The Galois approach to uniform structures

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Abstract

In the category of uniform spaces, entourages (neighbourhoods of the diagonal), being binary relations, may be easily described by polarities (i.e. Galois connections between power sets) and axialities (i.e. residuated pairs between power sets). Here we consider this in the pointfree context of the category of locales, establishing characterizations of uniformities in terms of either the quantale of Galois connections and the quantale of residuated pairs. In virtue of this we conclude that locale uniformities may be thought of as either a family of polarities or as a family of axialities. The latter is precisely Fletcher-Hunsaker's approach [4] to locale uniformities, showing that Galois connections rather than entourages are the root of this approach. Then we prove that the equivalence between the somewhat different approaches to uniform locales of [4] and [11] is a consequence of a very general phenomenon on Galois connections. Besides, it is presented a categorical description for the quantale of Galois connections that justifies the correctness of its definition.

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Introduction

In [8] J. Isbell initiated the theory of uniform locales, by formulating in the pointfree context the classical approach of Tukey to uniform spaces via covers. As in the case of spaces, there are other different ways of describing locale uniformities, such as the entourage uniformities of Fletcher and Hunsaker [4] and the Weil uniformities of Picado [11]. The latter approach is the (categorical) translation into the category of locales of Weil's classical approach to uniform spaces. There

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remains the question of understanding better the nature of Fletcher-Hunsaker's approach. This is the question that gave rise to this paper.

In all entourage-like approaches to uniform structures on a locale L there is a certain quantalic structure $(Q(L), \cdot)$ and a special subclass $Ent(L)$ of $Q(L)$ (the "entourages" of L) for which $E \leq E \cdot E$ for every $E \in Ent(L)$. The uniformity $\mathcal{E}(L)$ is then a non-empty symmetric admissible filter of $Ent(L)$ satisfying the square-refinement axiom. This suggests us that the quantale of (contravariant) Galois connections introduced by the second author in [14] may provide a new characterization for locale uniformities. On the other hand, there is also the well known quantale of residuated mappings ([2], [3]), that is, covariant Galois connections.

The purpose of this paper is twofold: On one hand, to understand better the nature of Fletcher-Hunsaker's approach [4], in order to explain why this approach is equivalent to the one of Picado [11]; and, on the other hand, to describe uniform structures in terms of those quantales of Galois connections. These questions turned out to be closely related. Indeed, the answer to the latter shed light on the former: since the characterization in terms of the quantale of residuated pairs coincides with Fletcher-Hunsaker's approach, Galois connections rather than entourages are at the root of this approach.

This done, we have brought to an end the sistematic study of locale uniformities, initiated by the second author in [11] (see [12] for a survey), and proceeded in [13], with the goal of completing the picture for locale uniformities: they may be thought of as either a family of covers, a family of entourages (binary relations), a family of metrics (gauges), a family of polarities (Galois connections) or as a family of axialities (residuated pairs). The new descriptions via Galois connections and residuated pairs that we present in this paper embody the descriptions of classical Weil entourages in terms of polarities and axialities [3].

The paper is organized as follows. We begin with a brief survey on the background required (Section 1), collecting several facts about locales, frames, quantales, locale uniformities and Galois connections, and then proceed to present (Section 2) the quantalic structure of Galois connections. We show that it can be described in a natural categorical way, justifying the "correctness" of its definition.

Next we consider locale uniformities (Section 3), characterizing them via Galois connections (both contravariantly and covariantly). The covariant approach gives us precisely Fletcher-Hunsaker's entourages [4], illuminating their nature. We show that the equivalence between both approaches (that is, the isomorphism between the corresponding categories) is a special case of a purely lattice-theoretical fact about Galois connections; indeed, there is an interesting correspondence between Galois entourages and residuated pairs, preserving the quantalic structures, which becomes one-to-one precisely when one assumes the symmetry and the square-refinement axioms.

We conclude, in Section 4, with a diagram summarizing all possible ways of describing locale uniformities.

For technical reasons we have decided not to be quite consistent in formulating the context in which we work. In Section 2 we prefer to use the localic (geometric) perspective as it makes the categorical motivation more explicite. When dealing with uniformities, in Sections 3 and 4, we use the dual frame (algebraic) approach (which is, indeed, where all the fundamental facts are situated).

1. Background and preliminaries

1.1. Galois connections. Originally Galois connections were expressed in a contravariant form with transformations that reverse order [1]. A *Galois connection* between the partially ordered sets A and B is a pair (f,g) of order-reversing (antitone) maps $f : A \to B$ and $g : B \to A$ such that

$$a \leq gf(a) \text{ for all } a \in A \text{ and } b \leq fg(b) \text{ for all } b \in B. \tag{1.1.1}$$

We denote by $Gal(A,B)$ the set of all order-reversing maps $f : A \to B$ for which there exists $f^+ : B \to A$ such that (f, f^+) is a Galois connection, with the pointwise partial order.

As it is well known (and very easy to prove), condition (1.1.1) is equivalent to

$$a \leq g(b) \Leftrightarrow b \leq f(a) \text{ for every } a \in A \text{ and } b \in B.$$

Maps f and g uniquely determine each other; indeed

$$f(a) = \sup(\{b \in B \mid g(b) \geq a\})$$

and

$$g(b) = \sup(\{a \in A \mid f(a) \geq b\}).$$

If A and B are complete lattices then $f \in Gal(A,B)$ if and only if $f(\bigvee S) = \bigwedge f(S)$ for every $S \subseteq A$ (in particular, for $S = \emptyset$, $f(0) = 1$), that is, if and only if f is a complete join-homomorphism from A to B^{op}, the dual of B. In this case $Gal(A, B)$ is also a complete lattice (meets are defined pointwisely).

Galois connections appear in many areas of mathematics (cf. the survey [3]). For a more categorical approach to Galois connections we refer to [7].

