Reduction of Jacobi-Nijenhuis manifolds

J. M. Nunes da Costa * and Fani Petalidou †

Departamento de Matemática
Universidade de Coimbra
Apartado 3008
3001-454 Coimbra - Portugal
e-mail : jmcosta@mat.uc.pt, fpetalid@mat.uc.pt

Abstract

A reduction theorem for Jacobi-Nijenhuis manifolds is established and its relation with the reduction of homogeneous Poisson-Nijenhuis structures is shown. Reduction under Lie group actions is also studied.

Key words : Jacobi-Nijenhuis manifold, homogeneous Poisson-Nijenhuis manifold, reduction.

Mathematics Subject Classification (2000): 37J15, 53D10, 53D17, 53D20.

Introduction

The notion of Jacobi-Nijenhuis structure was introduced by J. Marrero *et al.* in [7]. Recently, the authors gave, in [13], a more strict definition of that structure which generalizes, in a natural way, the notion of Poisson-Nijenhuis manifold introduced by F. Magri and C. Morosi ([6], [3]) for better understanding the completely integrable hamiltonian systems.

In this paper we intend to study the reduction of Jacobi-Nijenhuis structures. Mainly, we define a foliation on a submanifold of a Jacobi-Nijenhuis manifold, in such a way that the manifold of the leaves is also endowed with a Jacobi-Nijenhuis structure. Since a Jacobi-Nijenhuis manifold carries a Jacobi structure and, on the other hand, there is a close relation between Jacobi-Nijenhuis manifolds and homogeneous Poisson-Nijenhuis manifolds, we were inspired in some technical arguments used in the reduction methods of both Jacobi ([10], [9]) and Poisson-Nijenhuis manifolds ([14]), in order to achieve our goal.

The paper is organized as follows. In section 1, we review some basic facts about Jacobi manifolds, including the reduction method. In section 2 we give a reduction theorem for homogeneous Poisson-Nijenhuis manifolds, which is adapted from the Poisson-Nijenhuis reduction theorem of Vaisman ([14]). Section 3 is devoted to Jacobi-Nijenhuis manifolds.

*This work was partially supported by CMUC-FCT and PRAXIS.
†This work was partially supported by CMUC-FCT.

We recall the essential definitions and the notions of associated homogeneous Poisson-Nijenhuis manifold and conformal equivalence. In section 4, we establish a reduction theorem for Jacobi-Nijenhuis manifolds, we study the reduction of conformally equivalent Jacobi-Nijenhuis structures and we show how the homogeneous Poisson-Nijenhuis reduction is related with the Jacobi-Nijenhuis reduction. Last section concerns the reduction of Jacobi-Nijenhuis structures under Lie group actions. The two cases presented are examples of the reduction theorem of previous section. In the first case, we obtain a Jacobi-Nijenhuis structure on the space of the orbits of a Lie group action. In the second, the action has a momentum map and the Jacobi-Nijenhuis structure is defined on a quotient of a level set of that momentum map.

Notation: In the following, we will denote by M a C^∞-differentiable manifold of finite dimension, by $C^\infty(M)$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$, $k \in \mathbb{N}$, the space of k-forms on M, and by $\mathcal{V}^k(M)$, $k \in \mathbb{N}$, the space of skew-symmetric contravariant k-tensors on M.

1 Jacobi manifolds

We consider a manifold M endowed with a 2-tensor Λ and a vector field E. The following bracket on $C^\infty(M)$,

$$\{f,g\} = \Lambda(df,dg) + \langle fdg - gdf, E\rangle, \quad f,g \in C^\infty(M), \tag{1}$$

is bilinear and skew-symmetric, and satisfies the Jacobi identity if and only if

$$[\Lambda,\Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E,\Lambda] = 0, \tag{2}$$

where $[\,,\,]$ denotes the Schouten bracket ([4]). When conditions (2) are verified, the pair (Λ, E) defines a *Jacobi structure* on M and (M, Λ, E) is called a *Jacobi manifold.* The bracket (1) is the *Jacobi bracket* and $(C^\infty(M), \{,\})$ is a local Lie algebra in the sense of Kirillov (cf. [2]). If the vector field E identically vanishes on M, conditions (2) reduce to $[\Lambda,\Lambda] = 0$, and M is endowed with a *Poisson structure.*

We denote by $\Lambda^\# : T^*M \to TM$ and $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$, the vector bundle maps associated with Λ and (Λ, E), respectively; i.e., for all α, β sections of T^*M and $f \in C^\infty(M)$,

$$\langle \beta, \Lambda^\#(\alpha)\rangle = \Lambda(\alpha,\beta) \tag{3}$$

and

$$(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E\rangle). \tag{4}$$

These vector bundle maps can be considered as homomorphisms of $C^\infty(M)$-modules, $\Lambda^\# : \Omega^1(M) \to \mathcal{V}^1(M)$ and $(\Lambda, E)^\# : \Omega^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$, respectively.

For any $f \in C^\infty(M)$, the vector field on M

$$X_f = \Lambda^\#(df) + fE, \tag{5}$$

is called the *hamiltonian vector field* associated with f.

The space $\Omega^1(M) \times C^\infty(M)$ possesses a Lie algebra structure whose bracket $\{\,,\}$ is defined as follows, (cf. [1]): for all $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M)$,

$$\{(\alpha, f), (\beta, g)\} := (\gamma, h), \tag{6}$$

where

$$\begin{aligned} \gamma &:= L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha,\beta)) + fL_E\beta - gL_E\alpha - i_E(\alpha\wedge\beta), \\ h &:= -\Lambda(\alpha,\beta) + \Lambda(\alpha, dg) - \Lambda(\beta, df) + \langle fdg - gdf, E\rangle, \end{aligned}$$

(L is the Lie derivative operator).

Let $a \in C^\infty(M)$ be a function which vanishes nowhere on M. For all $f, g \in C^\infty(M)$, we may define

$$\{f, g\}^a := \frac{1}{a}\{af, ag\}. \tag{7}$$

This new bracket $\{,\}^a$ on $C^\infty(M)$ defines another Jacobi structure (Λ^a, E^a) on M, which is said to be *a-conformal* to the initially given one. The two Jacobi structures (Λ, E) and (Λ^a, E^a) are said to be *conformally equivalent* and

$$\Lambda^a = a\Lambda \quad , \quad E^a = \Lambda^\#(da) + aE. \tag{8}$$

A *homogeneous Poisson manifold* (M, Λ, T) is a Poisson manifold (M, Λ) with a vector field $T \in \mathcal{V}^1(M)$ such that

$$L_T\Lambda = [T, \Lambda] = -\Lambda. \tag{9}$$

Homogeneous Poisson manifolds are closely related to Jacobi manifolds. With each Jacobi manifold (M, Λ, E) we may associate a homogeneous Poisson manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$, with

$$\tilde{M} = M \times \mathbb{R}, \quad \tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E) \text{ and } \tilde{T} = \frac{\partial}{\partial t}, \tag{10}$$

where t is the usual coordinate on $\mathbb{R}$, [5]. The manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ is called the *Poissonization* of (M, Λ, E).

