On the pullback stability of a quotient map with respect to a closure operator

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Abstract

Given a category equipped with a Dikranjan-Giuli closure operator c, the pullback stability of a c-quotient along some special morphisms is established. In particular, it is shown that, under mild conditions, the hereditary c-quotients are just the c-final morphisms; this generalizes a well-known result on topological spaces, since c-final for c the Kuratowski closure operator means pseudo-open. The universal c-quotients, that is, the c-quotients preserved by pullbacks along arbitrary morphisms, are also studied.

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0 Introduction

The notion of quotient map has taken the attention of numerous mathematicians, not only in topology but in several other contexts. In the recent paper [6], the authors analysed the notions of c-final morphism and c-quotient morphism, where c is a closure operator, as candidates for a general concept of a quotient map with respect to a closure operator. In particular, they obtained nice properties on the behaviour of c-final morphisms under pullbacks.

In this paper, we pay attention to c-quotients and show that they also enjoy nice properties. In particular, under mild conditions, a c-quotient is hereditary if and only if it is c-final, what generalizes a well-known result in the category of topological spaces. We begin, in Section 2, by presenting a new characterization of c-quotients, which, in some sense, gives a measure of the distance between the concepts of c-quotient and c-final morphism. Let $f : X \to Y$ be a c-quotient and let

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1}$$

be a pullback diagram. In Section 3, we show that the notion of c-quotient ascends along c-closed and c-open monomorphisms, that is, if p' is a c-closed or c-open monomorphism then f' is a c-quotient. Section 5 is concerned with the characterization of the universal c-quotients, that is, those c-quotients which are preserved by any pullback. The universal quotients in $\mathcal{T}op$, i.e., universal k-quotients, where k is the Kuratowski closure operator, were characterized in [13] and [7] in terms of filters and open sets. Here we present another

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characterization by means of a condition on the closure operator. Furthermore, we show that the sufficiency of this condition stays valid in a more general setting.

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1 Preliminaries

1.1 Throughout this paper, we shall be working in a finitely complete category $\mathcal{X}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms (see, for example, [1, 10, 14]). We shall further assume that $\mathcal{X}$ is $\mathcal{M}$*-complete*, that is, $\mathcal{M}$ is pullback-stable and the intersection of a (possibly large) family of morphisms in $\mathcal{M}$ with common codomain exists in $\mathcal{X}$ and belongs to $\mathcal{M}$. By a *subobject* of an object X we mean an $\mathcal{M}$-morphism with codomain in X. We use $\mathrm{sub}X$ to indifferently denote the class of all subobjects of X or the usual class of equivalence classes of isomorphic $\mathcal{M}$-morphisms with codomain in $\mathcal{X}$. The $\mathcal{M}$-completeness of $\mathcal{X}$ implies that, for every object $X \in \mathcal{X}$, $\mathrm{sub}X$ is a complete (pre)ordered class.

1.2 Every morphism $f : X \to Y$ gives the image-preimage adjunction

$$f(-) \dashv f^{-1}(-) : \mathrm{sub}Y \longrightarrow \mathrm{sub}X\,.$$

Thus, we have that:
(a) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$, for all $f : X \to Y$, $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$;
(b) $f^{-1}\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} f^{-1}(n_i)$ and $f\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} f(m_i)$.
Moreover, we have the following properties (cf. [11]):
(c) $m \cong f^{-1}(f(m))$, if f is monic and $\mathcal{E}$ is stable under pullbacks;
(d) $f(f^{-1}(n)) \cong n$, if and only if every pullback of f along an $\mathcal{M}$-morphism belongs to $\mathcal{E}$.

1.3 Let the diagram (1) in Introduction be a pullback diagram. Then, for every $m \in \mathrm{sub}X$, by image-preimage adjunction, $f'(p'^{-1}(m)) \leq p^{-1}(f(m))$; we say that (1) satisfies the *Beck-Chevalley Property (BCP)* if, moreover, we have $f'(p'^{-1}(m)) \cong p^{-1}(f(m))$. We shall make use of the following result (cf. [12, 5]):

Every pullback in $\mathcal{X}$ satisfies (BCP) if and only if $\mathcal{E}$ is stable under pullback.

1.4 A *closure operator* c of $\mathcal{X}$ with respect to $(\mathcal{E}, \mathcal{M})$ is given by a family of functions $c_X : \mathrm{sub}X \to \mathrm{sub}X$ $(X \in \mathcal{X})$ such that, for $m, n \in \mathrm{sub}X$, $m \leq c_X(m)$, $c_X(m) \leq c_X(n)$ if $m \leq n$, and all $f : X \to Y$ in $\mathcal{X}$ are c-continuous, that is, $f(c_X(m)) \leq c_Y(f(m))$. A subobject $m \in \mathrm{sub}X$ is *c-closed* if $m \cong c_X(m)$ and the operator c is called *idempotent* if $c_X(m)$ is c-closed for all $m \in \mathrm{sub}X$, $X \in \mathcal{X}$. A morphism $f : X \to Y$ is said to be *c-closed* if, for each $m \in \mathrm{sub}X$, $f(c_X(m)) \cong c_Y(f(m))$, and it is said to be *c-open* if, for each $n \in \mathrm{sub}Y$, $f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$. A morphism $f : X \to Y$ is said to be a *c-quotient* provided that it belongs to $\mathcal{E}$ and, for each $n \in \mathrm{sub}Y$, the c-closedness of $f^{-1}(n)$ implies the c-closedness of n. It is said to be *c-final* if, for each $n \in \mathrm{sub}Y$, $c_Y(n) \cong f(c_X(f^{-1}(n)))$.

