus \{0\}$ such that $f(x) \geq b$ and $g(b) \geq y$. Then, using Proposition 3.2, $(g \circ f)(x) \geq p_{x,y}(x) \geq y$.

Now let $h \in Gal(A,C)$ satisfying $h \geq g \cdot f$. We need to show that $h \geq g \circ f$ in order to conclude the proof. Again by Proposition 3.2, $h \geq g \circ f$ means that $h(x) \geq p_{a,c}(x)$ for every $x \in A$ and every pair (a,c) satisfying $f(a) \geq b$ and $g(b) \geq c$ for some nonzero $b \in B$. This is equivalent to saying that

$$\Big(f(a) \geq b \neq 0,\ g(b) \geq c,\ 0 < x \leq a\Big) \Rightarrow h(x) \geq c,$$

which is a true assertion since $h \geq g \cdot f$ means

$$\Big(f(a) \geq b \neq 0,\ g(b) \geq c\Big) \Rightarrow h(a) \geq c$$

and, moreover, $h(x) \geq h(a)$ whenever $x \leq a$. $\square$

In order to conclude the associativity of the antitone composition we have to restrict ourselves to frames.

Lemma 4.2. *Let $f \in Ant(A,B), g \in Ant(B,C)$ and $h \in Ant(C,D)$. If A and D are frames then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.*

Proof. We prove that $(h \cdot g) \cdot f \geq h \cdot (g \cdot f)$ (the reverse inequality may be shown similarly). It suffices to ensure that $((h \cdot g) \cdot f)(x) \geq y$ whenever $f(x) \geq b \neq 0$ and $(h \cdot g)(b) \geq y$. So let $f(x) \geq b \neq 0$ and $h \cdot g(b) \geq y$. Then, since D is a frame, we have

$$\begin{aligned} y &= y \wedge (h \cdot g)(b) \\ &= y \wedge \bigvee\{d \in D \mid \exists c \in C \setminus \{0\} : g(b) \geq c \neq 0, h(c) \geq d\} \\ &= \bigvee\{y \wedge d \mid \exists c \in C \setminus \{0\} : g(b) \geq c \neq 0, h(c) \geq d\}. \end{aligned}$$

This means that y may be written as a join $\bigvee_{i\in I} y_i$ of a family $\{y_i \mid i \in I\} \subseteq C$ for which there exists $c_i \in C\setminus\{0\}$ satisfying $g(b) \geq c_i$ and $h(c_i) \geq y_i$. Since $(g\cdot f)(x) \geq c_i \neq 0$ and $h(c_i) \geq y_i$ for every $i \in I$, we may conclude that $((h \cdot g) \cdot f)(x) \geq y_i$ for every $i \in I$; consequently, $((h \cdot g) \cdot f)(x) \geq \bigvee_{i\in I} y_i = y$. □

For each $f \in Ant(A, B)$ let us define

$$\begin{array}{rcl} f_0 : A & \to & B \\ x & \mapsto & \bigvee\{y \in B \mid \exists S \subseteq A : \bigvee S = x \text{ and } f(s) \geq y \text{ for every } s \in S\}. \end{array}$$

This is an antitone map whenever A is a frame.

Lemma 4.3. *Let $f \in Ant(A, B)$ and $g \in Ant(B, C)$. If B is a frame then $< g > \circ < f >= g \circ f$.*

Proof. By Lemma 4.1, $< g > \circ < f >=<< g > \cdot < f >>$. So it suffices to show that $< g > \cdot < f >\leq g \circ f$. For this, consider the set

$$\mathcal{E} = \{h \in Ant(A, B) \mid f \leq h \leq< f >, g \cdot h \leq g \circ f\}.$$

This is a non-empty set since $f \in \mathcal{E}$. Let us prove that $h_0 \in \mathcal{E}$ whenever $h \in \mathcal{E}$. Consider $y \in C$ satisfying $h_0(x) \leq b$ and $g(b) \leq y$ for some nonzero $b \in B$. Then

$$\begin{array}{rcl} b & = & b \wedge h_0(x) \\ & = & b \wedge \bigvee\{\beta \in B \mid \exists S \subseteq A : \bigvee S = x \text{ and } h(s) \geq \beta \text{ for every } s \in S\} \\ & = & \bigvee\{b \wedge \beta \mid \beta \in B, \exists S \subseteq A : \bigvee S = x \text{ and } h(s) \geq \beta \text{ for every } s \in S\}. \end{array}$$

The fact $b \neq 0$ implies the existence of $\beta \in B$ and $S \subseteq A$ such that $x = \bigvee S$, $h(s) \geq \beta$ for all $s \in S$ and $b\wedge\beta \neq 0$. Then $g(b\wedge\beta) \geq g(b) \geq y$ and, for each $s \in S$, $h(s) \geq \beta \geq b\wedge\beta \neq 0$. This implies $(g \cdot h)(s) \geq y$ for every $s \in S$. But $g \cdot h \leq g \circ f$ so $y \leq (g \circ f)(s)$ for every $s \in S$. Hence

$$y \leq \bigwedge_{s\in S} (g \circ f)(s) = (g \circ f)(\bigvee S) = (g \circ f)(x).$$