Let us now recall the reduction procedure for Jacobi manifolds.

Theorem 1.1 ([10], [9]) *Let (M, Λ, E) be a Jacobi manifold, S a submanifold of M and F a vector sub-bundle of T_SM, which satisfy the following conditions:*

(i) *the distribution $TS \cap F$ is completely integrable and the foliation of S defined by this distribution is simple, that is, all the leaves have the same dimension and the set $\hat{S}$ of leaves has the structure of a differentiable manifold for which the canonical projection $\pi : S \to \hat{S}$ is a submersion;*

(ii) *for any $f, h \in C^\infty(M)$ with differentials df and dh, restricted to S, vanishing on F, the differential $d\{f, h\}$, restricted to S, vanishes on F;*

(iii) *if $F^0 \subset T^*_SM$ denotes the annihilator of F, then $(\Lambda|_S)^\#(F^0) \subset TS + F$, and the restriction of E to S is a differentiable section of $TS + F$.*

Then, there exists on $\hat{S}$ a unique Jacobi structure $(\hat{\Lambda}, \hat{E})$ whose associated bracket is given, for any $\hat{f}, \hat{h} \in C^\infty(\hat{S})$ and any differentiable extensions f of $\hat{f} \circ \pi$ and h of $\hat{h} \circ \pi$ with differentials df and dh, restricted to S, vanish on F, by

$$\{\hat{f}, \hat{h}\} \circ \pi = \{f, h\} \circ i, \tag{11}$$

where i is the canonical injection of S in M.

The Jacobi manifold $(\hat{S}, \hat{\Lambda}, \hat{E})$ is said to have been obtained from (M, Λ, E) by reduction via (S, F).

Let $\lambda : T_S M \to TS$ be a (projection) vector bundle map such that its restriction to TS is the identity map and $F \subset Ker\lambda$. Then, the Jacobi structures (Λ, E) on M and $(\hat{\Lambda}, \hat{E})$ on $\hat{S}$ are related by the formulæ:

$$\hat{\Lambda}^{\#}_{\pi(x)} = T_x\pi \circ \lambda_x \circ \Lambda^{\#}_{i(x)} \circ {}^t\lambda_x \circ {}^tT_x\pi, \quad x \in S, \tag{12}$$

$$\hat{E} \circ \pi = T\pi \circ \lambda \circ E \circ i. \tag{13}$$

We remark that the transpose of λ, ${}^t\lambda : T^*S \to T^*_S M$, is the injection that extends each linear form on S to a linear form on M that vanishes on $Ker\lambda$.

2 Reduction of homogeneous Poisson-Nijenhuis manifolds

This section is devoted to Poisson-Nijenhuis and homogeneous Poisson-Nijenhuis manifolds. We give a reduction theorem for homogeneous Poisson-Nijenhuis manifolds.

A *Nijenhuis operator* on a differentiable manifold M is a tensor field N of type $(1,1)$ which has a vanishing Nijenhuis torsion:

$$T(N)(X,Z) = [NX, NZ] - N[NX, Z] - N[X, NZ] + N^2[X, Z] = 0, \quad X, Z \in \mathcal{V}^1(M).$$

A *Poisson-Nijenhuis manifold* (M, Λ_0, N) is a Poisson manifold (M, Λ_0) with a Nijenhuis tensor N which is compatible with Λ_0, i.e. : **i)** $N\Lambda_0^{\#} = \Lambda_0^{\#}\,{}^tN$, where ${}^tN$ is the transpose of N, and **ii)** the map $\Lambda_0^{\#} \circ C(\Lambda_0, N) : \Omega^1(M) \times \Omega^1(M) \to \mathcal{V}^1(M)$ identically vanishes on M. $C(\Lambda_0, N)$ is the *Magri-Morosi concomitant* of Λ_0 and N ([6]) defined, for all $(\alpha, \beta) \in \Omega^1(M) \times \Omega^1(M)$, by

$$C(\Lambda_0, N)(\alpha, \beta) = \{\alpha, \beta\}_1 - \{{}^tN\alpha, \beta\}_0 - \{\alpha, {}^tN\beta\}_0 + {}^tN\{\alpha, \beta\}_0, \tag{14}$$

where $\{\,,\,\}_i$ is the bracket associated with Λ_i, $\Lambda_i^{\#} = N^i\Lambda_0^{\#}$, $i = 0, 1$, that defines a Lie algebra structure on $\Omega^1(M)$, [3]. N is called the *recursion operator* of (M, Λ_0, N).

In what concerns the reduction procedure, remark that, when a Jacobi manifold is Poisson, Theorem 1.1 is the Marsden-Ratiu Poisson reduction theorem, [8]. This last one was refined by Vaisman ([14]) in order to include the Poisson-Nijenhuis case.

Theorem 2.1 ([14]) *Let (M, Λ, N) be a Poisson-Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ verifying conditions i) and ii) of Theorem 1.1.* [1] *Moreover, if $N|_S(TS) \subset TS$, $N|_S(F) \subset F$, $N|_S$ sends projectable vector fields to projectable vector fields, and $(\Lambda|_S)^{\#}(F^0) \subset TS$, then there exists on $\hat{S}$ a Poisson-Nijenhuis structure $(\hat{\Lambda}, \hat{N})$, obtained from (Λ, N) by* reduction via (S, F).

The Poisson tensor $\hat{\Lambda}$ on $\hat{S}$ is associated with the vector bundle map $\hat{\Lambda}^{\#}$ given by (12) and the tensor $\hat{N}$ of type $(1,1)$ on $\hat{S}$ is given by

$$\hat{N} = T\pi \circ \lambda \circ N|_S \circ \lambda_h^{-1} \circ (T\pi)_h^{-1}, \tag{15}$$

where λ_h is the restriction of λ to $TS \subset T_S M$, which is the identity map, and $(T\pi)_h$ is the restriction of $T\pi$ to the horizontal vector sub-bundle of TS, with respect to the decomposition $TS \equiv T\hat{S} \oplus (TS \cap F)$.