By looking posets $(P, \leq)$ as *thin categories*, where the objects are the elements of P and

$$\begin{cases} |P(a,b)| = 1 & \text{if } a \leq b \\ \\ |P(a,b)| = 0 & \text{otherwise,} \end{cases}$$

the concept of Galois connection is just the categorical concept of dual adjunction applied to these categories: a pair $f : A \to B$, $g : B \to A$ of order-reversing maps (i.e. contravariant functors) is a Galois connection if and only if $f \dashv g$; maps $A \to B$ that are the first half of a Galois connection are precisely the left adjoint functors $A \to B^{op}$. For bounded A, there is a natural isomorphism $A^{op} \to H(A, 2) = A^*$ (where $H(A, B)$ is the set of all left adjoint functors $A \to B$, with the pointwise order, and 2 is the two-element chain), so that $(-)^*$ is a self-duality of the category of all bounded partially ordered sets and left adjoint maps. In this terminology, $Gal(A, B) \cong H(A, B^*)^*$.

Nowadays many authors prefer to work with Galois connections in the covariant form by its convenience (the survey [3] contains a list of references to this form). We refer to this dualized form as a residuated pair: a *residuated pair* between the partially ordered sets A and B is a pair (f, g) of order-preserving maps $f : A \to B$ and $g : B \to A$ such that

$$gf(a) \leq a \text{ for all } a \in A \text{ and } b \leq fg(b) \text{ for all } b \in B. \tag{1.1.2}$$

The map $f : A \to B$ in a residuated pair (f, g) is called *residuated* and the map $g : B \to A$ is called *residual.* Thus f is residuated (resp. residual) if and only if $f \in Gal(A, B^{op})$ (resp. $f \in Gal(A^{op}, B)$). If A and B are complete lattices then $f : A \to B$ is residuated if and only if it is a join-homomorphism.

As Blyth and Janowitz remarked in their monograph on Residuation Theory ([2], p. 19), the reason for preferring residuated mappings rather than Galois connections can be found in the fact that compositions of residuated pairs are handled more easily: two residuated mappings may be composed to yield a new residuated mapping and this is not the case with order-reversing mappings. Indeed, if

$f : A \to B$ and $g : B \to C$ are residuated maps, then $g \cdot f$ is clearly residuated with $(g \cdot f)^+ = f^+ \cdot g^+$ [2]. This composition endows $Gal(A, A^{op})$ with a quantalic structure (the *quantale of residuated maps* [3]).

However it is also possible to define a nice composition for Galois connections, as the second author recently proved in [14], if one restricts to locales (which is not a serious restriction since the right order-theoretic context for doing uniform topology is the category of locales).

We need to recall from Shmuely [19] that $Gal(A, B)$ is the tensor product of A and B in the category of complete join-semilattices. As Shmuely also proved ([19], Theorem 1.3) the elements of this tensor product stand in a one-to-one correspondence with certain semi-ideals of $A \times B$. Let us say that $E \subseteq A \times B$ is a *G-ideal* of $A \times B$ when

(1) E is a *down-set* of $A \times B$, that is,

$$\downarrow E := \{(x, y) \mid (x, y) \leq (a, b) \text{ for some } (a, b) \in E\}$$

coincides with E;

(2) $\Big(\{x\} \times S \subseteq E \Rightarrow (x, \bigvee S) \in E\Big)$ and $\Big(S \times \{y\} \subseteq E \Rightarrow (\bigvee S, y) \in E\Big)$.

We denote by $A \otimes B$ the set of all G-ideals of $A \times B$, partially ordered by inclusion. This is clearly a complete lattice, since any intersection of G-ideals is always a G-ideal. Furthermore $A \otimes B \cong Gal(A, B)$ [19]. For every $a \in A$ and $b \in B$,

$$\begin{array}{rcl} p_{a,b} : A & \longrightarrow & B \\ x & \longmapsto & \begin{cases} 1 & \text{if } x = 0 \\ b & \text{if } 0 < x \leq a \\ 0 & \text{if } x \not\leq a. \end{cases} \end{array}$$

is a Galois map. These maps generate $Gal(A, B)$ by joins, that is, every $f \in Gal(A, B)$ is a join of some family $\{p_{a_i,b_i}\}_{i \in I} \subseteq Gal(A, B)$ [19]. Under the isomorphism $Gal(A, B) \cong A \otimes B$, the Galois map $p_{a,b}$ corresponds to the G-ideal

$$a \otimes b := \downarrow(a, b) \cup \downarrow(1, 0) \cup \downarrow(0, 1).$$

(Note that the case $S = \emptyset$ in (2) implies that every G-ideal contains the set $\downarrow(1, 0) \cup \downarrow(0, 1)$.)

1.2. Frames, locales and quantales. Recall that a *frame* is a complete lattice L satisfying the distributive law

$$a \wedge \bigvee S = \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ is a map preserving finite meets including the top (usually denoted by 1) and arbitrary joins including the bottom (usually denoted by 0). The resulting category will be denoted by Frm. Frames are exactly complete Heyting algebras with the Heyting implication given by

$$a \to b = \bigvee\{x \in L \mid x \wedge a \leq b\}.$$

The motivating example for frames is the frame $\mathcal{O}(X)$ of open sets of a topological space X and the frame homomorphism $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ obtained from a continuous map $f : X \to Y$ by putting $\mathcal{O}(f)(x) = f^{-1}(x)$. Thus a contravariant functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$ is obtained; the dual of Frm is called the category of locales [8], usually denoted by Loc, and $\mathcal{O}$ becomes a covariant functor $\mathsf{Top} \to \mathsf{Loc}$.

We adopt the usual convention that if $h : L \to M$ is a continuous map of locales, we shall write $h^* : M \to L$ for the corresponding frame homomorphism, and $h_* : L \to M$ for the right adjoint of h^*.