1.5 Recall that a commutative diagram as in (1) in Introduction is said to be a $\mathcal{E}$-weak pullback whenever the canonical morphism $e : U \to X \times_Y V$ belongs to $\mathcal{E}$. The (BCP) property is enjoyed by all pullbacks in $\mathcal{X}$ iff it is enjoyed by all $\mathcal{E}$-weak pullbacks (see [12, 5, 11]). All results on pullbacks presented in sections 3 and 4 are also true for $\mathcal{E}$-weak pullbacks. This simply follows from the fact that all proofs work if, instead of pullbacks, we have $\mathcal{E}$-weak pullbacks.

2 A characterization of c-quotients

In this section we give a characterization of c-quotient which does not involve the notion of c-closed subobject. Furthermore, it sheds light on the difference between c-quotient morphisms and c-final morphisms.

Given a morphism $f : X \to Y$ in $\mathcal{X}$, we have the function

$$F : \mathrm{sub}Y \longrightarrow \mathrm{sub}Y$$

which assigns to each $n \in \mathrm{sub}Y$ the $\mathcal{M}$-subobject $f(c_X(f^{-1}(n)))$.

F is clearly extensive and monotone. Moreover, a morphism $f : X \to Y$ is c-final exactly whenever $c_Y \leq F$, that is, $c_Y(n) \leq F(n)$ for each $n \in \mathrm{sub}Y$. We call F the *c-final function associated to f*. We are going to show that c-quotient morphisms can also be characterized by means of the corresponding c-final function.

We obtain an ascending ordinal chain of functions from sub$Y$ into sub$Y$,

$$(F^\alpha)_{\alpha \in Ord},$$

by putting

$$F^0 = Id_{\mathrm{sub}Y},\ F^\alpha = FF^{\alpha-1},\ F^\beta = \bigvee_{\gamma<\beta} F^\gamma$$

for every successor ordinal α and for every limit ordinal β. We denote by F^∞ the function which, to each $n \in \mathrm{sub}Y$, assigns $\bigvee_{\alpha \in Ord} F^\alpha(n)$.

Theorem 2.1 *Let $\mathcal{X}$ be $\mathcal{M}$-wellpowered. Then a morphism $f : X \to Y$ is a c-quotient if and only if $c_Y \leq F^\infty$.*

Proof. Let us assume that $c_Y \leq F^\infty$. By definition of F^α, it is clear that, for each ordinal α, $F^\alpha(1_Y) \leq f(1_X)$. Thus, we have that $1_Y \cong c_Y(1_Y) \leq F^\infty(1_Y) \leq f(1_X)$; consequently, $f(1_X)$ is an isomorphism, so that $f \in \mathcal{E}$. Let $n \in \mathrm{sub}Y$ be such that $f^{-1}(n)$ is c-closed, that is, $c_X(f^{-1}(n)) \cong f^{-1}(n)$. Then, we have that $F(n) = f(c_X(f^{-1}(n)) \cong f(f^{-1}(n)) \leq n$. From this, it immediately follows that $F^\alpha(n) \leq n$ for each α. Thus $c_Y(n) \leq n$, that is, n is c-closed.

Conversely, let $f : X \to Y$ be a c-quotient. Since $\mathcal{X}$ is $\mathcal{M}$-wellpowered, the chain $(F^\gamma)_{\gamma \in Ord}$ is stationary, that is, there is some ordinal α such that $F^\beta \cong F^\alpha$ for all $\beta \geq \alpha$. Let $n \in \mathrm{sub}Y$; we are going to prove that $c_Y(n) \leq F^\alpha(n)$. For that, we show that $F^\alpha(n)$ is c-closed.

In fact, we have that

$$\begin{aligned} f^{-1}(F^\alpha(n)) &\cong f^{-1}(F^{\alpha+1}(n)) \\ &= f^{-1}(f(c_X(f^{-1}(F^\alpha(n))))) \\ &\geq c_X(f^{-1}(F^\alpha(n))), \end{aligned}$$

from what follows that $f^{-1}(F^\alpha(n))$ is c-closed. Thus, since f is a c-quotient, $F^\alpha(n)$ is c-closed. Now, using the fact that $n \leq F^\alpha(n)$, we obtain that $c_Y(n) \leq F^\alpha(n)$. $\square$

Remark 2.2 Assume that $\mathcal{X}$ is $\mathcal{M}$-wellpowered and that $\mathcal{E}$ is pullback stable along $\mathcal{M}$-morphisms. Then, as a consequence of the above theorem, we have that: If the closure operator c is idempotent, then a morphism $f : X \to Y$ is c-quotient if and only if $c_Y \cong F^\infty$, if and only if there is some $\alpha \in Ord$ such that $c_Y \cong F^\alpha$. In fact, let c be idempotent and let $f : X \to Y$ belong to $\mathcal{E}$. Then, for each $n \in \text{sub}Y$ and each $\alpha \in Ord$, we have that $F(n) = f(c_X(f^{-1}(n)) \leq f(f^{-1}(c_Y(n)) \cong c_Y(n)$ (by (c) of 1.2) and, assuming $F^\alpha(n) \leq c_Y(n)$, it follows that $F^{\alpha+1}(n) \cong F(F^\alpha(n)) \leq f(c_X(f^{-1}(c_Y(n)))) \leq f(f^{-1}(c_Y^2(n)) \cong c_Y^2(n) \cong c_Y(n)$. Thus, it it is clear that $F^\alpha(n) \leq c_Y(n)$ for all $\alpha \in Ord$.

3 Pullbacks of c-quotients

¿From now on, we assume that $\mathcal{E}$ is stable under pullback and so, by 1.3, that every pullback in $\mathcal{X}$ satisfies (BCP).

3.1 Let

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2}$$

be a pullback diagram. In [11] it was proved that:

If f is c-final and p' c-initial then f' is c-final and p c-initial.

We are going to study the parallel situation corresponding to replacing c-final by c-quotient and c-initial by c-closed or c-open monomorphism.