Let us introduce a tensor field N_S of type $(1,1)$ on the submanifold S, by setting

$$N_S = \lambda \circ N|_S \circ \lambda_h^{-1}. \tag{16}$$

Then, (15) can be written as

$$\hat{N} = T\pi \circ N_S \circ (T\pi)_h^{-1}. \tag{17}$$

Definition 2.2 *A* homogeneous Poisson-Nijenhuis manifold (M, Λ, N, T) *is a Poisson-Nijenhuis manifold (M, Λ, N) with a vector field $T \in \mathcal{V}^1(M)$ such that*

$$L_T\Lambda = -\Lambda \quad \text{and} \quad L_T N = 0. \tag{18}$$

Remark 2.3 *Conditions (18) assure that, for all $k \in \mathbb{N}$, $L_T\Lambda_k = -\Lambda_k$, where Λ_k is the Poisson tensor associated with $\Lambda_k^{\#} = N^k\Lambda$. That is, all the members of the hierarchy $(\Lambda_k, k \in \mathbb{N})$ of pairwaise compatible Poisson tensors on M are homogeneous with respect to the vector field T.*

Theorem 2.1 can easily be adapted to include homogeneous Poisson-Nijenhuis reduction case.

Theorem 2.4 *Let (M, Λ, N, T) be a homogeneous Poisson-Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ such that all the conditions of Theorem 2.1 are verified, and denote by $(\hat{S}, \hat{\Lambda}, \hat{N})$ the Poisson-Nijenhuis manifold obtained from (M, Λ, N) by reduction via (S, F). If the vector field $T \in \mathcal{V}^1(M)$ is tangent to S, $T|_S \notin Ker\lambda$ and $\lambda(T|_S) = T_S \in \mathcal{V}^1(S)$ is a projectable vector field with projection $\hat{T} \in \mathcal{V}^1(\hat{S})$, then $(\hat{S}, \hat{\Lambda}, \hat{N}, \hat{T})$ is a homogeneous Poisson-Nijenhuis manifold.*

Proof. We only have to prove that $L_{\hat{T}}\hat{\Lambda} = -\hat{\Lambda}$ and $L_{\hat{T}}\hat{N} = 0$.

It is easy to verify that the tensor field $L_{T_S}N_S$ on S is projectable and its projection is $L_{\hat{T}}\hat{N}$, i.e.,

$$L_{\hat{T}}\hat{N} = T\pi \circ L_{T_S}N_S \circ (T\pi)_h^{-1}, \tag{19}$$

[1] Obviously, the bracket considered in ii) is the Poisson bracket on (M, Λ).

where N_S is given by (16). Since T is tangent to S and $N|_S(TS) \subset TS$, (19) can be written as

$$L_{\hat{T}}\hat{N} = T\pi \circ \lambda \circ (L_{T|_S}N|_S) \circ \lambda_h^{-1} \circ (T\pi)_h^{-1}. \tag{20}$$

Taking into account that $L_T N = 0$, from (20) we obtain $L_{\hat{T}}\hat{N} = 0$.

On the other hand, for all $\hat{\alpha}, \hat{\beta} \in \Omega^1(\hat{S})$,

$$(L_{T|_S}\Lambda|_S)({}^t(T\pi \circ \lambda)(\hat{\alpha}),\, {}^t(T\pi \circ \lambda)(\hat{\beta})) = -\Lambda|_S({}^t(T\pi \circ \lambda)(\hat{\alpha}),\, {}^t(T\pi \circ \lambda)(\hat{\beta})). \tag{21}$$

The second member of (21) equals $-\hat{\Lambda}(\hat{\alpha}, \hat{\beta})$. Using the facts that T is tangent to S, the two $1-$forms ${}^t\lambda(L_{T_S}({}^tT\pi(\hat{\alpha})))$ and $L_{T|_S}({}^t(T\pi\circ\lambda)(\hat{\alpha}))$ coincide on TS, and $(\Lambda|_S)^{\#}((TS)^0) \subset F$, we conclude that the first member of (21) equals $(L_{\hat{T}}\hat{\Lambda})(\hat{\alpha}, \hat{\beta})$. So, $L_{\hat{T}} = -\hat{\Lambda}$, because $\hat{\alpha}$ and $\hat{\beta}$ are arbitrary.

$\diamondsuit$

3 Jacobi-Nijenhuis manifolds

The initial definition of Jacobi-Nijenhuis manifold was introduced by Marrero and *al.* in [7]. In [13], the authors gave a more strict definition of this concept. In this section we review the essential results concerning this structure, needed throughout this article.

Let M be a C^∞-differentiable manifold and $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$ a $C^\infty(M)$-linear map defined, for all $(X, f) \in \mathcal{V}^1(M) \times C^\infty(M)$, by

$$\mathcal{N}(X, f) = (NX + fY, \langle\gamma, X\rangle + gf), \tag{22}$$

where N is a tensor field of type $(1, 1)$ on M, $Y \in \mathcal{V}^1(M)$, $\gamma \in \Omega^1(M)$ and $g \in C^\infty(M)$. $\mathcal{N} := (N, Y, \gamma, g)$ can be considered as a vector bundle map, $\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}$. Since the space $\mathcal{V}^1(M) \times C^\infty(M)$ endowed with the bracket

$$[(X, f), (Z, h)] = ([X, Z], X \cdot h - Z \cdot f), \tag{23}$$

$((X, f), (Z, h)) \in \left(\mathcal{V}^1(M) \times C^\infty(M, \mathbf{R})\right)^2$, is a real Lie algebra, we may define the *Nijenhuis torsion* $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$. It is a $C^\infty(M)$-bilinear map $\mathcal{T}(\mathcal{N}) : \left(\mathcal{V}^1(M) \times C^\infty(M)\right)^2 \to \mathcal{V}^1(M) \times C^\infty(M)$ given by

$$\begin{aligned} \mathcal{T}(\mathcal{N})\left((X, f), (Z, h)\right) &= [\mathcal{N}(X, f), \mathcal{N}(Z, h)] - \mathcal{N}[\mathcal{N}(X, f), (Z, h)] - \\ &- \mathcal{N}[(X, f), \mathcal{N}(Z, h)] + \mathcal{N}^2[(X, f), (Z, h)], \end{aligned} \tag{24}$$

$((X, f), (Z, h)) \in \left(\mathcal{V}^1(M) \times C^\infty(M)\right)^2$.

Definition 3.1 *A $C^\infty(M)$-linear map $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$ is a* Nijenhuis operator *on M, if it has a vanishing Nijenhuis torsion.*

Suppose now that M is equipped with a Jacobi structure (Λ_0, E_0) and a Nijenhuis operator $\mathcal{N}$. Then, we may define a tensor field Λ_1 of type $(2,0)$ and a vector field E_1 on M, by setting
$$(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda_0, E_0)^\#. \tag{25}$$

Recall that two Jacobi structures (Λ_0, E_0) and (Λ_1, E_1), defined on the same differentiable manifold, are said to be *compatible* if their sum $(\Lambda_0 + \Lambda_1, E_0 + E_1)$ is again a Jacobi structure, (cf. [12]).