Moreover, for any non-void $\mathcal{F} \subseteq \mathcal{E}$, $\bigvee_{h\in\mathcal{F}} h$ belongs to $\mathcal{E}$, since

$$g \cdot (\bigvee_{h\in\mathcal{F}} h) \leq \bigvee_{h\in\mathcal{F}} (g \cdot h).$$

In fact, consider $b \in B \setminus \{0\}$ such that $b \leq \bigvee_{h\in\mathcal{F}} h(x)$. Then

$$b = b \wedge \bigvee_{h\in\mathcal{F}} h(x) = \bigvee_{h\in\mathcal{F}} (b \wedge h(x)).$$

This implies the existence of $h \in \mathcal{F}$ for which $b \wedge h(x) \neq 0$. Denoting this element $b \wedge h(x)$ by b' we have $g(b) \leq g(b')$ and $0 \neq b' \leq h(x)$. This proves $g \cdot (\bigvee_{h \in \mathcal{F}} h) \leq \bigvee_{h \in \mathcal{F}} (g \cdot h)$.

Therefore $\mathcal{E}$ has a largest element $k = \bigvee_{h \in \mathcal{E}} h$. But $k_0 \in \mathcal{E}$ thus $k = k_0$ i.e. $k \in Gal(A, B)$. Then $< f >= k \in \mathcal{E}$. Therefore $g\cdot < f > \leq g \circ f$. By symmetry, $< g > \cdot f \leq g \circ f$. Hence

$$\begin{aligned} < g > \cdot < f > &\leq g \circ < f > \\ &= < g \cdot < f >> \\ &\leq < g \circ f > \\ &= g \circ f, \end{aligned}$$

as claimed. □

We are now ready to prove the associativity of $\circ$.

Theorem 4.4. *Let A, B, C and D be frames. Then, for every $f \in Ant(A, B), g \in Ant(B, C)$ and $h \in Ant(C, D)$,*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof. It is an immediate consequence of the preceding lemmas:

$$\begin{aligned} (h \circ g) \circ f &= < h \cdot g > \circ f && \text{(by Lemma 4.1)} \\ &= (h \cdot g) \circ f && \text{(by Lemma 4.3)} \\ &= < (h \cdot g) \cdot f > && \text{(by Lemma 4.1)} \\ &= < h \cdot (g \cdot f) > && \text{(by Lemma 4.2)} \\ &= h \circ (g \cdot f) \\ &= h \circ < g \cdot f > \\ &= h \circ (g \circ f). \end{aligned}$$

□

5. The quantale of Galois connections

In the sequel let L be a frame. We now prove that $\big(Gal(L, L), \circ\big)$ is a quantale.

From the proof of Lemma 4.3 follows, in particular, that, in the complete lattice $Ant(L, L)$, $g \cdot (\bigvee_{i \in I} f_i) \leq \bigvee_{i \in I} (g \cdot f_i)$. Since the reverse inequality is, obviously, also true, we have

$$g \cdot (\bigvee_{i\in I} f_i) = \bigvee_{i\in I} (g \cdot f_i).$$

Similarly,

$$(\bigvee_{i\in I} g_i) \cdot f = \bigvee_{i\in I} (g_i \cdot f).$$

Therefore $(Ant(L, L), \cdot)$ is a quantale.

In the sequel, in order to avoid confusion, we use the symbol $\bigsqcup$ to denote joins in the complete lattice $Gal(L, L)$. So, for any family $\{f_i\}_{i\in I}$ in $Gal(L, L)$, $\bigvee_{i\in I} f_i$ denotes its join in $Ant(L, L)$ and $\bigsqcup_{i\in I} f_i$ its join in $Gal(L, L)$. Evidently, for any family $\{f_i\}_{i\in I}$ in $Ant(L, L)$,

$$< \bigvee_{i\in I} f_i >= \bigsqcup_{i\in I} < f_i > .$$

Theorem 5.1. *If L is a frame then $(Gal(L, L), \circ)$ is a quantale.*

Proof. It remains to show that $\circ$ distributes over arbitrary joins. Consider f and $\{g_i\}_{i\in I}$ in $Gal(L, L)$. Then

$$\begin{aligned}
g \circ \textstyle\bigsqcup_{i\in I} f_i &= g\circ < \textstyle\bigvee_{i\in I} f_i > \\
&= g \circ \textstyle\bigvee_{i\in I} f_i && \text{(by Lemma 4.3)} \\
&= < g \cdot \textstyle\bigvee_{i\in I} f_i > && \text{(by Lemma 4.1)} \\
&= < \textstyle\bigvee_{i\in I} (g \cdot f_i) > \\
&= \textstyle\bigsqcup_{i\in I} < g \cdot f_i > \\
&= \textstyle\bigsqcup_{i\in I} (g \circ f_i).
\end{aligned}$$

The other distributive law may be proved in a similar manner. □

6. Applications

For any set X, $\mathcal{P}(X \times X)$ equipped with the usual composite relation $R \circ S$ is a quantale. This quantalic structure of binary relations of X is the basic structure on which the theory of uniform spaces of Weil [19] relies on.

