Algebraically central and categorically central extensions

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Abstract

In any exact Maltsev category with coequalizers, the notion of central extension in the sense of categorical Galois theory is proved to be equivalent to the notion of central extension considered in universal algebra. This equivalence extends various results in this direction obtained for Ω-groups [16], Maltsev varieties [17] and for semi-abelian categories [6]. Moreover, when the category has a zero object, we prove that there is an equivalence of categories between the category of central extensions and the category of internal connected groupoids.

Keywords: central extension, Maltsev category, internal groupoid.

Introduction

In this article we prove that two different notions of central extension are equivalent in exact Maltsev categories.

The first notion arises from the categorical theory of central extensions by Janelidze and Kelly [15], which extends several classical results in homological algebra of groups, rings and other important algebraic structures. Within this theory, which is one of the applications of categorical Galois theory, the notion of central extension can be defined in any exact category $\mathcal{C}$, and depends on the choice of an admissible reflective subcategory $\mathcal{X}$ of $\mathcal{C}$:

$$\mathcal{X} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}.$$

The classical central extensions of groups appear as the very special case when $\mathcal{C}$ is the variety of groups and $\mathcal{X}$ is the subvariety of abelian groups:

$$Ab \underset{U}{\overset{Ab}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} Grp.$$

More generally, this notion of *categorically central* extension includes the one introduced by Frölich [10] and Lue [20] for varieties of Ω-groups.

The second notion of central extension arises from the generalization of the theory of commutators developed by Pedicchio in exact Maltsev categories [22] [23]. The main properties of the commutator of equivalence relations can be derived in this general context similarly to the well known special case of Maltsev varieties [24]. For this, the syntactic assumption of the existence of a ternary operation satisfying the equalities $p(x,y,y) = x$ and $p(x,x,y) = y$ is replaced by its semantic formulation asserting that any reflexive relation is an equivalence relation [8]. An extension $f\colon A \to B$ is then called *algebraically central* if its kernel equivalence R satisfies the property $[R, A \times A] = 0_A$, where 0_A is the smallest equivalence on A.

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Several results in the direction of unifying the two notions of categorically central and algebraically central extensions were proved by Janelidze and Kelly in [16] for varieties of Ω-groups, in [17] for Maltsev varieties, and by Bourn and Gran in [6] for semi-abelian categories. Ω-groups, Maltsev varieties and semi-abelian categories are all examples of exact Maltsev categories, and it was an open problem whether it was possible to find a more general result which could cover all these different cases. The conceptual simplicity of our proof confirms that the categorical language is essential to understand the property of centrality. Moreover, it suggested the construction of a functor from the category of central extensions $Centr(\mathcal{C})$ to the category $ConnGrpd(\mathcal{C})$ of connected internal groupoids in $\mathcal{C}$ (see also [4]); this functor actually provides an equivalence of categories between $Centr(\mathcal{C})$ and $ConnGrpd(\mathcal{C})$.
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1 Centrality in Maltsev categories

In this section we fix the notations and briefly recall a few properties of centrality of equivalence relations in exact Maltsev categories. More details can be found in [9], [22] and [23].
A category $\mathcal{C}$ is regular if it is finitely complete, every kernel pair has a coequalizer and regular epimorphisms are stable under pullbacks. If $\mathcal{C}$ is regular, any arrow $f: A \to B$ can be factorised as $f = i \circ p$ with p a regular epimorphism and i a monomorphism. Regular categories are naturally provided with a good calculus of relations, the composition of equivalence relations being then associative. When, moreover, the commutativity of the composition of equivalence relations is required, one obtains the notion of a Maltsev category:

Theorem 1.1. *[8] Let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

1. *the composition of equivalence relations is commutative*
2. *the composite relation of two equivalence relations is an equivalence relation*
3. *any reflexive relation in $\mathcal{C}$ is an equivalence relation*

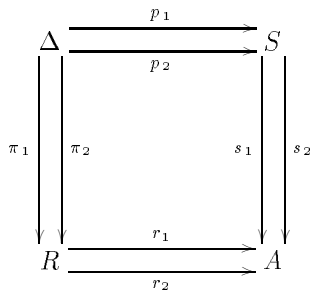
The last formulation of the Maltsev property can be expressed in any finitely complete category, and is now the standard definition of a Maltsev category [9].

A (Barr) exact category is a regular category in which any equivalence relation is effective [1], i.e. it is the kernel pair of a map. Exact Maltsev categories provide an appropriate context in which one can extend some aspects of homological algebra of groups [8]; moreover, there is a simple description of internal structures [9], [11] and a very efficient commutator theory [22], [23].
Examples
A variety in the sense of universal algebra has the Maltsev property precisely when its theory contains a ternary operation $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x$, $p(x, x, y) = y$ [21]; for instance in the variety of groups such a term $p(x, y, z)$ is given by $xy^{-1}z$. Among Maltsev varieties are then those of groups, abelian groups, modules over a fixed ring, rings, commutative rings, associative algebras and Lie algebras, quasi-groups, crossed modules. An example of different nature is given by the variety of Heyting algebras: in this case a Maltsev operation is given by $p(x, y, z) = ((x \to y) \to z) \wedge ((z \to y) \to x)$. Among the non-varietal examples of exact Maltsev categories we just recall that any abelian category is exact Maltsev, as is the dual of the category of sets and, more generally, the dual of any elementary topos. If $\mathcal{C}$ is exact Maltsev, so is the category $Grpd(\mathcal{C})$ of internal groupoids in $\mathcal{C}$. Finally, the category of topological groups is regular Maltsev, as is more generally the category of topological Maltsev algebras.