If one looks for the conditions that assure the pair (Λ_1, E_1), given by (25), defines a new Jacobi structure on M, compatible with (Λ_0, E_0), one finds (cf. [7]):

1. Λ_1 is skew-symmetric if and only if $\mathcal{N} \circ (\Lambda_0, E_0)^\# = (\Lambda_0, E_0)^\# \circ {}^t\mathcal{N}$, where ${}^t\mathcal{N}$ is the transpose of $\mathcal{N}$. This condition is equivalent to $NE_0 = \Lambda_0^\#(\gamma) + gE_0$, $N\Lambda_0^\# - Y \otimes E_0 = \Lambda_0^\# \, {}^tN + E_0 \otimes Y$ and $\langle \gamma, E_0 \rangle = 0$. Then,
$$\Lambda_1^\# = N\Lambda_0^\# - Y \otimes E_0 = \Lambda_0^\# \, {}^tN + E_0 \otimes Y \tag{26}$$
and
$$E_1 = NE_0 = \Lambda_0^\#(\gamma) + gE_0. \tag{27}$$

2. When Λ_1 is skew-symmetric, (Λ_1, E_1) defines a Jacobi structure on M if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$,
$$\begin{aligned} &\mathcal{T}(\mathcal{N}) \Big((\Lambda_0, E_0)^\#(\alpha, f), (\Lambda_0, E_0)^\#(\beta, h) \Big) = \\ &= \mathcal{N} \circ (\Lambda_0, E_0)^\# \left(\mathcal{C}\left((\Lambda_0, E_0), \mathcal{N} \right) ((\alpha, f), (\beta, h)) \right), \end{aligned}$$
where $\mathcal{C}\left((\Lambda_0, E_0), \mathcal{N}\right)$ is the *concomitant* of (Λ_0, E_0) and $\mathcal{N}$ which is given, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$, by
$$\begin{aligned} \mathcal{C}\left((\Lambda_0, E_0), \mathcal{N}\right)((\alpha, f), (\beta, h)) &= \{(\alpha, f), (\beta, h)\}_1 - \{{}^t\mathcal{N}(\alpha, f), (\beta, h)\}_0 - \\ &- \{(\alpha, f), {}^t\mathcal{N}(\beta, h)\}_0 + {}^t\mathcal{N}\{(\alpha, f), (\beta, h)\}_0, \end{aligned}$$
($\{\,,\}_i$ is the bracket (6) associated with the Jacobi structure (Λ_i, E_i), $i = 0, 1$).

3. In the case where (Λ_1, E_1) is a Jacobi structure, it is compatible with (Λ_0, E_0) if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$,
$$(\Lambda_0, E_0)^\# \left(\mathcal{C}\left((\Lambda_0, E_0), \mathcal{N}\right)((\alpha, f), (\beta, h)) \right) = 0.$$

Definition 3.2 ([13]) *A* Jacobi-Nijenhuis manifold $(M, (\Lambda_0, E_0), \mathcal{N})$ *is a Jacobi manifold* (M, Λ_0, E_0) *with a Nijenhuis operator* $\mathcal{N}$ *which is compatible with* (Λ_0, E_0)*, i.e.:* **i)** $\mathcal{N} \circ (\Lambda_0, E_0)^\# = (\Lambda_0, E_0)^\# \circ {}^t\mathcal{N}$ *and* **ii)** *the map* $(\Lambda_0, E_0)^\# \circ \mathcal{C}\left((\Lambda_0, E_0), \mathcal{N}\right) : (\Omega^1(M) \times C^\infty(M))^2 \to \mathcal{V}^1(M) \times C^\infty(M)$ *identically vanishes on* M*.* $\mathcal{N}$ *is called the* recursion operator *of* $(M, (\Lambda_0, E_0), \mathcal{N})$.

Theorem 3.3 ([7]) *Let $((\Lambda_0, E_0), \mathcal{N})$ be a Jacobi-Nijenhuis structure on a differentiable manifold M. Then, there exists a hierarchy $((\Lambda_k, E_k), k \in \mathbb{N})$ of Jacobi structures on M, which are pairwaise compatible. For all $k \in \mathbb{N}$, (Λ_k, E_k) is the Jacobi structure associated with the vector bundle map $(\Lambda_k, E_k)^\#$ given by $(\Lambda_k, E_k)^\# = \mathcal{N}^k \circ (\Lambda_0, E_0)^\#$. Moreover, for all $k, l \in \mathbb{N}$, the pair $((\Lambda_k, E_k), \mathcal{N}^l)$ defines a Jacobi-Nijenhuis structure on M.*

Next propostition shows the relation between Jacobi-Nijenhuis manifolds and homogeneous Poisson-Nijenhuis structures.

Proposition 3.4 ([13]) *With each Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, a homogeneous Poisson-Nijenhuis manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{N}, \tilde{T})$ can be associated, where $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ is the Poissonization (10) of (M, Λ, E) and $\tilde{N}$ is the Nijenhuis tensor field on $\tilde{M}$ given by*

$$\tilde{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt. \tag{28}$$

Finally, we recall the notion of conformal equivalence of Jacobi-Nijenhuis structures on a differentiable manifold M.

Proposition 3.5 ([13]) *Let $((\Lambda_0, E_0), \mathcal{N})$ be a Jacobi-Nijenhuis structure on M, (Λ_1, E_1) the Jacobi structure associated with $(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda_0, E_0)^\#$, $a \in C^\infty(M)$ a function which vanishes nowhere, and (Λ_0^a, E_0^a) (resp. (Λ_1^a, E_1^a)) the Jacobi structure a-conformal to (Λ_0, E_0) (resp. (Λ_1, E_1)). Then, there exists a Nijenhuis operator $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ such that $(\Lambda_1^a, E_1^a)^\# = \mathcal{N}^a \circ (\Lambda_0^a, E_0^a)^\#$, with*

$$N^a = N - Y \otimes \frac{da}{a}, \qquad Y^a = Y, \tag{29}$$

$$\gamma^a = \gamma + {}^tN\frac{da}{a} - (g + \frac{1}{a}L_Y a)\frac{da}{a}, \qquad g^a = g + \frac{1}{a}L_Y a. \tag{30}$$

The Jacobi-Nijenhuis structure $((\Lambda_0^a, E_0^a), \mathcal{N}^a)$ is said to be a-conformal *to $((\Lambda_0, E_0), \mathcal{N})$.*

4 Reduction of Jacobi-Nijenhuis manifolds

In this section we present the main result of this paper: a reduction theorem for Jacobi-Nijenhuis manifolds. We also study the reduction of conformally equivalent Jacobi-Nijenhuis structures and the relation between the Jacobi-Nijenhuis and homogeneous Poisson-Nijenhuis reduction.