3.2 First, we need the following definition:

By a chain of subobjects of X, we mean a family $(n_i)_{i\in I}$ of subobjects of X such that $(I, \leq)$ is a linear ordered set and $n_i \leq n_j$ whenever $i \leq j$. Given a morphism $g : X \to Y$, we say that *$\mathcal{M}$-unions of chains of subobjects of Y are preserved by the inverse image under g* provided that, for each chain $(n_i)_{i\in I}$ of subobjects of Y, it holds that

$$g^{-1}(\bigvee_{i\in I} n_i) \cong \bigvee_{i\in I} g^{-1}(n_i)\,.$$

We point out that the preservation by inverse images under monomorphisms of $\mathcal{M}$-unions of chains is fulfield by numerous categories; for instance, all monotopological categories over Set for $\mathcal{M}$ the class of embeddings, the categories of vector spaces, groups and rings, for $\mathcal{M}$ the injective homomorphisms.

Let us consider the pullback diagram (2), where p' and p are monomorphisms. Then, we prove the following two assertions:

Theorem 3.3 *If f is a c-quotient and p' c-closed, then f' is a c-quotient; furthermore, p is c-closed if c is idempotent.*

Proof. Let $n \in \text{sub}V$ be such that ${f'}^{-1}(n)$ is a c-closed subobject of U. We show that n is c-closed, thus f' is a c-quotient. Since p' is a c-closed morphism, we obtain that $p'({f'}^{-1}(n)) \cong p'(c_U({f'}^{-1}(n)) \cong c_X(p'({f'}^{-1}(n)))$, that is, $p'({f'}^{-1}(n))$ is c-closed.

Thus, since, by (BCP), $p'({f'}^{-1}(n)) \cong f^{-1}(p(n))$, we conclude that $f^{-1}(p(n))$ is c-closed. As f is a c-quotient, it follows that $p(n)$ is c-closed, i.e.,

$$c_Y(p(n)) \cong p(n)\,. \tag{3}$$

Then,

$$\begin{array}{lll} c_V(n) & \leq p^{-1}(p(c_V(n))), & \text{by 1.2(a)} \\ & \leq p^{-1}(c_Y(p(n))), & \text{by } c\text{-continuity} \\ & \cong p^{-1}(p(n)), & \text{from (3)} \\ & \cong n, & \text{by 1.2(c)} \end{array}$$

Therefore, n is c-closed.

Now, if c is idempotent, in order to show that p is c-closed, it suffices to show that p maps c-closed subobjects to c-closed subobjects (see [11]). Let n be a c-closed subobject of V. The idempotency of c implies that, then, ${f'}^{-1}(n)$ is also c-closed, since

$$c_U({f'}^{-1}(n)) \cong c_U({f'}^{-1}(c_V(n))) \leq {f'}^{-1}(c_V^2((n))) \cong {f'}^{-1}(c_V(n)) \cong {f'}^{-1}(n)\,.$$

Consequently, $p'({f'}^{-1}(n))$ is c-closed and, from the (BCP) property, the same happens to $f^{-1}(p(n))$. So, the fact that f is c-quotient implies that $p(n)$ is c-closed. $\square$

Theorem 3.4 *Let $\mathcal{X}$ be $\mathcal{M}$-wellpowered and let $\mathcal{M}$-unions of chains of subobjects of Y be preserved by inverse images under monomorphisms. If f is a c-quotient and p' is c-open, then f' is a c-quotient; furthermore, p is c-open if c is idempotent.*

Proof. We are going to make use of the characterization of c-quotients given in Section 2. Let F' be the c-final function associated to f'. First we show that, for each $n \in \mathrm{sub}Y$ and each ordinal β, we have that

$$p^{-1}(F^\beta(n)) \cong {F'}^\beta(p^{-1}(n))\,. \tag{4}$$

For $\beta = 0$, it is clear. Assuming the property for all ordinals less then β, we have:
For $\beta = \alpha + 1$,

$$\begin{array}{lll} p^{-1}(F^\beta(n)) & = p^{-1}(f(c_X(f^{-1}(F^\alpha(n))))) & \\ & \cong f'({p'}^{-1}(c_X(f^{-1}(F^\alpha(n))))), & \text{by (BCP)} \\ & \cong f'(c_U({p'}^{-1}(f^{-1}(F^\alpha(n))))), & \text{since } p' \text{ is } c\text{-open} \\ & \cong f'(c_U({f'}^{-1}(p^{-1}(F^\alpha(n))))), & \text{by (BCP)} \\ & \cong f'(c_U({f'}^{-1}({F'}^\alpha(p^{-1}(n))))), & \text{by inductive hypothesis} \\ & = {F'}^\beta(p^{-1}(n)) & \end{array}$$

For β a limit ordinal,

$$\begin{array}{lll} p^{-1}(F^\beta(n)) & = p^{-1}(\bigvee_{\gamma<\beta} F^\gamma(n)) & \\ & \cong \bigvee_{\gamma<\beta} p^{-1}(F^\gamma(n)), & \text{by the preservation of } \mathcal{M}\text{-unions of} \\ & & \text{chains by inverse image} \\ & \cong \bigvee_{\gamma<\beta} {F'}^\beta(p^{-1}(n)), & \text{by inductive hypothesis} \end{array}$$

Let now $m \in \mathrm{sub}V$; we want to prove that, for some $\alpha \in Ord$, $c_V(m) \leq {F'}^\alpha(m)$. We have that

$$\begin{array}{lll} c_V(m) & \leq p^{-1}(p(c_V(m))), & \text{by 1.2(a)} \\ & \leq p^{-1}(c_Y(p(m))), & \text{by } c\text{-continuity} \\ & \leq p^{-1}(F^\alpha(p(m))), & \text{for some } \alpha \in Ord, \text{ since } f \text{ is } c\text{-} \\ & & \text{quotient and } \mathcal{X} \text{ is } \mathcal{M}\text{-wellpowered,} \\ & \cong {F'}^\alpha(p^{-1}(p(m))), & \text{by (4)} \\ & \cong {F'}^\alpha(m), & \text{by 1.2(c).} \end{array}$$

Therefore, F' is a c-quotient.

