ON THE EXISTENCE OF BERTRAND PAIRS

by

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In this note we introduce the Bertrand Group for a curve. While there is no mention of this group in it, [6] contains a statement which would mean that for a simple, closed, twisted curve that group would be trivial. However there appears to be a flaw in the proof given there.

Here we are able to show that for a simple, twisted curve that group is either trivial or Z_2. If this latter group can occur we do not know. We also show that for some classes of curves the non-triviality of that group forces the curve to be plane.

1. In what follows X will stand for R or S^1 and we will be dealing with smooth space curves, that is, C^∞ immersions $f : X \to R^3$. Moreover the curvature k_f of f is assumed not to vanish. This way we have a well defined *Frenet-Serret* frame (T_f, N_f, B_f) everywhere. We will not assume parametrization by arc-length and the velocity of f will be denoted by v_f.

Two curves $f, g : X \to R^3$ are *Bertrand mates* [3] if, for every $x \in X$, the principal normal lines of f and g at x coincide.

Definition 1 : The **Bertrand Group** $B(f)$ of $f : X \to R^3$ is the subgroup of **Diff** (X) formed by the diffeomorphisms $\delta : X \to X$ such that f and $f \circ \delta$ are Bertrand mates.

Proposition 1 : Let $f : X \to R^3$ be a smooth curve with non-vanishing curvature. Then $B(f)$ is

a) cyclic of finite order if $X = S^1$.

b) trivial or not finite if $X = R$.

Proof: Denote by A_1^3 the open Grassmannian of affine lines in R^3 and define $\widetilde{N} : X \to A_1^3$, where $\widetilde{N}(x)$ is the principal normal line at x. Since we are assuming non-vanishing curvature $\widetilde{N}$ is an immersion. This fact implies that the action $\phi : B(f) \times X \to X$, with $\phi(\delta, x) = \delta(x)$, is properly discontinuous.

In fact let $\delta \in B(f)$ and suppose that $x \in X$ is such that $\delta(x) = x$. Since $\widetilde{N}$ is an immersion there is an open neighbourhood U of x such that $\widetilde{N} \mid U$ is injective. Then, for $y \in U \cap \delta^{-1}(U)$, $\delta(y) = y$. Therefore the fixed point set Δ of δ is open. Since Δ is also closed it follows that either δ has no fixed points or is the identity. Consequently the action of $B(f)$ on X is free.

Furthermore if $U \cap \delta(U) \neq \emptyset$ then δ is the identity and $B(f)$ acts in a properly discontinuous way. Hence the projection $p : X \to X/B(f)$ is a covering projection and $\pi_1(X/B(f), p(x))/p_*(\pi_1(X, x)) \approx B(f)$ [5].

If $X = S^1$ then $X/B(f)$ is diffeomorphic to S^1 and it follows that $B(f)$ is cyclic of finite order.

Assume now that $X = R$ and that $B(f)$ is finite. Then $X/B(f)$ is either R or S^1. Since $B(f) \approx \pi_1(X/B(f))$ it follows that it must be trivial. $\bowtie$

If $\delta \in B(f)$ then it follows that $\| f(x) - f(\delta(x)) \|$ does not depend on x.

Proposition 2 : Let $\delta \in B(f)$ be such that $\| f(x) - f(\delta(x)) \| > 0$. Then the torsion τ_f never vanishes or vanishes everywhere.

Proof: There exist constants c, c_1, c_2 such that, for every x,

$$f(\delta(x)) = f(x) + cN_f(x) \text{ and } T_f(\delta(x)) = c_1 T_f(x) + c_2 B_f(x).$$

Now use the Frenet-Serret formulas to conclude that, for every x,

$$c\tau_f(x) = (c_2 \delta'(x) v_f(\delta(x)))/v_f(x).$$

Then either $c_2 = 0$ which implies that τ_f vanishes everywhere or τ_f never vanishes. $\bowtie$

For some important classes of curves the non-triviality of the Bertrand group implies severe restrictions on the curves.

Proposition 3: Let $f : S^1 \to R^3$ be a simple curve. If f is transnormal or spherical and $B(f)$ is non-trivial then f is a plane curve.