Theorem 4.1 *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi-Nijenhuis manifold, S a submanifold of M, $i : S \hookrightarrow M$ the canonical injection, and F a vector sub-bundle of T_SM, which satisfy the conditions: i) and ii) of Theorem 1.1 and also*

- *iii) $(\Lambda|_S)^\#(F^0) \subset TS$ and $E|_S$ is a section of TS;*
- *iv) $N|_S(TS) \subset TS$, $N|_S(F) \subset F$ and N_S, given by (16), sends projectable vector fields to projectable vector fields;*

v) Y is tangent to S and $Y_S = \lambda(Y|_S) \in \mathcal{V}^1(S)$ is a projectable vector field, where $\lambda : T_SM \to TS$ is a (projection) vector bundle map such that its restriction to TS is the identity map and $F \subset Ker\lambda$;

vi) $\gamma|_S$ is a section of $(TS \cap F)^0$ and, for all sections Z of $TS \cap F$, $i_Z d({}^t(Ti)(\gamma|_S)) = 0$;

vii) $g|_S$ is constant on the leaves of S.

Under these conditions, there exists on $\hat{S}$ a Jacobi-Nijenhuis structure $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, where $(\hat{\Lambda}, \hat{E})$ is given by (12) and (13), $\hat{N}$ is given by (17), $\hat{Y} = T\pi \circ \lambda \circ Y|_S$, $\hat{\gamma} \in \Omega^1(\hat{S})$ is such that ${}^tT\pi(\hat{\gamma}) = {}^t(Ti)(\gamma|_S)$, and $\hat{g} \in C^\infty(\hat{S})$ is given by $\hat{g} \circ \pi = g|_S$. The Jacobi-Nijenhuis manifold $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ is said to have been obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (S, F).

Proof. Since all the conditions of Theorem 1.1 hold, $\hat{S}$ is endowed with a (reduced) Jacobi structure $(\hat{\Lambda}, \hat{E})$, given by (12) and (13). It remains to show that the Nijenhuis operator $\mathcal{N} := (N, Y, \gamma, g)$ also reduces to a Nijenhuis operator $\hat{\mathcal{N}}$ compatible with $(\hat{\Lambda}, \hat{E})$.

As in the case of Theorem 2.1, condition *iv)* guarantees the existence of a tensor field $\hat{N}$ of type $(1,1)$ on $\hat{S}$, given by (17). From condition *v)*, the vector field $Y_S = \lambda(Y|_S) \in \mathcal{V}^1(S)$ is projectable and we denote by $\hat{Y} \in \mathcal{V}^1(\hat{S})$ its projection. Also, by hypothesis *vi)*, the 1-form $\gamma_S = {}^t(Ti)(\gamma|_S)$ on S is projectable and we denote by $\hat{\gamma} \in \Omega^1(\hat{S})$ its projection. Finally, from condition *vii)*, there exists a function $\hat{g} \in C^\infty(\hat{S})$ such that $\hat{g} \circ \pi = g|_S$. Thus, we obtain a $C^\infty(\hat{S})$-linear map, $\hat{\mathcal{N}} : \mathcal{V}^1(\hat{S}) \times C^\infty(\hat{S}) \to \mathcal{V}^1(\hat{S}) \times C^\infty(\hat{S})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, defined as in (22). Using the properties of the restriction $\mathcal{N}|_S := (N|_S, Y|_S, \gamma|_S, g|_S)$ of $\mathcal{N}$ to the submanifold S, a straightforward calculation shows that $\hat{\mathcal{N}}$ has a vanishing Nijenhuis torsion.

In order to conclude that $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ defines a Jacobi-Nijenhuis structure on $\hat{S}$, we have to prove that $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^\# = (\hat{\Lambda}, \hat{E})^\# \circ {}^t\hat{\mathcal{N}}$ and that $(\hat{\Lambda}, \hat{E})^\# \circ \mathcal{C}\left((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}}\right) = 0$. Let $\hat{\alpha} \in \Omega^1(\hat{S})$, $\hat{f} \in C^\infty(\hat{S})$, and consider ${}^t(T\pi \circ \lambda)(\hat{\alpha})$, which is a section of T_S^*M, and $f \in C^\infty(M)$ an extension of $\hat{f} \circ \pi$, i.e., $f|_S = \hat{f} \circ \pi$. Then,

$$\mathcal{N}|_S((\Lambda|_S, E|_S)^\#({}^t(T\pi \circ \lambda)(\hat{\alpha}), f|_S) = (\Lambda|_S, E|_S)^\# ({}^t\mathcal{N}|_S({}^t(T\pi \circ \lambda)(\hat{\alpha}), f|_S)). \tag{31}$$

Since $(\Lambda|_S)^\#({}^t(T\pi \circ \lambda)(\hat{\alpha}))$ is a section of $(\Lambda|_S)^\#(F^0) \subset TS$ and $E|_S$ is a section of TS, the image by $(T\pi \circ \lambda)$ of the term vector field of the first member of (31) is equal to

$$\hat{N}(\hat{\Lambda}^\#(\hat{\alpha})) + \hat{f}\hat{N}(\hat{E}) - < \hat{\alpha}, \hat{E} > \hat{Y}. \tag{32}$$

Because ${}^t\lambda({}^tN_S({}^tT\pi(\hat{\alpha}))) -{}^t N|_S({}^t(T\pi \circ \lambda)(\hat{\alpha}))$ is a section of $(TS)^0$ and $(\Lambda|_S)^\#((TS)^0) \subset F$, we get

$$T\pi \circ \lambda((\Lambda|_S)^\#({}^tN|_S({}^t(T\pi \circ \lambda)(\hat{\alpha})))) = \hat{\Lambda}^\#({}^t\hat{N}(\hat{\alpha})),$$

and we may conclude that the image by $(T\pi \circ \lambda)$ of the term vector field of the second member of (31) is equal to

$$\hat{\Lambda}^\#({}^t\hat{N}(\hat{\alpha})) + \hat{f}\hat{\Lambda}^\#(\hat{\gamma}) + < \hat{\alpha}, \hat{Y} > \hat{E} + \hat{f}\hat{g}\hat{E}. \tag{33}$$

From (32) and (33), we obtain

$$\hat{N}(\hat{\Lambda}^\#(\hat{\alpha})) + \hat{f}\hat{N}(\hat{E}) - < \hat{\alpha}, \hat{E} > \hat{Y} = \hat{\Lambda}^\#({}^t\hat{N}(\hat{\alpha})) + \hat{f}\hat{\Lambda}^\#(\hat{\gamma}) + < \hat{\alpha}, \hat{Y} > \hat{E} + \hat{f}\hat{g}\hat{E},$$

which means that the term vector field of $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^{\#}(\hat{\alpha}, \hat{f})$ coincide with the term vector field of $(\hat{\Lambda}, \hat{E})^{\#} \circ {}^t\hat{\mathcal{N}}(\hat{\alpha}, \hat{f})$. In a similar way, one can prove that the term function of $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^{\#}(\hat{\alpha}, \hat{f})$ is equal to the term function of $(\hat{\Lambda}, \hat{E})^{\#} \circ {}^t\hat{\mathcal{N}}(\hat{\alpha}, \hat{f})$. Since $\hat{\alpha} \in \Omega^1(\hat{S})$ and $\hat{f} \in C^\infty(\hat{S})$ are arbitrary, we obtain $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^{\#} = (\hat{\Lambda}, \hat{E})^{\#} \circ {}^t\hat{\mathcal{N}}$. Applying the same kind of technical arguments as before, we can deduce, after a hard computation, that $(\hat{\Lambda}, \hat{E})^{\#} \circ \mathcal{C}\left((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}}\right) = 0$.