A *sublocale* (*part* in [8]) $M \hookrightarrow L$ of a locale L is, in frame terms, a surjective frame homomorphism $L \to M$. The elements $a \in L$ induce the *open*, respectively *closed*, sublocales $\hat{a} :\downarrow \{a\} \hookrightarrow L$, resp. $\check{a} :\uparrow \{a\} \hookrightarrow L$, given by $\hat{a}^*(x) = a \wedge x$ and $\check{a}^*(x) = a \vee x$, complementing each other. Sublocales of a locale L are conveniently described by nuclei on L [9]. A *nucleus* on L is a closure operator j on L which preserves binary meet. Given a regular monomorphism $f : M \hookrightarrow L$, the composition $L \xrightarrow{f^*} M \xrightarrow{f_*} L$ is a nucleus on L. In particular, for any open sublocale $\hat{a}$ of L the corresponding nucleus is the map $u(a) : L \to L$ given by the relative pseudocomplement $a \to b$ for any $b \in L$.

The tensor product in the category of complete join-semilattices, described above, provides the construction of binary products in Loc. Indeed, for locales L and M, the complete lattice $L \otimes M$ is again a locale; defining $p_1 : L \otimes M \to L$ and $p_2 : L \otimes M \to M$ by, respectively, $p_1^*(x) = x \otimes 1$ and $p_2^*(y) = 1 \otimes y$, $(L \otimes M, p_1, p_2)$ is the product of L and M in Loc.

For more details on frames and locales we refer the reader to [9] or [20]; for category theory to [6] and [10].

A *quantale* is a complete lattice Q with an associative binary multiplication $\cdot$ which distributes over arbitrary suprema:

$$a \cdot \bigvee S = \bigvee\{a \cdot s \mid s \in S\} \text{ and } (\bigvee S) \cdot a = \bigvee\{s \cdot a \mid s \in S\}.$$

More information on quantales can be found in the monograph [18] of Rosenthal.

1.3. Uniformities. The theory of uniform spaces was initiated by Weil in terms of entourages (i.e. reflexive binary relations). Uniform structures were first extended to locales using covers ([8], [17], [15]), but they may also be characterized in terms of Weil entourages [11]. A *Weil entourage* of a locale L is just an element E of $L \otimes L$ for which the diagonal $\Delta_L : L \to L \otimes L$ factorizes through the (open) sublocale $\hat{E} :\downarrow\{E\} \to L \otimes L$:

$$\begin{tikzcd} & L \arrow[dl, "\exists f"'] \arrow[d, "\Delta_L"] \\ \downarrow\{E\} \arrow[r, tail, "\hat{E}"'] & L \otimes L \end{tikzcd}$$

A *Weil uniformity* [11] on a frame L is a non-empty filter $\mathcal{E}$ of Weil entourages of L satisfying the following axioms:

[Square-refinement] For each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F^2 \subseteq E$;

[Symmetry] For every $E \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$;

[Admissibility] For every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$, where $y \overset{\mathcal{E}}{\lhd} x$ means that $\bigvee\{z \in L \mid (z, z) \in E, z \wedge x \neq 0\} \leq x$ for some $E \in \mathcal{E}$.

A *Weil uniform frame* is just a frame L with a specified Weil uniformity $\mathcal{E}$. For more details about the corresponding category WUFrm of Weil uniform frames and Weil homomorphisms consult [12].

Fletcher and Hunsaker [4] have given an alternative (equivalent) functional approach to uniform frames (see [5] for a survey).

Note that the notion of Weil entourage is the exact counterpart for locales of Weil's notion of entourage; indeed, $E \subseteq X \times X$ is an entourage of a set X if the diagonal $\Delta_X : X \to X \times X$ factorizes through the inclusion $E \hookrightarrow X \times X$:

$$\begin{tikzcd} & X \arrow[dl, "\exists f"'] \arrow[d, "\Delta_X"] \\ E \arrow[r, tail] & X \times X \end{tikzcd}$$

For any set X, $\mathcal{P}(X \times X)$ equipped with the usual composition relation $R \circ S$ is a quantale. This quantalic structure of binary relations of X is the basic structure on which the theory of uniform spaces of Weil relies on.

2. The quantale of Galois connections, categorically

Let L be a complete lattice. In [14] it is presented a way of composing Galois connections directly, in the following manner. For any order-reversing maps $f, g : L \to L$, let $g \cdot f : L \to L$ be defined by

$$\begin{aligned}(g \cdot f)(x) &= \bigvee\{g(y) \mid y \in L \setminus \{0\}, y \leq f(x)\} \\ &= \bigvee\{b \in L \mid \exists c \in L \setminus \{0\} : f(x) \geq c \text{ and } g(c) \geq b\}.\end{aligned}$$

This is an order-reversing map (the *antitone composition* of f with g). Then the *Galois composition* $g \circ f : L \to L$ is the Galois map $< g \cdot f >$ generated by $g \cdot f$. Equivalently, $g \circ f(x) = \bigvee\{y \in L \mid (x, y) \in E_{f,g}\}$, where $E_{f,g}$ denotes the G-ideal

$$\bigvee\{a \otimes b \in L \otimes L \mid \exists c \in L \setminus \{0\} : f(a) \geq c \text{ and } g(c) \geq b\}.$$

It is proved that $\big(Gal(L, L), \circ\big)$ is a quantale whenever L is a locale. In this section we are going to show that the operation $\circ$ has a very natural categorical description, which ensures its "correctness".

We begin by recalling that, in any cartesian category $\mathcal{C}$ with images, binary relations of an object X are defined as subobjects of the product $X \times X$. One can compose them; given two relations

$$\begin{tikzcd} R_1 \arrow[r, tail, "m_1"] & X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} R_2 \arrow[r, tail, "m_2"] & X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

the construction goes as follows [6]: consider the pullback

$$\begin{tikzcd} R_1 \circ R_2 \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & R_2 \arrow[d, "p_1 m_2"] \\ R_1 \arrow[r, "p_2 m_1"'] & X \end{tikzcd}$$

Then there exists a unique map $R_1 \circ R_2 \xrightarrow{f} X \times X$ such that the diagram

$$\begin{tikzcd}
 & & X \times X \arrow[ddll, "p_1"'] \arrow[ddrr, "p_2"] & & \\
 & & R_1 \circ R_2 \arrow[u, "f"] \arrow[dl] \arrow[dr] & & \\
 & R_1 \arrow[dl] \arrow[dr] & & R_2 \arrow[dl] \arrow[dr] & \\
X & & X & & X
\end{tikzcd}$$

is commutative. The image

$$f(R_1 \circ R_2) \overset{m}{\rightarrowtail} X \times X$$

of f is then the composition of the relations m_1 and m_2.