Let us now show that, assuming that c is idempotent, p is c-open. Since $\mathcal{X}$ is $\mathcal{M}$-wellpowered, and f and f' are c-quotients, there is some ordinal α such that $c_Y \cong F^\alpha$ and $c_V \cong F'^\alpha$ (see Remark 2.2). Thus, it follows that:

$$\begin{aligned} p^{-1}(c_Y(n)) &\cong p^{-1}(F^\alpha(n)) \\ &\cong F'^\alpha(p^{-1}(n)), \text{ by } (4) \\ &\cong c_V(p^{-1}(n)) \,. \end{aligned}$$

□

Remark 3.5 The idempotency of c is necessary for p to be c-closed in Theorem 3.3. The same happens to the c-openess of p in Theorem 3.4. To show that, let us consider the topological category $\mathcal{SG}ph$ of spatial graphs, that is, the category of graphs (X, E) such that the relation E is reflexive, and maps which preserve the edges. Let $\mathcal{M}$ be the class of embeddings of subgraphs and let c be the up-closure, that is, $c = \uparrow$ with $\uparrow_X (M) = \{x \in X \,|\, (\exists a \in M) a \to x\}$, for each $M \subseteq X$. It is well-known that the up-closure is not idempotent in $\mathcal{SG}ph$ (cf. [8]). In the diagram (2), let $U = V = X$ be the spatial graph $0 \to 1 \to 2$ and let Y be the spatial graph obtained from X by joining the edge $0 \to 2$. Let the underlying map of each one of the morphisms f, f', p and p' be the identity. This way, the diagram (2) is a pullback, f is $\uparrow$-quotient, p' is simultaneously $\uparrow$-closed and $\uparrow$-open, but p is not $\uparrow$-closed since $p(\uparrow_V (\{0\}) = \{0,1\} \neq \{0,1,2\} = \uparrow_Y (p(\{0,1\}))$; and p is not $\uparrow$-open because $p^{-1}(\uparrow_Y (\{0\}) = \{0,1,2\} \neq \{0,1\} = \uparrow_V (p^{-1}(\{0\}))$.

4 Hereditary c-quotients

From now on, we assume that the closure operator c is *hereditary*, i.e., for all $m : M \to Y$, $y : Y \to X$ in $\mathcal{M}$, it holds that $y \wedge c_X(y \cdot m) \cong y \cdot c_Y(m)$, or, equivalently, $c_Y(m) \cong y^{-1}(c_X(y \cdot m))$.

4.1 It is clear that the heredity of c is equivalent to every morphism in $\mathcal{M}$ being c-initial (cf. [11]).

4.2 A *hereditary c-quotient* is a morphism whose pullback along any $\mathcal{M}$-morphism is a c-quotient. Clearly any hereditary c-quotient is a c-quotient. It is well-known that in $\mathcal{T}op$, for the Kuratowski closure operato k, the hereditary k-quotients coincide with the k-final morphisms. In this section, we are going to show that this is a particular example of a general property. Indeed, next we prove that, under convenient conditions, the hereditary c-quotients actually coincide with c-final morphisms.

Theorem 4.2 *Let sub$Y$ be a Boolean algebra. Then a morphism with codomain in Y is a hereditary c-quotient if and only if it is c-final.*

Proof. From 3.1 and 4.1, it immediately follows that every c-final morphism is a hereditary c-quotient.

Conversely, let $f : X \to Y$ be a hereditary c-quotient and let $n : N \to Y$ be a subobject of Y. We want to prove that $c_Y(n) \cong f(c_X(f^{-1}(n)))$. Let $m : M \to Y$ be the complement of $f(c_X(f^{-1}(n)))$ and put $r = m \vee n : R \to Y$. Let $n_r : N \to R$ be the $\mathcal{M}$-morphism for which $r \cdot n_r = n$ and consider the pullback

$$\begin{tikzcd}
f^{-1}(R) \arrow[r, "g"] \arrow[d, "f^{-1}(r)"'] & R \arrow[d, "r"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

By hypothesis, g is a c-quotient; we show that $g^{-1}(n_r)$ is c-closed, so that n_r is c-closed. To this end, since c is hereditary, it suffices to show that $f^{-1}(r) \wedge c_X(f^{-1}(r) \cdot g^{-1}(n_r)) \cong f^{-1}(r) \cdot g^{-1}(n_r))$, that is, that $f^{-1}(r) \wedge c_X(f^{-1}(n)) \cong f^{-1}(n)$. In fact, this isomorphism occurs since, on the one hand, it is clear that $f^{-1}(n) \leq f^{-1}(r) \wedge c_X(f^{-1}(n))$; and, on the other hand, we have that

$$\begin{aligned}
f^{-1}(r) \wedge c_X(f^{-1}(n)) &\leq f^{-1}(r) \wedge f^{-1}(f(c_X(f^{-1}(n)))), \quad \text{by 1.2(a)} \\
&\cong f^{-1}(r \wedge f(c_X(f^{-1}(n)))), \quad \text{by 1.2(b)} \\
&\cong f^{-1}((m \vee n) \wedge f(c_X(f^{-1}(n)))) \\
&\cong f^{-1}(m \wedge f(c_X(f^{-1}(n))) \vee (n \wedge f(c_X(f^{-1}(n))))) \\
&\cong f^{-1}(0_X \vee n), \quad \text{by 1.2(d)} \\
&\cong f^{-1}(n)\,.
\end{aligned}$$