In the following $\mathcal{C}$ will denote an exact Maltsev category with coequalizers. If $f: A \to B$ is an arrow in $\mathcal{C}$ and R is an equivalence relation on A, we shall use the notation $f(R)$ for the regular

image of R along f. In any regular Maltsev category, if $f: A \to B$ is a regular epimorphism, then $f(R)$ is an equivalence relation. The smallest and the largest equivalence relations on an object $A \in \mathcal{C}$ are denoted by 0_A (or by 0, if there is no ambiguity) and $A \times A$, respectively.

Definition 1.1. *Let R and S be equivalence relations on an object $A \in \mathcal{C}$. A double relation on R and S is an object $\Delta \in \mathcal{C}$ equipped with morphisms $\Delta \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} R$ and $\Delta \overset{p_1}{\underset{p_2}{\rightrightarrows}} S$ such that*



1. $r_1 \circ \pi_1 = s_1 \circ p_1$, $r_2 \circ \pi_1 = s_1 \circ p_2$, $r_1 \circ \pi_2 = s_2 \circ p_1$, $r_2 \circ \pi_2 = s_2 \circ p_2$

2. $\Delta \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} R$ *is an equivalence relation on* R

3. $\Delta \overset{p_1}{\underset{p_2}{\rightrightarrows}} S$ *is an equivalence relation on* S

An element of a double relation Δ can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where aSb, cSd, aRc and bRd.

Definition 1.2. *A centralising relation for R and S is a double relation Δ on R and S such that any (and then all) commutative square in the diagram above is a pullback.*

In any Maltsev category, if there is a centralising relation on R and S, it is necessarily unique [9]. Given two equivalence relations R and S on an object A in an exact Maltsev category with coequalizers, there is a natural way to define a double relation on R and S, denoted by Δ_R^S:

Definition 1.3. *Let A be an object in $\mathcal{C}$, $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} A$ and $S \overset{s_1}{\underset{s_2}{\rightrightarrows}} A$ two equivalence relations on A and $e_S: A \to S$ the arrow giving the reflectivity of the equivalence relation S. The relation Δ_R^S on S is defined as the kernel pair of the coequalizer of the arrows $e_S \circ r_1$ and $e_S \circ r_2$.*

By definition Δ_R^S is an equivalence relation on S. The Maltsev assumption implies that it is also an equivalence relation on R and Δ_R^S is a double relation on R and S. If $\mathcal{C}$ is a variety, Δ_R^S is the smallest equivalence relation on S under which all the elements of the form (a, a) and (b, b) are equivalent if (a, b) is in R. An important property of symmetry is that $\Delta_R^S = \Delta_S^R$. The definition of the commutator of the equivalence relations R and S is the following:

Definition 1.4. *Let P denote the kernel pair of s_1 and let $P \cap \Delta_R^S$ denote the intersection of P and Δ_R^S as relations on S; we write i for the inclusion of $P \cap \Delta_R^S$ in Δ_R^S and ν for the morphism $\nu = (s_2 \circ p_1, s_2 \circ p_2): \Delta_R^S \to A \times A$. The commutator $[R, S]$ of the equivalence relations R and S is the regular image of $\nu \circ i$.*

In terms of elements this means that (a, b) is in $[R, S]$ if and only if there exists an element $c \in A$ with $\begin{pmatrix} c & a \\ c & b \end{pmatrix}$ in Δ_R^S. By definition the commutator $[R, S]$ is a reflexive relation on A; since the category $\mathcal{C}$ is Maltsev, $[R, S]$ is actually an equivalence relation on A. This definition of the commutator of equivalence relations coincides with the classical one used in universal algebra given, for example, in [13].

The relationship between the commutator and centralising relations can be stated as follows [22]: given two equivalence relations R and S on A, then there is a centralising relation on R and S if and only if $[R, S] = 0_A$. When there exists such a centralising relation D on R and S, it is necessarily isomorphic to Δ_R^S; one then says that R and S centralise each other or that R and S are connected [5].

The following important property will be needed:

Proposition 1.1. *Let R, S be equivalence relations on an object $A \in \mathcal{C}$ and let $f\colon A \to B$ be a regular epimorphism. If R and S centralise each other, then $f(R)$ and $f(S)$ centralise each other.*

2 Central extensions

In this section we briefly recall the main definitions of the categorical theory of central extensions [15]. We then prove a new result that holds in any exact Maltsev category.