If we perform this composition in the category of locales, with open sublocales, we get precisely the operation $\circ$ defined above, as we shall now prove. We begin by observing that $Gal(L, L)$ may be described by the open sublocales of $L \otimes L$ in the following way: each Galois map $f : L \to L$ corresponds to the open sublocale $\hat{f} : \downarrow\{f\} \hookrightarrow L \otimes L$. So, each Galois map may be regarded as a binary relation on L. Note that $\hat{f}$ is the open sublocale corresponding to the nucleus

$$\begin{array}{ccc} L \otimes L & \to & L \otimes L \\ g & \mapsto & (f \to g). \end{array}$$

Theorem 2.1. *For every locale L and every $f, g \in Gal(L, L)$, the composition $g \circ f$ is the categorical composition in* Loc *of f with g, when elements of $Gal(L, L)$ are regarded as relations on L.*

Proof. The categorical composition of

$$\hat{g} : \downarrow\{g\} \to L \otimes L$$

with

$$\hat{f} : \downarrow\{f\} \to L \otimes L$$

is the image $m : M' \to L \otimes L$ of the unique map $\varphi : M \to L \otimes L$ such that $p_1\varphi = p_1\hat{g}\varphi_1$ and $p_2\varphi = p_2\hat{f}\varphi_2$, where $(M, \varphi_1, \varphi_2)$ is the pullback of $(p_2\hat{g}, p_1\hat{f})$.

$$
\begin{tikzcd}
L \otimes L \arrow[rrr, "p_2"] \arrow[dddd, "p_1"'] & & & L \\
& M' \arrow[ul, "m"'] & & \\
& M \arrow[u, "q"'] \arrow[uul, dashed, "\varphi"] \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \downarrow\{f\} \arrow[uur, "p_2\hat{f}"'] \arrow[d, "p_1\hat{f}"] & \\
& \downarrow\{g\} \arrow[r, "p_2\hat{g}"'] \arrow[dl, "p_1\hat{f}"] & L & \\
L & & &
\end{tikzcd}
$$

We have to show that the sublocale $m : M' \to L \otimes L$ corresponds to the Galois map

$$
\begin{array}{rccl}
g \circ f : & L & \to & L \\
 & x & \mapsto & \bigvee\{y \in L \mid (x, y) \in E_{f,g}\}.
\end{array}
$$

Since the sublocale $m : M' \to L \otimes L$ may be described by the nucleus $m_* m^*$, it suffices to check that $m_* m^*(F) = E_{f,g} \to F$ for every $F \in L \otimes L$. In order to do that we need to construct $\varphi : M \to L \otimes L$.

Claim 1. *The pullback of* $(p_2', p_1') = (p_2\hat{g}, p_1\hat{f})$ *is the locale*

$$M = \downarrow\{I_{f,g}\} = \{E \in L \otimes L \otimes L \mid E \subseteq I_{f,g}\},$$

where

$$I_{f,g} = \Big\{(x, y, z) \in L \times L \times L \mid x = 0 \text{ or } y = 0 \text{ or } z = 0 \text{ or } (y \neq 0, f(x) \geq y, g(y) \geq z)\Big\},$$

together with maps $\varphi_1 : M \to \downarrow\{g\}$ *and* $\varphi_2 : M \to \downarrow\{f\}$ *defined by*

$$\varphi_1^*(h) = \bigvee\{x \otimes y \otimes z \in L \otimes L \otimes L \mid h(x) \geq y \text{ and } g(y) \geq z\}$$

and

$$\varphi_2^*(h) = \bigvee\{x \otimes y \otimes z \in L \otimes L \otimes L \mid f(x) \geq y \text{ and } h(y) \geq z\}.$$

Proof of Claim 1.

- It is very easy to see that φ_1 and φ_2 are locale homomorphisms (that is, φ_1^*, φ_2^* are morphisms of Frm).
- $\varphi_1^* p_2'(a) = \varphi_1^*(p_{1,a} \wedge g) = \bigvee\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, y \leq a\}$ and $\varphi_2^* p_1'(a) = \varphi_2^*(p_{a,1} \wedge f) = \bigvee\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, y \leq a\}$. Thus $\varphi_1^* p_2' = \varphi_2^* p_1'$.
- If $\psi_1 : N \to \downarrow\{g\}$ and $\psi_2 : N \to \downarrow\{f\}$ satisfy $p_2'\psi_1 = p_1'\psi_2$, there exists a unique locale homomorphism $\psi : N \to M$ such that $\varphi_1\psi = \psi_1$ and $\varphi_2\psi = \psi_2$. Indeed, $\psi : N \to M$ defined by

  $$\psi^*(F) = \bigvee\{\psi_1(x \otimes y) \wedge \psi_2(y \otimes z) \mid (x, y, z) \in F\}$$

  is the unique such map, as can be easily checked.