Therefore, $c_R(n_r) \cong n_r$ and so, $r \cdot c_R(n_r) \cong n$. Thus, since the heredity of c assures that $r \wedge c_Y(n) \cong r \cdot c_R(n_r)$, we have that $r \wedge c_Y(n) \cong n$ and, consequently, that $m \wedge c_Y(n) \leq n$. Then, we obtain that

$$c_Y(n) \wedge m \leq n \wedge m \leq f(c_X(f^{-1}(n))) \wedge m \cong 0_Y\,, \tag{5}$$

and that

$$c_Y(n) \vee m \cong f(f^{-1}(c_Y(n))) \vee m \geq f(c_X(f^{-1}(n))) \vee m \cong 1_Y\,. \tag{6}$$

Since m is the complement of $f(c_X(f^{-1}(n)))$, it follows from (5) and (6) that $c_Y(n) \cong f(c_X(f^{-1}(n)))$. □

5 On universal c-quotients

A morphism $f : X \to Y$ is said to be a *universal c-quotient* provided that every pullback of f along an arbitrary morphism is a c-quotient.

5.1 A quotient map in the category of topological spaces is just a k-quotient, with k the Kuratowski closure operator. The universal quotienst in $\mathcal{T}op$ were characterized in [13] and [7] as being just those continuous maps $f : X \to Y$ which fulfil one of the following equivalent conditions:

(a) For each $y \in Y$ and each open coverage $(G_I)_{i\in I}$ of $f^{-1}(y)$, there is some finite subset J of I such that $y \in \operatorname{int}(\bigcup_{j\in J} f(G_j))$.

(b) For each filter base $\mathcal{B}$ in Y, if $y \in Y$ adheres to $\mathcal{B}$, then some $x \in f^{-1}(y)$ adheres to $f^{-1}(\mathcal{B})$.

These two conditions do not give clear information about the behaviour of f with respect to the closure operator. However several properties on the category of topological spaces are particular cases of properties in a much general setting, namely a category equipped with a closure operator. So, one natural question is: how to characterize universal quotients in $\mathcal{T}op$ by means of the closure operator? The answer is given by the following

Theorem 5.1 *In $\mathcal{T}op$, a morphism $f : X \to Y$ is a universal c-quotient, for c the Kuratowski closure operator, if and only if, for each family $(n_i)_{i\in I}$ of subobjects of Y, the condition*

$$\text{(GF)} \qquad \bigwedge_{\substack{J\subseteq I\\ J \text{finite}}} c_Y(\bigwedge_{j\in J} n_j) \leq f(\bigwedge_{i\in I} c_X(f^{-1}(n_i)))$$

holds.

Proof. Along the proof, we denote the Kuratowski closure of a subset F of a space by $\overline{F}$. The condition (b) stated in 5.1 is clearly equivalent to the following condition:

(c) For each filter $\mathcal{F}$ in Y, $\cap_{F\in\mathcal{F}}\overline{F} \subseteq f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)})$.

We show that it is equivalent to (GF). Assume (GF) and let $\mathcal{F}$ be a filter in Y. Putting $\mathcal{F} = \{F_i : i \in I\}$ and denoting by $\mathcal{J}$ the set of all finite subsets of I, we have that

$$\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}F_i} \subseteq f(\cap_{i\in I}\overline{f^{-1}(F_i)})\,. \tag{7}$$

But, for all finite subsets J of I, $\cap_{j\in J}F_i \in \mathcal{F}$; then $\{\overline{\cap_{j\in J}F_i},\, J\in\mathcal{J}\} = \{\overline{F_i},\, i\in I\}$ and (c) follows.

Conversely, assume (c). Let $(N_i)_{i\in I}$ be a family of subsets of Y. If, for some finite subset J of I, $\cap_{j\in J}N_i = \emptyset$, then (GF) is trivially verified. Otherwise, let $\mathcal{F}$ be the filter generated by $(N_i)_{i\in I}$. From (c), we have that

$$\cap_{F\in\mathcal{F}}\overline{F} \subseteq f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)})\,. \tag{8}$$

But

$$\cap_{F\in\mathcal{F}}\overline{F} = \cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}N_i} \tag{9}$$

and

$$f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)}) = f(\cap_{J\in\mathcal{J}}\overline{f^{-1}(\cap_{j\in J}N_j)}) = f(\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}f^{-1}(N_j)})\,. \tag{10}$$

Since $\{\overline{\cap_{j\in J}f^{-1}(N_j)},\, J\in\mathcal{J}\} \supseteq \{\overline{f^{-1}(N_i)},\, i\in I\}$, we have that

$$\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}f^{-1}(N_j)} \subseteq \cap_{i\in I}\overline{f^{-1}(N_i)}\,. \tag{11}$$

Then, from (8), (9) (10) and (11), we conclude that $\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}N_i} \subseteq f(\cap_{i\in I}\overline{f^{-1}(N_i)})$. □

Remark 5.2

1. It is clear that in a category with a closure operator, any morphism f fulfilling the condition (GF) is, in particular, c-final.

2. For the up-closure $\uparrow$ in the category of spatial graphs $\mathcal{SG}ph$, we have that:

   The universal $\uparrow$-quotients coincide with the $\uparrow$-final morphisms.

   To show this, first we recall that, in $\mathcal{SG}ph$, the $\uparrow$-final morphisms are just those surjections $f : X \to Y$ which fulfil the following condition: $y_1 \to y_2 \Leftrightarrow \exists x_1, x_2 \in X$ $(x_1 \to x_2$ and $f(x_i) = y_i$, $i = 1, 2)$ (cf. [6]). Let

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

be a pullback diagram, where $f : X \to Y$ is $\uparrow$-final and s is an arbitrary morphism. We show that g is also $\uparrow$-final. Given (x_1, z_1) and (x_2, z_2) in W, $(x_1, z_1) \to (x_2, z_2)$ is an edge in W iff $x_1 \to x_2$ is an edge in X and $z_1 \to z_2$ is an edge in Z. So, let $z_1 \to z_2$ be an edge in Z; then $s(z_1) \to s(z_2)$ belongs to Y, from what follows that there is some edge $x_1 \to x_2$ in X such that $f(x_i) = s(z_i)$, $i = 1, 2$. Consequently, $(x_1, z_1) \to (x_2, z_2)$ belongs to W with $g(x_1, z_1) = z_1$ and $g(x_2, z_2) = z_2$.