Proof: If f is spherical then its torsion τ_f vanishes somewhere. This can be seen by considering $h : S^1 \to R$, given by $h(x) = (B_f(x) \mid f(x) - a)$, where a is the centre of the sphere containing $f(S^1)$. Naturally h has critical points and these are points where the torsion vanishes.

If f is transnormal then it is known [4] that every normal affine plane meets the image of f in exactly two points. If $\delta \in B(f)$ is non-trivial then, for the normal affine plane at $x \in S^1$, those two points are $f(x)$ and $f(\delta(x))$. It follows that $T_f(\delta(x)) = -T_f(x)$. From the proof of Proposition 2 above we see that τ_f vanishes everywhere. $\bowtie$

Also let us consider now a smooth map $f : S^1 \to R$ and define $F : S^1 \to R^3$ by $F(x) = (x, f(x))$. A short calculation shows that the torsion τ_F vanishes somewhere. Therefore we have

Proposition 4: Let $f : S^1 \to R$ be a smooth map and $F : S^1 \to R^3$ be defined by $F(x) = (x, f(x))$. If $B(F)$ is not trivial then f is constant.

We can also use Proposition 2 to establish

Proposition 5: Let $f : S^1 \to R^3$ be a simple curve such that its image is symmetrical with respect to a point $p \notin f(S^1)$. If $B(f)$ is not trivial then $f(S^1)$ is a circle with p as the centre.

Proof: Assume without loss of generality that the centre of symmetry p is the origin O in R^3. Then there is a diffeomorphism $\alpha : S^1 \to S^1$ such that $A \circ f = f \circ \alpha$, where A is the reflection with respect to the origin. Since we have $\tau_f(\alpha(t)) = \tau_{f\circ\alpha}(t) = \tau_{A\circ f}(t) = -\tau_f(t)$ it follows that τ_f vanishes everywhere and f is a plane curve.

For plane curves, as we will mention again below, to say that $B(f)$ is non-trivial is equivalent to say that f is self-parallel. Also, for plane curves, self-parallelism is equivalent to transnormality [1]. The result then follows by Proposition 8, § 4 in [2]. $\bowtie$

If τ_f vanishes everywhere then to say that $B(f)$ is non-trivial is equivalent to say that f is self-parallel and self-parallel curves were studied in [1] and [7]. In the case of embeddings (injective immersions which are homeomorphisms onto the image) $B(f)$ is either Z_2 or trivial according as X is S^1 or R.

2. It is easy to obtain examples of non-injective curves with cyclic or infinite Bertrand groups. Let us concentrate now on simple curves with non-vanishing torsion.

Proposition 1: Let $f : X \to R^3$ be a simple curve with non-vanishing torsion. Then $B(f)$ is trivial if $X = R$ and it is trivial or Z_2 if $X = S^1$.

Proof: Assume that $X = R$. If $B(f)$ is not trivial choose two distinct elements from $B(f) \setminus \{id_R\}$. We have then two distinct and non-zero constants a_1, a_2 such that, for $x \in S^1$,

$$f(\delta_1(x)) = f(x) + a_1 N_f(x),\ f(\delta_2(x)) = f(x) + a_2 N_f(x).$$

If follows [3] that the reparametrizations by arc-length of f are circular helices and we reach a contradiction for the Bertrand group of f would then be trivial.

Similar arguments allow us to conclude that $B(f)$ is either trivial or Z_2 when f is a simple closed curve. We only have to argue replacing δ_1, δ_2 by $\delta, \delta^2 = \delta \circ \delta$, where δ is the generator of $B(f)$. $\bowtie$

Remark: Let $f : S^1 \to R^3$ be a simple curve and assume that $B(f)$ is Z_2. If δ is the generator of $B(f)$ then there are constants c_1, c_2 such that, for $x \in S^1$, we have

$$T_f(\delta(x)) = c_1 T_f(x) + c_2 B_f(x) \ ,$$

$$B_f(\delta(x)) = c_2 T_f(x) - c_1 B_f(x) \ .$$

That is, $T_f(\delta(x))$ and $B_f(\delta(x))$ are obtained from $T_f(x)$ and $B_f(x)$ by a reflection in the vector subspace $< T_f(x), B_f(x) >$. This appears to be the point that has been overlooked in [6] when coming to the conclusion that the vectors $T_f(\delta(x)), T_f(x)$ are linearly dependent.

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