If $\mathcal{X}$ is a reflective full replete subcategory of an exact category $\mathcal{C}$, we denote by $H\colon \mathcal{X} \to \mathcal{C}$ the inclusion functor and by $I\colon \mathcal{C} \to \mathcal{X}$ its left adjoint. In any exact Maltsev category such a subcategory $\mathcal{X}$ is admissible in the sense of the categorical theory of central extensions precisely when it is closed in $\mathcal{C}$ under subobjects and quotients. One uses the term Birkhoff subcategory for such a subcategory $\mathcal{X}$, by analogy with the case of varieties: of course, when $\mathcal{C}$ is a variety, a Birkhoff subcategory is simply a subvariety.

By an extension of B we mean a regular epimorphism $f\colon A \to B$ with codomain B. $Ext_{\mathcal{C}}(B)$, or simply $Ext(B)$, will denote the full subcategory of the comma category $\mathcal{C}/B$ given by the extensions of B. For any $B \in \mathcal{C}$ the left adjoint $I\colon \mathcal{C} \to \mathcal{X}$ induces a functor $I^B\colon Ext_{\mathcal{C}}(B) \to Ext_{\mathcal{X}}(IB)$. This functor I^B has a right adjoint $H^B\colon Ext_{\mathcal{X}}(IB) \to Ext_{\mathcal{C}}(B)$ which associates, with any extension $\phi\colon X \to IB$, the extension $\overline{\phi}$ in the pullback

$$\begin{array}{ccc} B \underset{HIB}{\times} HX & \longrightarrow & HX \\ {\scriptstyle \overline{\phi}}\downarrow & & \downarrow{\scriptstyle H\phi} \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

where $\eta_B\colon B \to HIB$ is the B-component of the unit of the adjunction $I \dashv H$. The admissibility condition precisely asserts that the functor H^B is fully faithful, for any $B \in \mathcal{C}$.

An extension $f\colon A \to B$ in $\mathcal{C}$ is said to be trivial, if it lies in the replete image of the fully faithful $H^B\colon Ext_{\mathcal{X}}(IB) \to Ext_{\mathcal{C}}(B)$. Equivalently, an extension $f\colon A \to B$ is trivial if the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HIA \\ {\scriptstyle f}\downarrow & (1) & \downarrow{\scriptstyle HIf} \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

is a pullback.

An extension $f\colon A \to B$ is (E,p)-split, where $p\colon E \to B$ is itself an extension of B, when $(E \underset{B}{\times} A, \pi_E)$ in the following pullback is a trivial extension of E:

$$\begin{array}{ccc} E \underset{B}{\times} A & \xrightarrow{\pi_A} & A \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{p} & B. \end{array}$$

Definition 2.1. *An extension $f\colon A \to B$ in $\mathcal{C}$ is central when there exists an extension (E,p) of B such that (A,f) is (E,p)-split.*

An extension $f\colon A \to B$ is normal if it is (A,f)-split. If one denotes by $Triv(B)$, $Norm(B)$ and $Centr(B)$ the full subcategories of the category $Ext(B)$ given by the trivial, normal and central extensions of B, one has

$$Triv(B) \subseteq Norm(B) \subseteq Centr(B) \subseteq Ext(B).$$

It is well-known that the admissibility condition on the subcategory $\mathcal{X}$ corresponds to a certain left exactness property of the left adjoint I. The Maltsev assumption on the category $\mathcal{C}$ forces this functor I to preserve also some special kind of pullbacks. In order to prove this property, we first need the following simple

Lemma 2.1. *Let $f\colon A \to C$ and $g\colon B \to D$ be two split epimorphisms in any category $\mathcal{C}$. Then any regular epimorphism $(p,q)\colon (f,i) \to (g,j)$ in the category of split epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle i}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle j}\uparrow\downarrow{\scriptstyle g} \\ C & \xrightarrow{p} & D \end{array}$$

is a pushout.

Proof: If $\alpha\colon C \to E$ and $\beta\colon B \to E$ are two arrows with $\alpha \circ f = \beta \circ q$, then the arrow $\beta \circ j$ gives the desired factorisation (since $q \circ i = j \circ p$). □

Proposition 2.1. *Let $\mathcal{C}$ be an exact Maltsev category and $\mathcal{X}$ any Birkhoff subcategory of $\mathcal{C}$. In the adjunction*

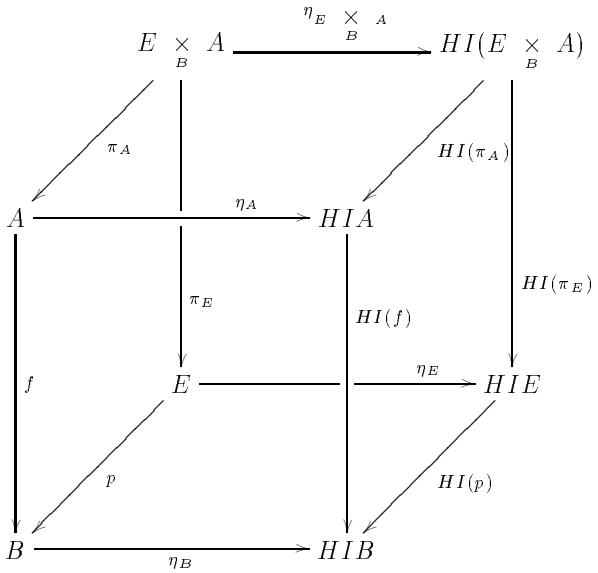
$$\mathcal{X} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}\ ,$$

the left adjoint I preserves pullbacks of extensions along split epimorphisms.