3. The condition (GF) does not give the desired characterization in the general setting. In fact, it fails to be a necessary condition in the category $\mathcal{SG}ph$ equipped with the up-closure. To show that, consider the following example: Let X be the spatial graph with underlying set $\bigcup\limits_{i \in \mathbb{N}} \{x_i, w_i\}$ and with non loop edges $x_i \to w_i$, $i \in \mathbb{N}$, and let Y be the spatial graph with underlying set $\bigcup\limits_{i \in \mathbb{N}} \{y_i\} \cup \{y\}$ and with non loop edges $y_i \to y$, $i \in \mathbb{N}$; let $f : X \to Y$ be defined by $f(x_i) = y_i$ and $f(w_i) = y$. Then, f is clearly $\uparrow$-final, so, from 2., a universal $\uparrow$-quotient. But it does not satisfy condition (GF). To see that, consider the family of all subobjects of Y of the form $N_i = \{y_k \mid k \geq i\}$; then $y \in \bigwedge\limits_{\substack{J \subseteq I \\ J \text{finite}}} \uparrow_Y (\bigwedge\limits_{j \in J} N_j)$ but $y \notin f(\bigwedge_{i \in I} \uparrow_X (f^{-1}(N_i)))$.

5.3 In this subsection, we show that in a suitable setting, which generalizes the one of the category of topological spaces, the condition (GF) is sufficient. For this purpose, we need to use the notion of point.

Let $\mathcal{X}$ have a terminal object 1; then, since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{X})$, each morphism with domain 1, being a split monomorphism, belongs to $\mathcal{M}$. The subobjects of an $\mathcal{X}$-object X with domain 1 are said to be *points of* X. The class of all points of X is denoted by pt$X$. The notion of point, in a sense which generalizes the one used here, was studied in detail in [3] (see also [4]). Let $m : M \to X$ be a subobject of X and let $x \in \mathrm{pt}X$. By pt(m) we denote the set of all points of X of the form mz, with $z \in \mathrm{pt}M$. We say that the category $\mathcal{X}$ *has enough points* provided that, for each $X \in \mathcal{X}$, $1_X \cong \bigvee \mathrm{pt}X$. This is equivalent to say that, for each $\mathcal{X}$-object X and each $m \in \mathrm{sub}X$, $m \cong \vee \mathrm{pt}(m)$. In fact, it is clear that the last assumption implies that $\mathcal{X}$ has enough points; conversely, we have that $\bigvee \mathrm{pt}(m) \cong \bigvee_{z \in \mathrm{pt}M} (mz) \cong m(\bigvee_{z \in \mathrm{pt}M} z) \cong m(1_M) \cong m$.

The following three lemmas will be useful to prove Theorem 5.3 below.

Lemma 5.3.1 *In a category with enough points, let m, m_i and n belong to sub$X$ and let $f : X \to Y$ be a morphism. Then:*

(a) $m \leq n \Leftrightarrow pt(m) \subseteq pt(n)$; and, thus, $m \cong n \Leftrightarrow pt(m) = pt(n)$.

(b) $pt(\bigwedge_{i \in I} m_i) = \bigcap_{i \in I} pt(m_i)$.

(c) $pt(m) = \emptyset \Rightarrow m \cong o_X$; furthermore, we have that $pt(m) = \emptyset \Leftrightarrow m \cong o_X$ whenever $o_X \ncong 1_X$.

(d) $x \in pt(m) \Rightarrow f(x) \in pt(f(m))$.

Proof. (a) and (b) are immediate.

(c) $\mathrm{pt}(m) = \emptyset \Rightarrow \bigvee \mathrm{pt}(m) \cong o_X \Leftrightarrow m \cong o_X$. To show that $o_X \ncong 1_X$ ensures that $\mathrm{pt}(o_X) = \emptyset$, let $z \in \mathrm{pt}(o_X)$. Then $z \leq o_X$, and so, $z \cong o_X$. But then, $O_X \cong 1$ and, consequently, for each $x \in \mathrm{pt}X$, it holds that $x \cong o_X$, from what follows that $1_X \cong \bigvee \mathrm{pt}X \cong o_X$.

(d) Let $m : M \to X$ be a subobject of X and let $z \in \mathrm{pt}M$ and $x \in \mathrm{pt}X$ be such that $x = mz$. Let $f(m) \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of the morphism fm. Then, we have that $f(x) \cong fx = fmz = f(m)ez$; this means that $f(x) \in \mathrm{pt}(f(m))$. $\square$

In the following, if sub$X$ is a Boolean algebra and $m \in \mathrm{sub}X$, we write m^c to denote the complement of m.

Lemma 5.3.2 *In a category with enough points, let subX and subY be Boolean algebras such that $o_Y \ncong 1_Y$, and let $f : X \to Y$ be a morphism. Then, for $m \in subX$ and $n, n_i \in subY$, we have that:*

(a) $y \in pt(n) \Leftrightarrow y \wedge n^c \cong o_Y \Leftrightarrow y \notin pt(n^c)$.

(b) $pt(\bigvee_{i \in I} n_i) = \bigcup_{i \in I} pt(n_i)$, for any finite set I.

(c) If complements are preserved under the inverse image by f, then:

(i) $m \wedge f^{-1}(n) \cong o_X \Leftrightarrow f(m) \wedge n \cong o_Y$.

(ii) $pt(f(m)) = \{f(x) \,:\, x \in pt(m)\}$.