$\diamondsuit$

Remark 4.2 *Under the assumptions of Theorem 4.1, if $(M, (\Lambda, E), \mathcal{N})$ is a Jacobi-Nijenhuis manifold which is reducible via (S, F) to $(\hat{S}, (\hat{\Lambda}_0, \hat{E}_0), \hat{\mathcal{N}})$, then, each member $(\hat{\Lambda}_k, \hat{E}_k)$ of the hierarchy $((\hat{\Lambda}_k, \hat{E}_k), k \in \mathbb{N})$ of Jacobi structures on $\hat{S}$, given by Theorem 3.3, is obtained, by reduction via (S, F), from the corresponding member (Λ_k, E_k) of the hierachy $((\Lambda_k, E_k), k \in \mathbb{N})$ of Jacobi structures on M.*

Next proposition establishes a relation between reduction and conformal equivalence of Jacobi-Nijenhuis structures.

Proposition 4.3 *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi-Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ which satisfy the conditions of Theorem 1.1, and $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, the Jacobi-Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (S, F). Let $a \in C^\infty(M)$ be a function which vanishes nowhere and such that da is a section of F^0, and $((\Lambda^a, E^a), \mathcal{N}^a)$ the Jacobi-Nijenhuis structure on M, a-conformal to $((\Lambda, E), \mathcal{N})$. Then $(M, (\Lambda^a, E^a), \mathcal{N}^a)$ is reducible via (S, F) and the reduced structure on $\hat{S}$ is conformally equivalent to $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$.*

Proof. Since da is a section of F^0, it is easy to check that, if the Jacobi structure (Λ, E) on M is reducible via (S, F), then the a-conformal Jacobi structure (Λ^a, E^a) on M is also reducible via (S, F). Furthermore, condition iii) of Theorem 4.1 holds. So, $\hat{S}$ is equipped with two (reduced) Jacobi structures $(\hat{\Lambda}, \hat{E})$ and $(\widehat{\Lambda^a}, \widehat{E^a})$ that are compatible (cf. [12]). But $\widehat{\Lambda^a} = \hat{\Lambda}^{\hat{a}}$ and $\widehat{E^a} = \hat{E}^{\hat{a}}$, where $\hat{a} \in C^\infty(\hat{S})$ is given by $\hat{a} \circ \pi = a|_S$; that is, the Jacobi structures $(\hat{\Lambda}, \hat{E})$ and $(\widehat{\Lambda^a}, \widehat{E^a})$ on $\hat{S}$ are conformally equivalent.

It remains to check that $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ verifies the conditions $iv)$ - $vii)$ of Theorem 4.1. Because Y is tangent to S and da vanishes on F, $N^a|_S(TS) \subset TS$ and $N^a|_S(F) \subset F$. Let $X \in \mathcal{V}^1(S)$ be a projectable vector field. Then, we have that $N^a_S(X) = N_S(X) - < \dfrac{da}{a}, X > Y_S$ and, for any section Z of $TS \cap F$,

$$L_Z(N^a_S(X)) = L_Z(N_S(X)) - < \frac{da}{a}, X > L_Z Y_S$$

is also a section of $TS \cap F$. So, $N^a_S(X) \in \mathcal{V}^1(S)$ and it is a projectable vector field. For the restriction $\gamma^a|_S$ of $\gamma^a \in \Omega^1(M)$ to the submanifold S, since ${}^tN|_S(F^0) \subset F^0$, we obtain that $\gamma^a|_S$ is a section of $(TS \cap F)^0$ and that $i_Z d({}^tTi(\gamma^a|_S)) = 0$, for all sections Z of $TS \cap F$. Finally,

$$L_Z g^a = L_Z g + (L_Z \frac{1}{a}) L_Y a + \frac{1}{a} L_Z(L_Y a) = 0,$$

for all sections Z of $TS \cap F$, which implies that g^a is constant on the leaves of S. From the definitions of $\mathcal{N}^a$ and $\hat{\mathcal{N}}$, it follows that $\widehat{\mathcal{N}^a} = \hat{\mathcal{N}}^{\hat{a}}$. $\diamondsuit$

Now we are going to present the relationship between the reduction of a Jacobi-Nijenhuis manifold and the reduction of the corresponding homogeneous Poisson-Nijenhuis manifold, in the sense of Proposition 3.4.

Let $(M,(\Lambda,E),\mathcal{N})$ be a Jacobi-Nijenhuis manifold, S a submanifold of M, F a vector sub-bundle of T_SM, and $(\tilde{M},\tilde{\Lambda},\tilde{N},\tilde{T})$ the corresponding homogeneous Poisson-Nijenhuis manifold, in the sense of Proposition 3.4. Consider the submanifold $\tilde{S}=S\times\mathbb{R}$ of $\tilde{M}=M\times\mathbb{R}$ and the vector sub-bundle $\tilde{F}$ of $T_{\tilde{S}}\tilde{M}$ given by $\tilde{F}=F\times\{0\}$. Then, $T\tilde{S}\cap\tilde{F}=(TS\cap F)\times\{0\}$. We denote by $\tilde{i}:\tilde{S}\hookrightarrow\tilde{M}$ the canonical injection and by $\tilde{\lambda}:T_{\tilde{S}}\tilde{M}\to T\tilde{S}$ a (projection) vector bundle map such that its restriction to $T\tilde{S}$ is the identity map and $\tilde{F}\subset Ker\tilde{\lambda}$. We should point out that the vector field $\tilde{T}=\dfrac{\partial}{\partial t}$ is tangent to $\tilde{S}$, $\tilde{T}|_{\tilde{S}}\notin Ker\tilde{\lambda}$ and $\tilde{\lambda}(\tilde{T}|_{\tilde{S}})\in\mathcal{V}^1(\tilde{S})$ is a projectable vector field. Under these assumptions, we can state the following result.