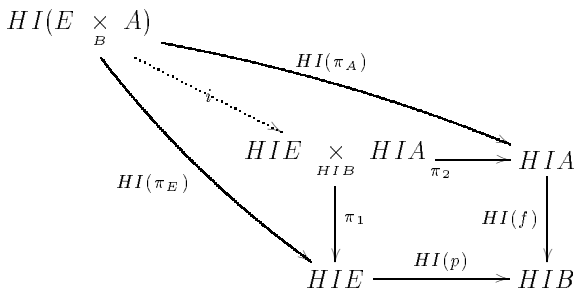
Proof: Let

$$\begin{array}{ccc} E \underset{B}{\times} A & \xrightarrow{\pi_A} & A \\ {\scriptstyle i_E}\uparrow\downarrow{\scriptstyle \pi_E} & (2) & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ E & \xrightarrow{p} & B \end{array}$$

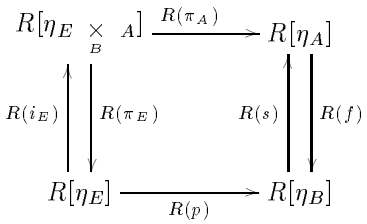
be the pullback of a split epimorphism f along a regular epi p. There is a commutative diagram



The vertical right hand square is a pushout of regular epimorphisms by Lemma 2.1. Since $\mathcal{C}$ is an exact Maltsev category, the comparison arrow i to the pullback of $HI(p)$ along $HI(f)$ is a regular epimorphism (by Theorem 5.7 in [7])



The arrow $\alpha : E \underset{B}{\times} A \to HIE \underset{HIB}{\times} HIA$ induced by the units of the adjunction η_A, η_B and η_E is a regular epi, being the composite $i \circ \eta_E \underset{B}{\times} A$. To prove that i is an isomorphism, it is sufficient to prove that the kernel relations $R[\alpha]$ of α and $R[\eta_E \underset{B}{\times} A]$ of $\eta_E \underset{B}{\times} A$ are isomorphic. Clearly, when we take the kernel pair $R[\alpha]$ of α, we obtain $R[\eta_E] \underset{R[\eta_B]}{\times} R[\eta_A]$. The induced factorisation $\gamma : R[\eta_E \underset{B}{\times} A] \to R[\alpha]$ is a monomorphism in any category. It is also a regular epimorphism, being the unique factorisation, in an exact Maltsev category, from the pushout of regular epimorphisms



to the pullback of $R(p)$ along $R(f)$. Hence $R[\alpha] = R[\eta_E \underset{B}{\times} {}_A]$ and the pullback (2) is preserved by the functor I. □

3 Main results

In this section $\mathcal{C}$ denotes an exact Maltsev category with coequalizers.

We denote by $\mathcal{C}_{Ab}$ the subcategory of the abelian objects in $\mathcal{C}$; $\mathcal{C}_{Ab}$ is the full subcategory of $\mathcal{C}$ given by those objects $A \in \mathcal{C}$ which have the property that $[A \times A, A \times A] = 0_A$. It was proved in [12] that the forgetful functor $U : \mathcal{C}_{Ab} \to \mathcal{C}$ has a left adjoint $I : \mathcal{C} \to \mathcal{C}_{Ab}$ associating with any object $C \in \mathcal{C}$ its quotient on the largest commutator $[C \times C, C \times C]$ on C; moreover, the category $\mathcal{C}_{Ab}$ is a naturally Maltsev [19] Birkhoff subcategory of $\mathcal{C}$. Accordingly, an extension $f : A \to B$ is *categorically central* if it is central in the sense of categorical Galois theory, i.e. according to Definition 2.1, with respect to the Birkhoff subcategory $\mathcal{C}_{Ab}$. On the other hand the term algebraically central will be used for an extension $f : A \to B$ which is central in the sense of commutator theory: if R is the kernel pair of f, then an extension f is *algebraically central* if $[R, A \times A] = 0_A$ [16]. The main purpose of this section consists in proving that these two notions coincide in any exact Maltsev category with coequalizers.

Proposition 3.1. *If $f : A \to B$ is categorically central, then $f : A \to B$ is algebraically central.*

Proof: The category $\mathcal{C}_{Ab}$ is naturally Maltsev and, accordingly, any two relations on the same object centralise each other. This means, in particular, that any extension in $\mathcal{C}_{Ab}$ is algebraically central. As in the algebraic case, to prove the result it will be then sufficient to show that in any pullback of extensions

$$\begin{array}{ccc} E \underset{B}{\times} A & \xrightarrow{\pi_A} & A \\ \downarrow{\scriptstyle \pi_E} & (*) & \downarrow{\scriptstyle f} \\ E & \xrightarrow{g} & B \end{array}$$

the extension $f : A \to B$ is algebraically central if and only if $\pi_E : E \underset{B}{\times} A \to E$ is algebraically central. As soon as centrality is expressed in terms of centralising double relations, the proof is conceptually simple in any category; here below we give the details.