Claim 2. $\varphi : M \to L \otimes L$ *is given by*

$$\varphi^*(F) = \bigvee\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, (x, z) \in F\};$$

in particular,

$$\varphi^*(a \otimes b) = \bigvee\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, x \leq a, z \leq b\}.$$

Proof of Claim 2. To prove this we need the following basic property of G-ideals: every G-ideal F may be written as $\bigvee_{(a,b)\in F}(a \otimes b)$. Applying it we get

$$\begin{aligned}
\varphi^*(F) &= \varphi^*\Big(\bigvee_{(a,b)\in F}(a \otimes b)\Big) \\
&= \bigvee_{(a,b)\in F}(\varphi^*(a \otimes 1) \wedge \varphi^*(1 \otimes b)) \\
&= \bigvee_{(a,b)\in F}((p_1'\varphi_2)^*(a) \wedge (p_2'\varphi_1)^*(b)).
\end{aligned}$$

Then, by Claim 1, we have

$$\begin{aligned}
\varphi^*(F) &= \bigvee\Big\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, x \leq a, z \leq b, (a, b) \in F\Big\} \\
&= \bigvee\{x \otimes y \otimes z \mid f(x) \geq y, g(y) \geq z, (x, z) \in F\}.
\end{aligned}$$

Let us finally prove the theorem. For any $F \in L \otimes L$ we have

$$\begin{aligned} m_* m^*(F) &= \bigvee\{E \in L \otimes L \mid m^*(E) \leq m^*(F)\} \\ &= \bigvee\{E \in L \otimes L \mid \varphi^*(E) \leq \varphi^*(F)\} \\ &= \bigvee\{a \otimes b \mid \varphi^*(a \otimes b) \leq \varphi^*(F)\}. \end{aligned}$$

By Claim 2, this join is equal to

$$\bigvee\Big\{a \otimes b \mid x, y, z \neq 0, x \leq a, z \leq b \text{ and } (f(x) \geq y, g(y) \geq z) \Rightarrow (x, z) \in F\Big\}.$$

But this is precisely the relative pseudocomplement $E_{f,g} \to F$, as straightforward checking shows. □

Remark. The associativity of the operation $\circ$ is the hardest thing to show in the proof, in [14], of the fact that $(Gal(L, L), \circ)$ is a quantale. Now one could think about concluding it by categorical tools. Indeed there is a well known theorem (cf. [6], p. 83) which asserts that, in any cartesian category $\mathcal{C}$ with images, composition of relations is associative if and only if $\mathcal{C}$ ir regular. But Loc is not regular thus composition of relations in locales is not associative. Note that this does not contradict the fact that $\circ$ is associative in $Gal(L, L)$ because here we only deal with compositions of special binary relations (more precisely, the open sublocales of $L \otimes L$). This seems to indicate that one cannot avoid the rather long and technical proof of the associativity of $\circ$, presented in [14].

3. Uniformities via Galois connections

The fact that an entourage of a set X is a subset E of X satisfying $\Delta_X(X) \subseteq E$ leads us to define a *polarity* of a frame L (the reason for the choice of the word "polarity" will be apparent in the next section) as a map $f \in Gal(L, L)$ such that $1 \leq \Delta_L^*(f)$ or, equivalently,

$$\bigvee\{x \in L \mid x \leq f(x)\} = 1.$$

We shall denote the set of all polarities of L by $Pol(L)$.

Proposition 3.1. *For every $f \in Pol(L)$, $f \leq f \circ f$.*

Proof. For each $x \in L$ we have

$$\begin{aligned} f(x) &= f(x) \wedge 1 \\ &= f(x) \wedge \bigvee \{y \in L \mid y \leq f(y)\} \\ &= \bigvee \{f(x) \wedge y \mid y \leq f(y), y \wedge f(x) \neq 0\}. \end{aligned}$$

Therefore, in order to show that

$$(x, f(x)) \in \bigvee \{a \otimes b \in L \otimes L \mid \exists c \in L \setminus \{0\} : f(a) \geq c \text{ and } f(c) \geq b\},$$

it suffices to check that

$$(x, f(x) \wedge y) \in \bigvee \{a \otimes b \in L \otimes L \mid \exists c \in L \setminus \{0\} : f(a) \geq c \text{ and } f(c) \geq b\}$$

whenever $f(y) \geq y$ and $y \wedge f(x) \neq 0$, which is indeed true: $f(x) \geq f(x) \wedge y \neq 0$ and $f(f(x) \wedge y) \geq f(y) \geq y \geq y \wedge f(x)$. $\square$

Now we have all the necessary tools to describe uniform structures. For any $x \in L$ and $f \in Gal(L, L)$ let

$$st(x, f) := \bigvee \{y \in L \mid y \leq f(y), y \wedge x \neq 0\}.$$

Definition 3.2. A non-empty filter $\mathcal{G}$ of $(Pol(L), \leq)$ is a *Galois uniformity* on L if it satisfies the following axioms:

[**Square-refinement**] For each $f \in \mathcal{G}$ there is $g \in \mathcal{G}$ such that $g \circ g \leq f$;

[**Symmetry**] For every $f \in \mathcal{G}$, $f^+ \in \mathcal{G}$;

[**Admissibility**] For every $x \in L$, $x = \bigvee \{y \in L \mid y \overset{\mathcal{G}}{\lhd} x\}$, where $y \overset{\mathcal{G}}{\lhd} x$ means that $st(y, f) \leq x$ for some $f \in \mathcal{G}$.

A *Galois uniform frame* is a frame together with a specified uniformity. Let $(L, \mathcal{G})$ and $(M, \mathcal{H})$ be Galois uniform frames. A *Galois homomorphism* $f : (L, \mathcal{G}) \to (M, \mathcal{H})$ is a frame homomorphism such that, for every $g \in \mathcal{G}$, there exists $h \in \mathcal{H}$ satisfying $fg \leq hf$.

This yields the category GUFrm of Galois uniform frames and Galois homomorphisms.

Now let us get a similar description using the quantale $\left(Gal(L, L^{op}), \cdot\right)$ of residuated maps. Here the operation $\cdot$ is just function composition.

Given $f \in Gal(L, L^{op})$, an element $y \in L$ is *f-small* if $y \leq f(x)$ whenever $y \wedge x \neq 0$. We say that the residuated map f is an *axiality* (the reason for the choice of this word will also be apparent in the next section) if

$$\bigvee\{y \in L \mid y \text{ is } f\text{-small}\} = 1.$$

We denote the set of all axialities of L by $Ax(L)$.