Proof. (a) Let $y \in \mathrm{pt}Y$. Then $y \in \mathrm{pt}(n) \Leftrightarrow y \leq n \Leftrightarrow y \wedge n^c \cong o_Y \Leftrightarrow \mathrm{pt}(y \wedge n^c) = \emptyset \Leftrightarrow \mathrm{pt}(y) \wedge \mathrm{pt}(n^c) = \emptyset \Leftrightarrow y \notin \mathrm{pt}(n^c)$, the last but two equivalence being a consequence of 5.3.1 (c).

(b) Using (a) above and (b) and (c) of 5.3.1, we conclude that $y \in \mathrm{pt}(n_1 \vee n_2)$ iff $y \notin \mathrm{pt}(n_1^c) \cap \mathrm{pt}(n_2^c)$. Again by (a), we get that the last condition is equivalent to $y \in \mathrm{pt}(n_1)$ or $y \in \mathrm{pt}(n_2)$.

(c) (i) If $m \wedge f^{-1}(n) \cong o_X$, then $m \leq (f^{-1}(n))^c \cong f^{-1}(n^c)$. Consequently, $f(m) \leq n^c$ and, then, $f(m) \wedge n \cong o_Y$. Conversely, if $m \wedge f^{-1}(n) \ncong o_X$, then, by 5.3.1(c), there is some $x \in \mathrm{pt}(m \wedge f^{-1}(n))$. Using Lemma 5.3.1, one concludes that $f(x) \in \mathrm{pt}(f(m)) \cap \mathrm{pt}(n)$ and, then, since $o_Y \ncong 1_Y$, that $f(m) \wedge n \ncong o_Y$.

(c) (ii) The inclusion $\{f(x) : x \in pt(m)\} \subseteq \mathrm{pt}(f(m))$ is clear, from 5.3.1 (d). On the other hand, for $y \in \mathrm{pt}Y$, if $y \in \mathrm{pt}(f(m))$ then $y \wedge f(m) \ncong o_Y$ and, from (i), $f^{-1}(y) \wedge m \ncong o_X$. Thus $\mathrm{pt}(f^{-1}(y)) \cap \mathrm{pt}(m) \neq \emptyset$. This means that, for some $x \in \mathrm{pt}X$, $x \in \mathrm{pt}(f^{-1}(y)) \cap \mathrm{pt}(m)$ and, thus, $y \in \{f(x) : x \in \mathrm{pt}(m)\}$. $\square$

Lemma 5.3.3 *For a subobject $m : M \to X$ of X, the following two first assertions are equivalent and they are equivalent to (c) in a category with enough points.*

(a) $m : M \to X$ is c-closed.

(b) $m \cong \bigwedge\{n \in subX \,:\, n$ c-closed and $m \leq n\}$.

(c) If $x \in pt(X)$ is such that $m \leq n \Rightarrow x \in pt(n)$ for all c-closed subobjects n of X, then $x \in pt(m)$.

Proof. (a)$\Leftrightarrow$ (b): It is straightforward.

(b)$\Leftrightarrow$ (c): It follows from (a) and (b) of Lemma 5.3.1. $\square$

5.3.4 Given a pullback

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "r"'] & W \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{12}$$

let $\mathcal{N}$ be the family of all subobjects of Z of the form $n = r^{-1}(n_1) \vee g^{-1}(n_2)$ for some c-closed subobjects $n_1 \in \mathrm{sub}X$ and $n_2 \in \mathrm{sub}W$. We say that the pullback (12) is *c-initial*, provided that the c-closed subobjects of Z are just those $m : M \to Z$ in sub$Z$ such that $m \cong \bigwedge\{n \in \mathrm{sub}X : n \in \mathcal{N}$ and $m \leq n\}$. If $\mathcal{X}$ has enough points, this is equivalent to say that m fulfils the following property: for any $z \in \mathrm{pt}Z$, if $z \in \mathrm{pt}(n)$, for all $n \in \mathcal{N}$ such that $m \leq n$, then $z \in \mathrm{pt}(m)$.

We point out that for $\mathcal{X} = \mathcal{T}op$, $\mathcal{M}$ the class of embeddings and c the Kuratowski closure operator, the c-initiality for pullbacks coincides with the usual initiality of pullbacks in $\mathcal{T}op$.

Theorem 5.3 *Let $\mathcal{X}$ have enough points and let c be an idempotent closure operator in $\mathcal{X}$ such that any finite union of c-closed subobjects is a c-closed subobject. Moreover, let the diagram (12) be a c-initial pullback, let subX, subY, subZ and subW be Boolean algebras, with $o_Y \ncong 1_Y$, and let all morphisms in (12) preserve complements under the inverse image. Then, if $f : X \to Y$ fulfils condition (GF), it is a universal c-quotient.*

Proof. If $o_X \cong 1_X$ or $o_W \cong 1_W$, it is trivially concluded that g is a c-quotient. Thus, besides the assumption of that $o_Y \ncong 1_Y$, we also assume, without loss of generality, that $o_X \ncong 1_X$ and $o_W \ncong 1_W$. If $f : X \to Y$ fulfils condition (GF), then it is clearly a c-quotient; in particular, it belongs to $\mathcal{E}$ and so $g \in \mathcal{E}$. Let $n : N \to W$ be a subobject of W such that $g^{-1}(n)$ is c-closed. In order to show that n is c-closed, we are going to use (c) of Lemma 5.3.3: we show that, for $w \in \mathrm{pt}W$, if $w \notin \mathrm{pt}(n)$, then there exists a c-closed subobject u of W such that $n \leq u$ and $w \notin \mathrm{pt}(u)$.