Let us first prove that algebraically central extensions are stable under pullbacks in $\mathcal{C}$. We denote by R and $\overline{R}$ the kernel equivalences of f and π_E, respectively. Let Δ be the centralising relation on R and $A \times A$:

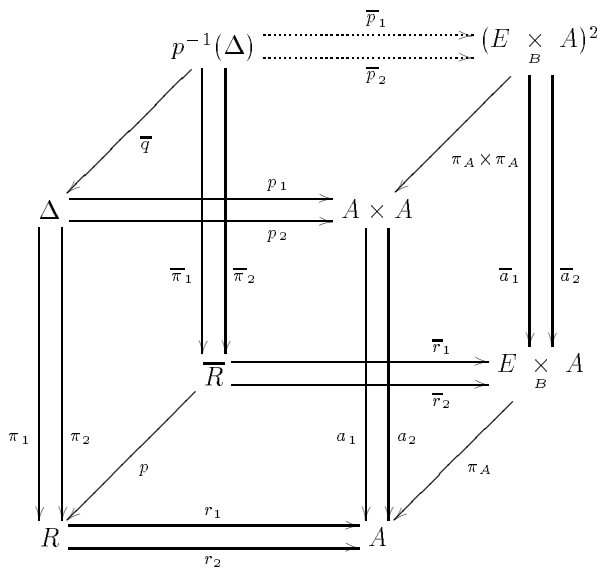
$$\begin{array}{ccc} \Delta & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & A \times A \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle \pi_2} & & {\scriptstyle a_1}\downarrow\uparrow{\scriptstyle a_2} \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & A \end{array}$$

Let $p : \overline{R} \to R$ denote the induced restriction of $\pi_A \times \pi_A : (E \underset{B}{\times} A)^2 \to A \times A$ to $\overline{R}$, (where we write $(E \underset{B}{\times} A)^2$ for the largest equivalence relation on $E \underset{B}{\times} A$). We shall prove that the relation

$p^{-1}(\Delta)$, defined by the pullback

$$\begin{array}{ccc} p^{-1}(\Delta) & \xrightarrow{\overline{q}} & \Delta \\ {\scriptstyle (\overline{\pi}_1,\overline{\pi}_2)}\downarrow & & \downarrow{\scriptstyle (\pi_1,\pi_2)} \\ \overline{R}\times\overline{R} & \xrightarrow[p\times p]{} & R\times R \end{array}$$

is a centralising relation on $\overline{R}$ and $(E \underset{B}{\times} A)^2$.



There are induced arrows $\overline{p}_1 = (\overline{r}_1 \times \overline{r}_1) \circ (\overline{\pi}_1, \overline{\pi}_2)$ with $(\pi_A \times \pi_A) \circ \overline{p}_1 = p_1 \circ \overline{q}$, and $\overline{p}_2 = (\overline{r}_2 \times \overline{r}_2) \circ (\overline{\pi}_1, \overline{\pi}_2)$ with $(\pi_A \times \pi_A) \circ \overline{p}_2 = p_2 \circ \overline{q}$. By the definition of $p^{-1}(\Delta)$ there is also an arrow $\varepsilon\colon (E \underset{B}{\times} A)^2 \to p^{-1}(\Delta)$ with $\overline{p}_1 \circ \varepsilon = 1 = \overline{p}_2 \circ \varepsilon$. There is accordingly a reflexive graph

$$p^{-1}(\Delta) \underset{\overline{p}_2}{\overset{\overline{p}_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\varepsilon}}}} (E \underset{B}{\times} A)^2$$

which is easily seen to be an equivalence relation. All the commutativity requirements in the definition of a double relation on $\overline{R}$ and $(E \underset{B}{\times} A)^2$ are satisfied by $p^{-1}(\Delta)$, and we only need to prove that the square

$$\begin{array}{ccc} p^{-1}(\Delta) & \xrightarrow[\overline{p}_1]{} & (E \underset{B}{\times} A)^2 \\ \downarrow{\scriptstyle \overline{\pi}_1} & & {\scriptstyle \overline{a}_1}\downarrow \\ \overline{R} & \xrightarrow[\overline{r}_1]{} & E \underset{B}{\times} A \end{array}$$

is a pullback. Certainly both the squares (1) and (2) here below are pullbacks:

$$\begin{array}{ccccc} p^{-1}(\Delta) & \xrightarrow{(\overline{\pi}_1,\overline{\pi}_2)} & \overline{R}\times\overline{R} & \xrightarrow{\overline{r}_1\times\overline{r}_1} & (E \underset{B}{\times} A)^2 \\ \downarrow{\scriptstyle \overline{q}} & (1) & \downarrow{\scriptstyle p\times p} & (2) & \downarrow{\scriptstyle \pi_A\times\pi_A} \\ \Delta & \xrightarrow[(\pi_1,\pi_2)]{} & R\times R & \xrightarrow[r_1\times r_1]{} & A\times A. \end{array}$$