Proposition 3.3. *For every $f \in Ax(L)$, $f \leq f \cdot f$.*

Proof. For each $x \in L$ we have

$$\begin{aligned} x &= x \wedge \bigvee\{y \in L \mid y \text{ is } f\text{-small}\} \\ &= \bigvee\{x \wedge y \mid x \wedge y \neq 0, y \text{ is } f\text{-small}\} \\ &\leq f(x). \end{aligned}$$

Then, immediately, $f(x) \leq f(f(x))$ for every $x \in L$. □

We have again all the necessary tools to describe uniformities. If we pursue this approach we obtain precisely Fletcher-Hunsaker's theory of entourage uniform frames [4] (our axialities being precisely their entourages). This indicates that Galois maps are the root for the entourage uniformities of [4].

It is quite surprising that, despite the dual order on the second frame, also $Gal(L, L^{op})$ completely describes frame uniformities. We conclude with a nice correspondence between Galois maps and residuated maps on frames, that clarifies this.

To begin with, for each $f \in Gal(L, L)$ let $\overline{f} : L \to L$ be defined by $\overline{f}(x) = st(x, f)$. Clearly, this is a residuated map and $f \leq g$ implies $\overline{f} \leq \overline{g}$. Furthermore, we have:

Proposition 3.4. *For any $f \in Gal(L, L)$ we have:*

(a) *If $x \leq f(x)$ then x is $\overline{f}$-small;*

(b) *If f is a polarity and x is $\overline{f}$-small then $x \leq f \circ f(x)$.*

Proof. (a) Consider $y \in L$ satisfying $x \wedge y \neq 0$, Then, by hypothesis,

$$x \leq \bigvee\{z \in L \mid z \wedge y \neq 0, z \leq f(z)\},$$

that is, $x \leq st(y, f) = \overline{f}(y)$.

(b) Consider $y \in L$ such that $y \leq f(y)$ and $y \wedge x \neq 0$. Then, by the $\overline{f}$-smallness of x,

$$x \leq \overline{f}(y) = \bigvee\{z \in L \mid z \wedge y \neq 0, z \leq f(z)\}.$$

But $\overline{f}(y) \leq f \circ f(y)$. Indeed, for every $z \in L$ satisfying $z \wedge y \neq 0$ and $z \leq f(z)$, we have $f(y) \geq y \geq y \wedge z \neq 0$ and $f(y \wedge z) \geq f(z) \geq z$, which shows that $z \leq f \circ f(y)$.

So far we have proved that $x \leq f \circ f(y)$ whenever $y \leq f(y)$ and $y \wedge x \neq 0$. It follows that

$$\begin{aligned} x &\leq \bigwedge\{f \circ f(y) \mid y \leq f(y), y \wedge x \neq 0\} \\ &= f \circ f\Big(\bigvee\{y \in L \mid y \leq f(y), y \wedge x \neq 0\}\Big). \end{aligned}$$

Combining this with the relation

$$x = x \wedge \bigvee\{y \in L \mid y \leq f(y)\} \leq \bigvee\{y \in L \mid y \leq f(y), y \wedge x \neq 0\}$$

(which is a consequence of f being a polarity) we see that

$$f \circ f\Big(\bigvee\{y \in L \mid y \leq f(y), y \wedge x \neq 0\}\Big) \leq f \circ f(x).$$

Hence $x \leq f \circ f(x)$. $\square$

Conversely, given $f \in Gal(L, L^{op})$ define map $\overset{\circ}{f}\colon L \to L$ by putting

$$\overset{\circ}{f}(x) = \bigvee\Big\{y \in L \mid (x, y) \in \bigvee\{a \otimes a \mid a \text{ is } f\text{-small}\}\Big\}.$$

Obviously, this is a Galois map and $f \leq g$ implies $\overset{\circ}{f} \leq \overset{\circ}{g}$. It is straightforward to check that, under the correspondences $f \mapsto \overline{f}$ and $f \mapsto \overset{\circ}{f}$, axialities correspond to polarities and vice-versa. The following provides a useful property of $\overset{\circ}{f}$:

Proposition 3.5. *For every $f \in Gal(L, L^{op})$ and every $x, z \in L$, we have:*

(a) $p_{z,z} \circ \overset{\circ}{f} \leq p_{f(z),z}$;

(b) $\overset{\circ}{f}(x) \leq f(z)$ *whenever* $x \wedge z \neq 0$.

Proof. (a) The map $\overset{\circ}{f}$ is clearly the Galois map generated by the order-reversing map $f_\cup$ defined by

$$f_\cup(x) = \bigvee\Big\{y \in L \mid (x, y) \in \bigcup\{a \otimes a \mid a \text{ is } f\text{-small}\}\Big\} \text{ (cf. [14]).}$$

Thus $p_{z,z}\circ \overset{\circ}{f} = p_{z,z}\circ <f_\cup>$. By Lemma 4.1 and Lemma 4.3 of [14], we have

$$p_{z,z}\circ <f_\cup> = p_{z,z}\circ f_\cup = <p_{z,z}\cdot f_\cup>,$$

where $p_{z,z}\cdot f_\cup$ is the antitone composition of $f_\cup$ with $p_{z,z}$, defined by

$$(p_{z,z}\cdot f_\cup)(x) = \bigvee\{p_{z,z}(a) \mid a \in L\setminus\{0\}, a \le f_\cup(x)\}.$$

Therefore it suffices to check that $p_{z,z}\cdot f_\cup \le p_{f(z),z}$. Since

$$\begin{aligned}(p_{z,z}\cdot f_\cup)(x) &= \bigvee\{p_{z,z}(a) \mid f_\cup(x) \ge a \ne 0\}\\ &= \bigvee\{z \in L \mid z \wedge f_\cup(x) \ne 0\},\end{aligned}$$

we only need to check that $z\wedge f_\cup(x) \ne 0$ implies $x \le f(z)$. To see this, note that $z\wedge f_\cup(x)\ne 0$ means that there exists $y \in L$ for which $y\wedge z \ne 0$ and $(x,y)\le(a,a)$, for some f-small element a. Then $a\wedge z \ge y\wedge z \ne 0$, thus $a \le f(z)$. Hence $x \le f(z)$.
(b) Consider $(x,y) \in \bigvee\{a\otimes a \mid a \text{ is } f\text{-small}\}$. We need to show that $y \le f(z)$. By symmetry, $(y,x) \in \bigvee\{a\otimes a \mid a \text{ is } f\text{-small}\}$. Then $\overset{\circ}{f}(y) \ge x\wedge z \ne 0$ and $p_{z,z}(x\wedge z) = z$; this means that $(p_{z,z}\circ \overset{\circ}{f})(y) \ge z$. Thus, from (a), it follows that $p_{f(z),z}(y) \ge z$, that is, $y \le f(z)$. □