Suppose that $w \notin \mathrm{pt}(n)$. Then, by (d) of Lemma 5.3.1, for each $z \in \mathrm{pt}(g^{-1}(w))$, we have that $z \notin \mathrm{pt}(g^{-1}(n))$. Consequently, using the c-initiality of the pullback (12), we obtain that, for each $z \in \mathrm{pt}(g^{-1}(w))$, there are c-closed subobjects n_{1z} and n_{2z} of X and W, respectively, such that

$$g^{-1}(n) \leq r^{-1}(n_{1z}) \vee g^{-1}(n_{2z}) \quad \text{and} \quad z \notin \mathrm{pt}(r^{-1}(n_{1z}) \vee g^{-1}(n_{2z}))\,. \tag{13}$$

Put $I = \mathrm{pt}(g^{-1}(w))$ and denote by $\mathcal{J}$ the class of all finite subsets of I. For each $z \in \mathrm{pt}(g^{-1}(w))$, let

$$m_z = (f(n_{1z}^c))^c\,.$$

Then, by (GF),

$$\bigwedge_{J\in\mathcal{J}} c_Y(\bigwedge_{z\in J} m_z) \leq f(\bigwedge_{z\in I} c_X(f^{-1}(m_z))\,. \tag{14}$$

First, we show that, for each $z \in I$, it holds that $f^{-1}(m_z) \leq n_{1z}$. Indeed,

$$\begin{aligned} m_z = (f(n_{1z}^c))^c \ &\Rightarrow m_z \wedge f(n_{1z}^c) \cong o_Y \\ &\Leftrightarrow f^{-1}(m_z) \wedge n_{1z}^c \cong o_X, \text{ by 5.3.2(c)(i)} \\ &\Rightarrow f^{-1}(m_z) \leq n_{1z} \end{aligned}$$

Consequently, since n_{1z} is c-closed, we have that $c_X(f^{-1}(m_z)) \leq n_{1z}$.

Now, we are going to show that $s(w) \notin \mathrm{pt}(c_Y(\bigwedge_{z\in J} m_z))$ for some $J \in \mathcal{J}$. For each $z \in \mathrm{pt}(g^{-1}(w))$, since $z \notin \mathrm{pt}(r^{-1}(n_{1z}))$, we have that $z \wedge r^{-1}(n_{1z}) \cong o_Z$; then, by 5.3.2(c)(i), $r(z) \wedge n_{1z} \cong o_X$. Since $c_X(f^{-1}(m_z)) \leq n_{1z}$, we obtain that $r(z) \wedge c_X(f^{-1}(m_z)) \cong o_X$. Consequently, for all $z \in \mathrm{pt}(g^{-1}(w))$, we have that $r(z) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$. Then, by Lemma 5.3.2(c)(ii), we conclude that $r(g^{-1}(w)) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$, that is, by (BCP), that $f^{-1}(s(w)) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$. By 5.3.2(c)(i), it follows that $s(w) \wedge f\left(\bigwedge_{z\in I} c_X(f^{-1}(m_z))\right) \cong o_Y$ and, consequently, using (14), that $s(w) \wedge (\bigwedge_{J\in\mathcal{J}}(c_Y(\bigwedge_{z\in J} m_z))) \cong o_Y$. This implies that $s(w) \notin \mathrm{pt}\,(c_Y(\bigwedge_{z\in J} m_z))$, for some $J \in \mathcal{J}$, and, thus, for this J, $w \notin \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)))$. On the other hand, for each $z \in J$, since $z \notin \mathrm{pt}(g^{-1}(n_{2z}))$, we have, by 5.3.1(c), that $z \wedge g^{-1}(n_{2z}) \cong o_Z$, what, combined with 5.3.2(c)(i) and the fact that $g(z) \cong w$, assures that $w \wedge n_{2z} \cong o_W$, that is, that $w \notin \mathrm{pt}(n_{2z})$. Now, using 5.3.2(b), we get that $w \notin \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z}))$.

The subobject $s^{-1}(c_Y(\bigwedge_{z\in J} m_z))$ is c-closed, since c is idempotent. Then, the subobject $s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z})$, being the finite union of c-closed subobjects of W, is, by assumption, c-closed. We are going to show that it has n as its subobject, what completes the proof. Let $t \in \mathrm{pt}(n)$. By (13) and using (BCP), we have that, for each $z \in J$, $t \leq s^{-1}(f(n_{1z})) \vee n_{2z}$. If $t \leq n_{2z}$ for some $z \in J$, then it is clear that $t \in \mathrm{pt}\left(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee (\bigvee_{z\in J} n_{2z}\right)$. Otherwise, since t is a point, for each $z \in J$, we get that $t \wedge n_{2z} \cong o_W$, and, by 5.3.1(d), that $g^{-1}(t) \wedge g^{-1}(n_{2z}) \cong o_X$. Consequently, as $g^{-1}(t) \leq r^{-1}(n_{1z}) \vee g^{-1}(n_{2z})$, and we are working in a Boolean algebra, it follows that $g^{-1}(t) \leq r^{-1}(n_{1z})$. Thus, we have that $r(g^{-1}(t)) \leq n_{1z}$, that is, $f^{-1}(s(t)) \leq n_{1z}$. This implies that $f^{-1}(s(t)) \wedge n_{1z}^c \cong o_X$. Then, using Lemma 5.3.2(c)(i) twice, we obtain in succession, that $s(t) \wedge f(n_{1z}^c) \cong o_Y$ and $t \wedge s^{-1}\left(f(n_{1z}^c)\right) \cong o_W$. Hence, $t \leq \left(s^{-1}\left(f(n_{1z}^c)\right)\right)^c$, and thus, $t \leq s^{-1}(m_z)$. Consequently, one concludes that $t \leq \bigwedge_{z\in J} s^{-1}(m_z) \leq s^{-1}(c_Y(\bigwedge_{z\in J} m_z))$. Therefore, we have that $t \in \mathrm{pt}\left(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee (\bigvee_{z\in J} n_{2z}\right)$. □

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