It follows then that (3) and (4) are pullbacks

$$\begin{array}{ccccc} p^{-1}(\Delta) & \xrightarrow{\overline{q}} & \Delta & \xrightarrow{\pi_1} & R \\ \downarrow{\scriptstyle \overline{p}_1} & (3) & \downarrow{\scriptstyle p_1} & (4) & \downarrow{\scriptstyle r_1} \\ (E \underset{B}{\times} A)^2 & \xrightarrow[\pi_A\times\pi_A]{} & A\times A & \xrightarrow[a_1]{} & A, \end{array}$$

so that (5) + (6) and (6) are pullbacks:

$$\begin{array}{ccccc} p^{-1}(\Delta) & \xrightarrow{\overline{\pi}_1} & \overline{R} & \xrightarrow{p} & R \\ \downarrow{\scriptstyle \overline{p}_1} & (5) & \downarrow{\scriptstyle \overline{r}_1} & (6) & \downarrow{\scriptstyle r_1} \\ (E \underset{B}{\times} A)^2 & \xrightarrow[\overline{a}_1]{} & (E \underset{B}{\times} A) & \xrightarrow[\pi_A]{} & A. \end{array}$$

Accordingly, the square (5) is a pullback and $p^{-1}(\Delta)$ a centralising relation on $\overline{R}$ and $(E \underset{B}{\times} A)^2$.

Conversely, let us assume that the extension π_E in the diagram $(*)$ is an algebraically central extension, namely that $\overline{R}$ and $(E \underset{B}{\times} A)^2$ centralise each other. The induced factorisations $p\colon \overline{R} \to R$ and $\pi_A \times \pi_A\colon (E \underset{B}{\times} A)^2 \to A \times A$ are clearly regular epimorphisms:

$$\begin{array}{ccccc} \overline{R} & \rightrightarrows & E \underset{B}{\times} A & \leftleftarrows & (E \underset{B}{\times} A)^2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_A\times\pi_A} \\ R & \rightrightarrows & A & \leftleftarrows & A\times A. \end{array}$$

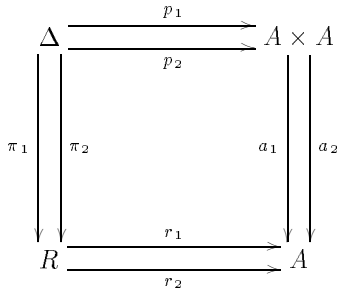
By Proposition 1.1 it follows that R and $A \times A$ centralise each other, since the equivalence relations R and $A \times A$ are the regular images along π_A of $\overline{R}$ and $(E \underset{B}{\times} A)^2$. The extension $f\colon A \to B$ is then algebraically central. □

Proposition 3.2. *If $f\colon A \to B$ is algebraically central, then $f\colon A \to B$ is categorically central.*

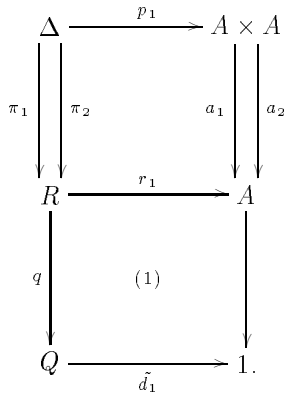
Proof: We shall prove that any algebraically central extension $f\colon A \to B$ is normal.
As usual we write

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$$

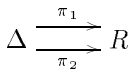
for the kernel pair of f. Our assumption that f is algebraically central asserts that there is a centralising relation Δ on R and $A \times A$:



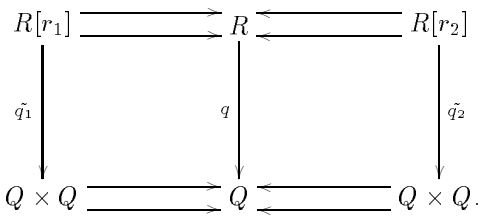
Now, by taking the coequalizer q of π_1 and π_2 we obtain the commutative diagram



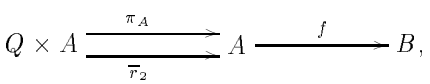
Since the category $\mathcal{C}$ is exact, the equivalence relation



is the kernel pair of its coequalizer q, and this latter is a pullback stable regular epi. By assumption the arrow $(p_1, r_1)\colon (\Delta, R) \to (A \times A, A)$ determines a discrete fibration of internal equivalence relations, so that by Corollary 2 in [2], the square (1) is a pullback. Remark then that the object Q is abelian: indeed, by Proposition 1.1 the fact that the equivalences $R[r_1]$ and $R[r_2]$ centralise each other implies that $Q \times Q$ and $Q \times Q$ centralise each other, the induced arrows $\tilde{q_1}$ and $\tilde{q_2}$ in the following diagram being regular epimorphisms:



This shows that the kernel equivalence R of f is isomorphic to



with Q in $\mathcal{C}_{Ab}$. Let $t \circ g$ be the regular epi-monomorphism factorisation of the unique arrow from Q to 1 and let I denote the regular image of this arrow. Clearly the arrow $g\colon Q \to I$ lies in $\mathcal{C}_{Ab}$, I being a subobject of 1. There is an induced arrow $i\colon A \to I$ with $i \circ \pi_A = g \circ \pi_Q$. The extension $\pi_A\colon Q \times A \to A$ appears then as the pullback of the trivial extension g along i: by the stability of trivial extensions under pullbacks the arrow π_A is a trivial extension and $f\colon A \to B$ is central. □

Proposition 3.3. *$Centr(B)$ is a reflective subcategory of $Ext(B)$.*

Proof: For any extension $f\colon A \to B$ there is the canonical quotient $\eta_A\colon A \to \frac{A}{[R,A\times A]}$, where R is the kernel pair of f. The centralisation functor $J\colon Ext(B) \to Centr(B)$ associates with $f\colon A \to B$ the unique extension $f'\colon \frac{A}{[R,A\times A]} \to B$ with $f' \circ \eta_A = f$. By using the properties of the categorical commutator one can see that f' is a central extension with the property that if $g\colon C \to B$ is another central extension with $g \circ t = f$ for some $t\colon A \to C$, then t factorizes through η_A. Accordingly, $Centr(B)$ is a reflective subcategory of $Ext(B)$ closed under subobjects. The category $Centr(B)$ is also closed in $Ext(B)$ under quotients, as it follows by Proposition 1.1. □

4 Connected internal groupoids

Let us denote by $ConnGrpd(B)$ the category of connected internal groupoids in $\mathcal{C}$ over B. These are the internal groupoids in $\mathcal{C}$ with B as object of objects

$$X_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} B$$

with the property that the induced arrow $(d_1, d_2)\colon X_1 \to B \times B$ is a regular epimorphism.

Proposition 4.1. *Let $\mathcal{C}$ be an exact Maltsev category with a zero object. The categories $ConnGrpd(B)$ and $Centr(B)$ are equivalent.*

Proof: There is a normalisation functor (see [4]) $N\colon ConnGrpd(B) \to Centr(B)$ associating with any internal groupoid

$$X_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} B$$

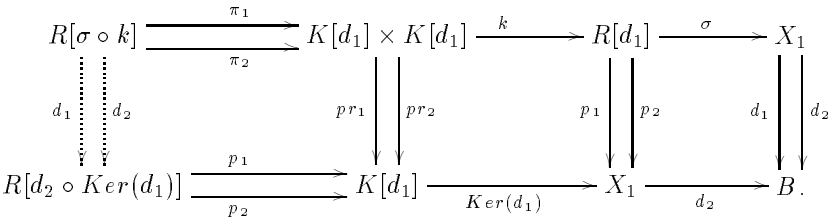
the arrow

$$K[d_1] \xrightarrow{Ker(d_1)} X_1 \xrightarrow{d_2} B.$$

where $Ker(d_1)\colon K[d_1] \to X_1$ is the kernel of d_1. When the groupoid is connected, this normalisation $d_2 \circ Ker(d_1)$ of the given internal connected groupoid is a regular epi: indeed, if $s_1 : B \to B \times B$ is the canonical arrow $s_1 = (\omega_B, 1_B)$ (where $\omega_B\colon B \to B$ is the zero arrow), then $d_2 \circ Ker(d_1)$ can be also obtained by the pullback

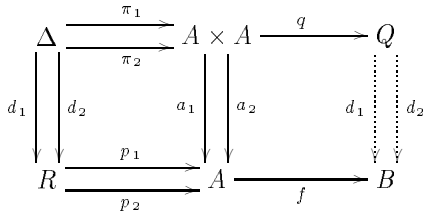
$$\begin{array}{ccc} K[d_1] & \xrightarrow{Ker(d_1)} & X_1 \\ {\scriptstyle d_2 \circ Ker(d_1)}\downarrow & & \downarrow{\scriptstyle (d_1,d_2)} \\ B & \xrightarrow[s_1]{} & B \times B. \end{array}$$

To see that it is a central extension, consider the following diagram:



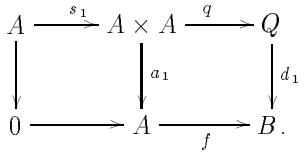
The arrow σ corresponds to the operation which associates, with any pair of arrows (α, β) with same domain, the arrow $\beta \circ \alpha^{-1}$. The equivalence relations $R[\sigma \circ k]$ and $R[d_2 \circ Ker(d_1)]$ are the obvious kernel pairs, while $d_1, d_2 \colon R[\sigma \circ k] \to R[d_2 \circ Ker(d_1)]$ are induced by the universal property of kernel pairs. The central and the right hand commutative squares are clearly discrete fibrations of internal groupoids. From this, it easily follows that $R[\sigma \circ k]$ is a centralising double relation on $R[d_2 \circ Ker(d_1)]$ and $K[d_1] \times K[d_1]$. This shows that the extension $d_2 \circ Ker(d_1)$ is central. It is also clear how to define the functor N on arrows.