Let us define a *symmetric Galois map* as a Galois map $f \in Gal(L,L)$ for which $f^+ = f$. Of course, this means that $f(f(x)) \ge x$ for every $x \in L$ or, equivalently, that $x \le f(y)$ if and only if $y \le f(x)$ for every $x,y \in L$. It is straightforward to check that $\overset{\circ}{f}$ is always a symmetric Galois map.

Similarly a *symmetric residuated map* is a residuated map f for which

$$f(x)\wedge y \ne 0 \Leftrightarrow x\wedge f(y)\ne 0 \text{ for every } x,y\in L.$$

The map $\overline{f}$ is always symmetric.

The following lemma is of crucial importance for our purpose.

Lemma 3.6. *For any $f \in Gal(L,L)$ we have:*

(a) *If $(x,y) \in \bigvee\{a\otimes a \mid a \le f(a)\}$ then $y \le f(x)$;*

(b) *If f is a symmetric polarity and x and y are non-zero elements satisfying $x \le f(y)$ then $x\vee y \le f\circ f(x\vee y)$.*

Proof. (a) From the properties of a Galois map f it follows immediately that $E_f := \{(x,y) \in L \times L \mid y \leq f(x)\}$ is a G-ideal: f being order-reversing implies that E_f is a down-set and the property $f(\bigvee_{i\in I} x_i) = \bigwedge_{i\in I} f(x_i)$ implies condition (2) of the definition of G-ideal.

Therefore the inclusion $\{(a,a) \in L \times L \mid a \leq f(a)\} \subseteq E_f$ implies

$$\bigvee\{a \otimes a \mid a \leq f(a)\} \subseteq E_f.$$

(b) By the symmetry of f, $z \leq f \circ f(z)$ whenever $f(z) \neq 0$.

So, let $x, y \in L$, $x, y \neq 0$, satisfying $x \leq f(y)$ and $y \leq f(x)$. Then $f \circ f(x) \geq x$ and $f \circ f(y) \geq y$. On the other hand, by Proposition 3.1, $f \circ f(x) \geq f(x) \geq y$ and $f \circ f(y) \geq f(y) \geq x$. Hence $f \circ f(x \vee y) = f \circ f(x) \wedge f \circ f(y) \geq x \vee y$. □

Theorem 3.7. *Let f be an axiality of L. Then:*

(a) $\overline{\overset{\circ}{f}} \leq f$;

(b) *If f is symmetric then* $f \leq \overline{\overset{\circ}{f^3}}$.

Proof. (a) It is an immediate consequence of Proposition 3.5(b).
(b) Since

$$f(x) = \bigvee\{f(x \wedge y) \mid y \text{ is } f\text{-small}, y \wedge x \neq 0\}$$

it suffices to check that $f(x \wedge y) \leq \overline{\overset{\circ}{f^3}}(x)$ whenever y is f-small and $y \wedge x \neq 0$. So let $y \in L$ be an f-small element such that $x \wedge y \neq 0$. Then $f(x \wedge y)$ is f^3-small. In fact, if $f(x \wedge y) \wedge \alpha \neq 0$ then, by symmetry, $x \wedge y \wedge f(\alpha) \neq 0$ and, consequently, by f-smallness, $y \leq f(x \wedge f(\alpha)) \leq f(f(\alpha))$; this implies $f(x \wedge y) \leq f^3(\alpha)$. Now, if $f(x \wedge y)$ is f^3-small then, obviously, $f(x \wedge y) \leq \overset{\circ}{f^3}(f(x \wedge y))$. But $x \wedge y \wedge f(x \wedge y) = x \wedge y \neq 0$ hence

$$f(x \wedge y) \leq \overline{\overset{\circ}{f^3}}(x \wedge y) \leq \overline{\overset{\circ}{f^3}}(x).$$

□

We have a similar result for polarities:

Theorem 3.8. *Let f be a polarity of L. Then:*

(a) $\overset{\circ}{\overline{f}} \leq f \circ f$;

(b) *If f is symmetric then* $f \leq \overset{\circ}{\overline{f \circ f}}$.

Proof. (a) By Proposition 3.4(b),

$$\bigvee\{a \otimes a \mid a \text{ is } \overline{f}\text{-small}\} \subseteq \bigvee\{a \otimes a \mid a \leq f \circ f(a)\}.$$

Therefore, by Lemma 3.6(a), $y \leq f \circ f(x)$ for every $(x, y) \in \bigvee\{a \otimes a \mid a \text{ is } \overline{f}\text{-small}\}$. This ensures that $\overset{\circ}{\overline{f}}(x) \leq f \circ f(x)$.
(b) Using Proposition 3.4(a),

$$\bigvee\Big\{y \in L \mid (x, y) \in \bigvee\{a \otimes a \mid a \leq f \circ f(a)\}\Big\} \leq \overset{\circ}{\overline{f \circ f}}(x).$$

Let us show that

$$f(x) \leq \bigvee\Big\{y \in L \mid (x, y) \in \bigvee\{a \otimes a \mid a \leq f \circ f(a)\}\Big\}.$$

The cases $x = 0$ and $f(x) = 0$ are obvious. If $x \neq 0$ and $f(x) \neq 0$ this is a consequence of Lemma 3.6(b). Indeed, the symmetry of f implies $x \leq f(f(x))$; applying Lemma 3.6(b) we may conclude that $x \vee f(x) \leq f \circ f(x \vee f(x))$, which implies that

$$(x, x \vee f(x)) \in \bigvee\{a \otimes a \mid a \leq f \circ f(a)\}. \qquad \square$$