On the other hand, binary relations can be described by *polarities* (i.e. Galois connections between power sets) [3]. In fact, there is a one-to-one correspondence between relations $R \subseteq X \times X$ and Galois connections on $\mathcal{P}(X)$, that assigns to each R the Galois connection $(R_\forall, R^\forall) : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$R_\forall(U) := \bigcap_{a\in U} \{b \in X \mid (a, b) \in R\} = \{b \in X \mid \forall a \in U \ (a, b) \in R\} \quad \text{for } U \subseteq X$$

and

$$R^{\forall}(V) := \bigcap_{b\in V} \{a \in X \mid (a,b) \in R\} = \{a \in X \mid \forall b \in V \ (a,b) \in R\} \quad \text{for } V \subseteq X.$$

In particular, Weil entourages E of X (that is, reflexive binary relations on X) correspond to Galois connections $(E_{\forall}, E^{\forall})$ satisfying $\{x\} \subseteq E_{\forall}(\{x\})$ for every $x \in X$ or, equivalently, $\bigcup\{A \subseteq X \mid A \subseteq E_{\forall}(A)\} = X$.

Similarly, binary relations may also be described by *axialities*, that is, residuated pairs between power sets (see [3] for the details).

Therefore the classical description of uniform spaces, introduced by Weil in terms of entourages, may be immediately formulated in terms of polarities and axialities. In all these entourage-like approaches to uniform structures the main tool is a certain quantalic structure.

The theory of uniform spaces of Weil can be generalized to the pointfree context of frames and locales (see [16] for a survey on this generalization). Of course, it seems desirable to have characterizations of uniform frames that are analogous to those of uniformities which are given by polarities and axialities. In order to pursue this goal one needs "good" quantalic structures for $Gal(L, L)$ and $Gal(L, L^{op})$. The former was presented in this paper and fortunately we could get it precisely in the context that we need it: when L is a frame. In [5] it is shown how to apply it to get a new (equivalent) approach to uniform frames. The latter is the well-known quantale of residuated maps [3]. By applying it to the study of uniform frames, as it is shown in [5], one gets precisely Fletcher-Hunsaker-Lindgren's approach [6]. This put some light on the nature of this approach: Galois connections rather than entourages are at its root.

We can say more; the equivalence between the aproaches to frame uniformities [6] and [16] is a special case of the following very general phenomenon on Galois connections, relating the elements of the quantales $Gal(L, L)$ and $Gal(L, L^{op})$. For each $f \in Gal(L, L)$ let $\overline{f} : L \to L$ be defined by

$$\overline{f}(x) = \bigvee\{y \in L \mid y \wedge x \neq 0, y \leq f(y)\}.$$

This is a residuated map on L. Conversely let $f \in Gal(L, L^{op})$ and define $\overset{\circ}{f}: L \to L$ by

$$\overset{\circ}{f}(x) = \bigvee\{y \in L \mid (x, y) \text{ is } f\text{-small}\},$$

where (x, y) is said to be *f-small* whenever $x \wedge \alpha \neq 0$ and $y \wedge \alpha \neq 0$ imply $x \vee y \leq f(\alpha)$. This way we get a Galois map $\overset{\circ}{f}$ on L. Since $\overline{\overset{\circ}{f}}$ and $\overset{\circ}{\overline{f}}$ may differ from f these correspondences do not establish a bijection. Nevertheless,

$$f \leq \overset{\circ}{\overline{f^3}} \text{ and } \overset{\circ}{\overline{f}} \leq f^3$$

for every $f \in Gal(L,L)$ such that $f^+ = f$ (these Galois connections are called *symmetric* in [5]) and

$$f \leq \overline{\overset{\circ}{f}{}^3} \text{ and } \overline{\overset{\circ}{f}} \leq f$$

for every symmetric $f \in Gal(L, L^{op})$ (an axiality is said to be *symmetric* if $f(x) \wedge y \neq 0$ if and only if $x \wedge f(y) \neq 0$ [5]).

This means that, in the presence of the square-refinement axiom (which asserts that for every f in the structure there exists g in the structure satisfying $g \circ g \leq f$)) and the axiom of symmetry, we get a bijection. More precisely, the correspondences $f \mapsto \overline{f}$ and $f \mapsto \overset{\circ}{f}$ establish an isomorphism between the filters of $(Gal(L,L), \circ)$ and $(Gal(L,L^{op}), \circ)$ satisfying the symmetry and square-refinement axioms. This fact about Galois connections explains why the somewhat different approaches to uniform frames [6] and [16] are equivalent. The details can be found in [5].

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