Proposition 4.4 *If the homogeneous Poisson-Nijenhuis manifold $(\tilde{M},\tilde{\Lambda},\tilde{N},\tilde{T})$ is reduced via $(\tilde{S},\tilde{F})$ to a homogeneous Poisson-Nijenhuis manifold $(\hat{\tilde{S}},\hat{\tilde{\Lambda}},\hat{\tilde{N}},\hat{\tilde{T}})$, then the Jacobi-Nijenhuis manifold $(M,(\Lambda,E),\mathcal{N})$ is reducible via (S,F) to a Jacobi-Nijenhuis manifold $(\hat{S},(\hat{\Lambda},\hat{E}),\hat{\mathcal{N}})$.*

Moreover, $(\hat{\tilde{S}},\hat{\tilde{\Lambda}},\hat{\tilde{N}},\hat{\tilde{T}})$ is the homogeneous Poisson-Nijenhuis manifold that corresponds to $(\hat{S},(\hat{\Lambda},\hat{E}),\hat{\mathcal{N}})$ in the sense of Proposition 3.4.

The following lemma is useful in the proof of Proposition 4.4.

Lemma 4.5 *A vector field $\tilde{X}\in\mathcal{V}^1(\tilde{S})$ is projectable by $\tilde{\pi}:\tilde{S}\to\hat{\tilde{S}}$ if and only if $\tilde{X}=X+\tilde{f}\dfrac{\partial}{\partial t}$, where $X\in\mathcal{V}^1(S)$ is projectable by $\pi:S\to\hat{S}$ and $\tilde{f}\in C^\infty(\tilde{S})$ is such that $L_Z\tilde{f}=0$, for all sections Z of $TS\cap F$.*

Proof. Taking into account that a vector field $\tilde{X}\in\mathcal{V}^1(\tilde{S})$ can be written as $\tilde{X}=X+\tilde{f}\dfrac{\partial}{\partial t}$, with $X\in\mathcal{V}^1(S)$ and $\tilde{f}\in C^\infty(\tilde{S})$, and that a section of $T\tilde{S}\cap\tilde{F}$ can be identified with a section of $TS\cap F$, the conclusion follows readily. $\diamondsuit$

Proof. (Of Proposition 4.4) It is known (cf. [10]) that, if the Poisson manifold $(\hat{\tilde{S}},\hat{\tilde{\Lambda}})$ is obtained from $(\tilde{M},\tilde{\Lambda})$ by reduction via $(\tilde{S},\tilde{F})$, then the Jacobi manifold $(\hat{S},\hat{\Lambda},\hat{E})$ is obtained from (M,Λ,E) by reduction via (S,F) and, as a consequence of $T\tilde{S}\cap\tilde{F}=(TS\cap F)\times\{0\}$, $\hat{\tilde{S}}=\hat{S}\times\mathbb{R}$. Moreover, since $\tilde{F}^0=F^0\times T^*\mathbb{R}$, $(\tilde{\Lambda}|_{\tilde{S}})^{\#}(\tilde{F}^0)\subset T\tilde{S}$ implies $(\Lambda|_S)^{\#}(F^0)\subset TS$ and that $E|_S$ is a section of TS. From $\tilde{N}|_{\tilde{S}}(\tilde{F})\subset\tilde{F}$, we obtain $N|_S(F)\subset F$ and also that $\gamma|_S$ is a section of $(TS\cap F)^0$, and from $\tilde{N}|_{\tilde{S}}(T\tilde{S})\subset T\tilde{S}$, we get $N|_S(TS)\subset TS$ and we may conclude that Y is tangent to S. Let $X\in\mathcal{V}^1(S)$ be a projectable vector field. Using the fact that $\tilde{X}=X+\dfrac{\partial}{\partial t}\in\mathcal{V}^1(\tilde{S})$ is a projectable vector field and hence $\tilde{N}_{\tilde{S}}(\tilde{X})=N_S(X)+Y_S+(<\gamma_S,X>+g_S)\dfrac{\partial}{\partial t}$ is also a projectable vector field, from Lemma 4.5 we conclude that $N_S(X)$ and Y_S are projectable vector fields on S.

In addition, $\tilde{N}_{\tilde{S}}(X) = N_S(X) + < {}^t(Ti)(\gamma|_S), X > \frac{\partial}{\partial t} \in \mathcal{V}^1(\tilde{S})$ is also a projectable vector field and, from Lemma 4.5, for all sections Z of $TS \cap F$,

$$L_Z < {}^t(Ti)(\gamma|_S), X >= 0. \tag{34}$$

Since (34) holds for all projectable vector fields X on S, taking into account that, for any $x \in S$, the projectable vector fields form a basis of T_xS, we deduce that $i_Z d({}^tTi(\gamma|_S)) = 0$, for all sections Z of $TS \cap F$. Finally, because $\tilde{N}_{\tilde{S}}(\frac{\partial}{\partial t}) = Y_S + g|_S \frac{\partial}{\partial t}$ is a projectable vector field on $\tilde{S}$, from Lemma 4.5 we have that $L_Z g|_S = 0$ for all sections Z of $TS \cap F$. Thus, we conclude that the Jacobi-Nijenhuis manifold $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ is obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (S, F).

The last part of the proposition is a consequence of the fact that $T\tilde{\pi} = (T\pi, id_{T\mathbb{R}})$ and $\tilde{\lambda} = (\lambda, id_{T\mathbb{R}})$. ◊

5 Reduction under Lie group actions

Let ϕ be a left action of a Lie group G on a Jacobi manifold (M, Λ, E). ϕ is said to be a *Jacobi action* if, for all $h \in G$, the map $\phi_h : M \to M$, $\phi_h(x) = \phi(h, x)$, is a Jacobi diffeomorphism, [11]. The action ϕ is *proper* if the space $\hat{M}$ of the orbits has the structure of a differentiable manifold for which the canonical projection $\pi : M \to \hat{M}$ is a submersion.

Let $\mathcal{G}$ denote the Lie algebra of G. For any $X \in \mathcal{G}$, let $X_M \in \mathcal{V}^1(M)$ be the fundamental vector field corresponding to X,

$$X_M(x) = \frac{d}{dt}(\phi(exp(-tX), x))|_{t=0}, \; x \in M.$$

If the Lie group G is connected, then ϕ is a Jacobi action if and only if $[X_M, \Lambda] = 0$ and $[X_M, E] = 0$, for all $X \in \mathcal{G}$.

Proposition 5.1 *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi-Nijenhuis manifold, G a connected Lie group that acts on M with a proper Jacobi action ϕ and F the vector sub-bundle of TM tangent to the orbits of ϕ. If for all $X \in \mathcal{G}$, $L_{X_M} N = 0$, $L_{X_M} Y = 0$, $L_{X_M}\gamma = 0$, $i_{X_M}\gamma = 0$, $L_{X_M} g = 0$, and $N(X_M) = (\xi(X))_M$, where $\xi : \mathcal{G} \to \mathcal{G}$ is an endomorphism, then, the space $\hat{M}$ of the orbits of ϕ is endowed with a structure of a Jacobi-Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (M, F).*

Proof. A straightforward calculation leads to the conclusion that all the conditions of Theorem 4.1 hold. ◊

Let us now suppose that the Jacobi action ϕ of the connected Lie group G on the Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$ admits a momentum map J; that is, a map $J : M \to \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual space of the Lie algebra $\mathcal{G}$ of G, such that for all $X \in \mathcal{G}$, $X_M = \Lambda^{\#}(d < J, X >) + < J, X > E$, where $< J, X > \in C^\infty(M)$ is given by $< J, X > (x) =< J(x), X >$, for any $x \in M$. In addition, we suppose that J is Ad^*-equivariant, i.e., $J \circ \phi_h = Ad^*_h \circ J$, for all $h \in G$, where Ad^* is the coadjoint action of G on $\mathcal{G}^*$.