For any filter $\mathcal{P}$ of $(Pol(L), \leq)$, $\{\overline{f} \mid f \in \mathcal{P}\}$ is a base for a filter of $(Ax(L), \leq)$. Let $\Phi(\mathcal{P})$ be the filter generated by it. Conversely, for any filter $\mathcal{A}$ of $(Ax(L), \leq)$, let $\Psi(\mathcal{A})$ be the filter of $(Pol(L), \leq)$ generated by $\{\overset{\circ}{f} \mid f \in \mathcal{A}\}$. Now we can readily derive, from Theorems 3.7 and 3.8, the following:

Corollary 3.9. *The correspondences Φ and Ψ establish a bijection between the filters of $Pol(L)$ satisfying the symmetry and the square-refinement properties and the filters of $Ax(L)$ satisfying the symmetry and square-refinement properties.* $\square$

From this corollary it follows immediately that the approaches to uniform frames via polarities and axialities are equivalent; more precisely, there is an isomorphism between the category GUFrm and the category of entourage uniform frames of [4], and consequently, between the categories GUFrm and WUFrm. This corollary reveals that the equivalence between the approaches of [4] and [11] relies on a purely lattice theoretical fact of Galois connections; despite the different orders on the second frame, both $Gal(L, L)$ and $Gal(L, L^{op})$ describe frame uniformities because of this fact. This answers the question that gave rise to this paper.

4. Concluding remarks

The following diagram summarizes the several (equivalent) approaches to uniform frames.

- FRAME UNIFORMITES
  - Covers [8], [15]
  - Weil entourages [11]
  - Gauges [16], [13]
  - Galois connections
    - Polarities
    - Axialities [4], [5]

It leads to the conclusion that Galois connections rather than entourages are the root of the approach in [4] and [5].

It is quite surprising that both $Gal(L, L)$ and $Gal(L, L^{op})$ describe Weil entourages, despite the different orders on the second frame. This is the frame counterpart of the following fact on binary relations; for sets A and B, all binary relations $R \subseteq A \times B$ are completely described by Galois connections between power sets in two particularly simple ways:

(1) A Galois connection between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is called a *polarity* [1]. Any relation $R \subseteq A \times B$ induces a polarity $(R_\forall, R^\forall)$ between $\mathcal{P}(A)$ and $\mathcal{P}(B)$, defined by

$$R_\forall(U) := \{b \in B \mid \forall a \in U\ (a, b) \in R\} \quad \text{for } U \subseteq A,$$

$$R^\forall(V) := \{a \in A \mid \forall b \in V\ (a, b) \in R\} \quad \text{for } V \subseteq B.$$

Conversely, every polarity (f, g) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induces a relation

$$(a, b) \in R_{(f,g)} \equiv b \in f(\{a\}) \text{ if and only if } a \in g(\{b\}).$$

Since

$$\Big((R_{(f,g)})_\forall, (R_{(f,g)})^\forall\Big) = (f, g)$$

and $R_{(R_\forall, R^\forall)} = R$ we have a bijection between all relations $R \subseteq A \times B$ and all Galois connections $(f, g) : \mathcal{P}(A) \to \mathcal{P}(B)$.

(2) The contravariant case of Galois connections between power sets also describe all binary relations $R \subseteq A \times B$. Indeed, any relation $R \subseteq A \times B$ induces an *axiality* [3] $(R_\exists, R^\forall)$ between $\mathcal{P}(A)$ and $\mathcal{P}(B)$, that is, a Galois connection between $\mathcal{P}(A)$ and $\mathcal{P}(B)^{op}$, defined by

$$R_\exists(U) := \{b \in B \mid \exists a \in U\ (a,b) \in R\} \quad \text{for } U \subseteq A,$$

$$R^\forall(V) := \{a \in A \mid \forall b \in B((a,b) \in R \Rightarrow b \in V)\} \quad \text{for } V \subseteq B.$$

If for an axiality (f,g) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ we define the relation

$$(a,b) \in R_{(f,g)} \equiv b \in f(\{a\})$$

then

$$\Big((R_{(f,g)})_\exists, (R_{(f,g)})^\forall\Big) = (f,g)$$

and $R_{(R_\exists, R^\forall)} = R$.

Under the bijection (1), entourages E correspond to the polarities $(E_*, E^*) : \mathcal{P}(X) \to \mathcal{P}(X)$ that are extensive on atoms, that is, $E_*(\{x\}) \supseteq \{x\}$ for every $x \in X$, or equivalently,

$$X = \bigcup \{A \subseteq X \mid A \subseteq E_*(A)\}$$

(we recognize here the specialization for sets of the definition of polarity for frames). One can then characterize uniform spaces in terms of these polarities. This is the well known result which asserts that any (quasi-)uniformity $\mathcal{U}$ on a set X is characterized by the collection of all maps

$$\begin{array}{rccc} f_E: & \mathcal{T}(\mathcal{U}) & \longrightarrow & \mathcal{T}(\mathcal{U}) \\ & U & \longmapsto & E_\forall(U) \end{array} \qquad (E \in \mathcal{B})$$

for some base $\mathcal{B}$ of $\mathcal{U}$ (where $\mathcal{T}(\mathcal{U})$ is the topology determined by $\mathcal{U}$).

A perfect analogous characterization in which the polarities are replaced by the axialities may be also obtained. This is the reason why, alternatively, any (quasi-)uniformity $\mathcal{U}$ on a set X is characterized by the collection of all maps

$$\begin{array}{rccc} f'_E: & \mathcal{T}(\mathcal{U}) & \longrightarrow & \mathcal{T}(\mathcal{U}) \\ & U & \longmapsto & E_\exists(U) \end{array} \qquad (E \in \mathcal{B})$$

for some base $\mathcal{B}$ of $\mathcal{U}$.

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