Conversely, let us define a functor $V \colon Centr(B) \to ConnGrpd(B)$ as follows: let $f \colon A \to B$ be a central extension and let Δ be the centralising relation associated with $A \times A$ and the kernel equivalence R of f. By taking the coequalizer $q \colon A \times A \to Q$ of the projections π_1 and π_2 on $A \times A$ one obtains a reflexive graph with B as objects of objects:



the category $\mathcal{C}$ being regular Maltsev, this reflexive graph actually is an internal groupoid in $\mathcal{C}$ [11]. It is connected, since $(d_1, d_2) \circ q = f \times f$, and $f \times f$ is a regular epi. The functor V is naturally defined on arrows, the category of internal groupoids in $\mathcal{C}$ being full in the category of reflexive graphs [9].

We shall then show that $(N \circ V)(f \colon A \to B) = (f \colon A \to B)$. This follows from the fact that $q \circ s_1$ is the kernel of d_1, both the squares in the following commutative diagram being pullbacks:



Then $d_2 \circ Ker(d_1) = d_2 \circ q \circ s_1 = f \circ a_2 \circ s_1 = f$. A similar argument shows that if $g \colon C \to B$ is another central extension of B and $h \colon A \to C$ an arrow in $Centr(B)$ (i.e. $g \circ h = f$), then $(N \circ V)(h) = h$.

The proof that $V \circ N = 1$ is also easy, and we leave it to the reader. □

The same arguments as above can be used to prove also the following result:

Corollary 4.1. *Let $\mathcal{C}$ be an exact Maltsev category with a zero object. Then the category of internal connected groupoids in $\mathcal{C}$ and the category of central extensions in $\mathcal{C}$ are equivalent.*

References

[1] **M. Barr**, Exact Categories, *LNM 236*, Springer-Verlag, Berlin, 1971, 1-120.

[2] **D. Bourn**, Normal subobjects and abelian objects in protomodular categories, *Journal of Algebra*, **228**, 2000, 143-164.

[3] **D. Bourn**, A categorical genealogy for the congruence distributive condition, Preprint, 2000.

[4] **D. Bourn**, 3×3 lemma and protomodularity, to appear in *Journal of Algebra*.

[5] **D. Bourn - M. Gran**, Centrality and normality in protomodular categories, Preprint, 2000.

[6] **D. Bourn - M. Gran**, Central extensions in semi-abelian categories, Preprint, 2000.

[7] **A. Carboni - G.M. Kelly - M.C. Pedicchio**, Some remarks on Maltsev and Goursat categories, *Appl. Categorical Structures*, **1**, 1993, 385-421.

[8] **A. Carboni - J. Lambek - M.C. Pedicchio**, Diagram chasing in Mal'cev categories, *J. Pure Appl. Algebra*, **69**, 1990, 271-284.

[9] **A. Carboni - M.C. Pedicchio - N. Pirovano**, Internal graphs and internal groupoids in Mal'cev categories, *Proc. Conference Montreal 1991*, 1992, 97-109.

[10] **A. Frölich**, Baer-invariants of algebras, *Trans. Amer. Math. Soc.*, **109**, 1963, 221-244.

[11] **M. Gran**, Internal categories in Mal'cev categories, *J. Pure Appl. Algebra,* **143**, 1999, 221-229.

[12] **M. Gran**, Central extensions and internal groupoids in Maltsev categories, *J. Pure Appl. Algebra*, **155**, 2001, 139-166.

[13] **H. P. Gumm**, *Geometrical Methods in Congruence Modular Varieties,* Mem. Amer. Math. Soc., **45** (286), 1983.

[14] **P.J. Higgins**, Groups with multiple operators, *Proc. London Math. Soc.*, **(3)**, 6, 1956, 366-416.

[15] **G. Janelidze - G.M. Kelly**, Galois theory and a general notion of central extension, *J. Pure Appl. Algebra*, **97**, 1994, 135-161.

[16] **G. Janelidze - G.M. Kelly**, Central extensions in universal algebra: a unification of three notions, *Alg. Universalis*, **44**, 2000, 123-128.

[17] **G. Janelidze - G.M. Kelly**, Central extensions in Mal'tsev varieties, *Theory and Applications of Categories*, Vol. 7, No. 10, 2000, 219-226.

[18] **G. Janelidze - M.C. Pedicchio**, Internal categories and groupoids in congruence modular varieties, *Journal of Algebra* **193**, 1997, 552-570.

[19] **P.T. Johnstone**, Affine categories and naturally Mal'cev categories, *J. Pure Appl. Algebra*, **61**, 1989, 251-256.

[20] **A.S.-T. Lue**, Baer-invariants and extensions relative to a variety, *Proc. Cambridge Philos. Soc.*, 63, 1967, 569-578.

[21] **A.I. Mal'cev**, On the general theory of algebraic systems, *Mat. Sbornik N. S.*, **35**, 1954, 3-20.

[22] **M.C. Pedicchio**, A categorical approach to commutator theory, *Journal of Algebra*, **177**, 1995, 647-657.

[23] **M.C. Pedicchio**, Arithmetical categories and commutator theory, *Appl. Categorical Structures*, **4**, 1996, 297-305.

[24] **J.D.H. Smith**, *Mal'cev Varieties, LNM 554*, Springer-Verlag, 1976.

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