Let $\mu \in \mathcal{G}^*$ be a weakly regular value of J. Then, $S = J^{-1}(\mu)$ is a submanifold of M and $T_x J^{-1}(\mu) = Ker(T_x J)$, for all $x \in J^{-1}(\mu)$. Denote by F the vector sub-bundle of $T_S M$ given by

$$F = \{X_M - < \mu, X > E,\ X \in \mathcal{G}\}. \tag{35}$$

Then $F \cap T(J^{-1}(\mu)) = \{X_M - < \mu, X > E,\ X \in \mathcal{G}_\mu\}$, where $\mathcal{G}_\mu$ is the Lie algebra of the isotropy group G_μ. In [11] we proved that $F \cap T(J^{-1}(\mu))$ is a completely integrable vector sub-bundle of $T(J^{-1}(\mu))$ and, if it has constant rank and defines a simple foliation of $J^{-1}(\mu)$, then $(\widehat{J^{-1}(\mu)}, \hat{\Lambda}, \hat{E})$ is a Jacobi manifold obtained from (M, Λ, E) by reduction via $(J^{-1}(\mu), F)$. In this reduction procedure, one verifies that $(\Lambda|_S)^{\#}(F^0) \in TS$ and $E|_S$ is a section of TS.

Keeping the notations of the previous sections, we may establish the following result for Jacobi-Nijenhuis structures.

Proposition 5.2 *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi-Nijenhuis manifold such that the vector field E is complete. Let G be a connected Lie group which acts on M with a left Jacobi action that admits a Ad^*-equivariant momentum map J. Let $\mu \in \mathcal{G}^*$ be a weakly regular value of J, $S = J^{-1}(\mu)$, and F the vector sub-bundle of $T_S M$ given by (35). Suppose that $TS \cap F$ has constant rank and defines a simple foliation of S and that the following conditions hold:*

a) $T_S J \circ N|_S = T_S J$;

b) $\forall X \in \mathcal{G},\ N|_S(X_M - < \mu, X > E) = (\xi(X))_M - < \mu, \xi(X) > E$, *where* $\xi : \mathcal{G} \to \mathcal{G}$ *is an endomorphism;*

c) $\forall X \in \mathcal{G}_\mu,\ L_{X_M} N_S = 0$ *and* $L_E N_S = 0$;

d) Y *is tangent to* $S = J^{-1}(\mu)$, $L_E Y = 0$, *and* $L_{X_M} Y = 0$, *for all* $X \in \mathcal{G}_\mu$;

e) $i_E(d\gamma_S) = 0$ *and, for all* $X \in \mathcal{G}_\mu$, $L_{X_M}\gamma_S = 0$ *and* $i_{X_M}(d\gamma_S) = 0$;

f) $g|_S$ *is a first integral of* E *and of* X_M, *for all* $X \in \mathcal{G}_\mu$.

Under these conditions, $(\widehat{J^{-1}(\mu)}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ is a Jacobi-Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via $(J^{-1}(\mu), F)$.

Proof. An easy computation shows that the condition $iv)$ of Theorem 4.1 follows from hypothesis a), b) and c). On the other hand, from d), e) and f), conditions $v)$-$vii)$ of Theorem 4.1 also hold. Taking into account the previous comments, the proof is conclued. $\diamondsuit$

As observed in [11], the vector sub-bundle $T(J^{-1}(\mu)) \cap F$ of $T(J^{-1}(\mu))$ is the tangent bundle to the orbits of the restriction to $G_\mu \times J^{-1}(\mu)$ of the action $\phi\prime$ of G_μ on M defined, for all $x \in M$ and $X \in \mathcal{G}_\mu$, by $\phi\prime(exp(tX), x) = \phi(exp(tX), \rho_{t<\mu,X>}(x))$, where $(\rho_t)_{t\in\mathbb{R}}$ is the flow of the vector field E. Thus, the Jacobi-Nijenhuis structure $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ obtained in Proposition 5.2 is in fact defined on the space $J^{-1}(\mu)/G_\mu$ of the orbits of the action $\phi\prime$ of G_μ on $J^{-1}(\mu)$.

References

[1] Y. Kerbrat et Z. Souici-Benhammadi, *Variétés de Jacobi et groupoïdes de contact*, C. R. Acad. Sci. Paris, Série I, **317** (1993) 81-86.

[2] A. Kirillov, *Local Lie algebras*, Russian Math. Surveys **31** (1976) 55-75.

[3] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. I.H.P. **53** (1990) 35-81.

[4] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in: *Élie Cartan et les mathématiques d'aujourd'hui*, Astérisque, numéro hors série (1985) 257-271.

[5] A. Lichnerowicz, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. Pures Appl. **57** (1978) 453-488.

[6] F. Magri and C. Morosi, *A geometric characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Università di Milano, Quaderno S 19, 1984.

[7] J. C. Marrero, J. Monterde, E. Padron, *Jacobi-Nijenhuis manifolds and compatible Jacobi structures*, C. R. Acad. Sci. Paris, Série I, **329** (1999) 797-802.

[8] J. Marsden and T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. **11** (1986) 161-169.

[9] K. Mikami, *Reduction of local Lie algebra structures*, Proc. Am. Math. Soc. **105** (1989) 686-691.

[10] J. M. Nunes da Costa, *Réduction des variétés de Jacobi*, C. R. Acad. Sci. Paris **308**, Série I (1989) 101-103.

[11] J. M. Nunes da Costa, *Une généralisation, pour les variétés de Jacobi, du théorème de Marsden-Weinstein*, C. R. Acad. Sci. Paris **310**, Série I (1990) 411-414.

[12] J. M. Nunes da Costa, *Compatible Jacobi manifolds : geometry and reduction*, J. Phys. A : Math. Gen. **31** (1998) 1025-1033.

[13] F. Petalidou et J. M. Nunes da Costa, *Structure locale de variétés de Jacobi-Nijenhuis*, preprint 00-29 Departamento de Matemática da Universidade de Coimbra, 2000, submited.

[14] I. Vaisman, *Reduction of Poisson-Nijenhuis manifolds*, J. Geom. Phys. **19** (1996) 90-98.