Commutators and Central Extensions in Universal Algebra

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Abstract

We investigate internal groupoids and pseudogroupoids in varieties of universal algebras, and give a new characterization of internal groupoids in congruence modular varieties. We then prove that in any congruence modular variety an algebraically central extension is categorically central. The converse implication being already known, it follows that there is a perfect agreement between these two notions in any congruence modular variety. This theorem extends various partial results in this direction proved, so far, for Ω-groups, for Maltsev varieties and semi-abelian categories.

Keywords: Congruence modular varieties, internal groupoids and pseudogroupoids, connector, central extensions.

Introduction

The main purpose of this paper consists in proving that two independent notions of central extension, the first one arising from the recent categorical theory of central extension [10], the second one arising from the theory of commutators in universal algebra [7], are equivalent in any congruence modular variety.

The first notion of central extension, usually refered to as *categorically central extension*, comes from a general theory considering any exact category $\mathcal{C}$ and any "admissible" subcategory $\mathcal{X}$ of $\mathcal{C}$, an example of this being given by any congruence modular variety $\mathcal{C}$ and any subvariety $\mathcal{X}$. This theory gives a complete classification of the central extensions of an object B in $\mathcal{C}$, which includes, as a very special case, the classical description of central extensions of a perfect group B when $\mathcal{C}$ is the variety of groups and $\mathcal{X}$ is the subvariety of abelian groups.

The second notion we consider is the one of *algebraically central extension*, by which is meant a surjective homomorphism $f: A \to B$ in a congruence modular variety $\mathcal{C}$ with the kernel congruence $Eq(f) = \{(x, y) \in A \times A \mid f(x) = f(y)\}$ contained in the centre $Z(A)$ of A. If we denote by Δ_A and ∇_A the smallest and the largest equivalence relations on A, an extension $f: A \to B$ is then algebraically central if the commutator $[Eq(f), \nabla_A]$ in the universal algebraic sense is trivial: $[Eq(f), \nabla_A] = \Delta_A$.

It was proved in [11] that when $\mathcal{C}$ is any congruence modular variety and $\mathcal{X}$ is its admissible subvariety of abelian algebras in $\mathcal{C}$, which is formed by those algebras A in $\mathcal{C}$ with $[\nabla_A, \nabla_A] = \Delta_A$, then any categorically central extension is algebraically central. Various partial results aiming to establish whether the converse result was also true were proved in [11] for Ω-groups, in [12] for Maltsev varieties, in [4] for semi-abelian categories.

The final step of this unification presented here is simple, once the precise relationship between the internal structures of connectors [3] (called pregroupoids in [15], [16] and [17]) and of

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pseudogroupoids [14] has been made clear. The solution of this problem seems to confirm the viewpoint announced in [16], [13], [14], [5], according to which commutator theory is based on internal categorical structures.

A new characterization of internal groupoids in varieties of universal algebras is also given, which is clearly related to the results of [13]. In the context of congruence modular varieties it can be stated very simply as follows: a reflexive graph

$$X_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

is an internal groupoid if and only if $[Eq(d), Eq(c)] = \Delta_{X_1}$ and $Eq(d) \circ Eq(c) = Eq(c) \circ Eq(d)$.

1 Pseudogroupoids and commutators

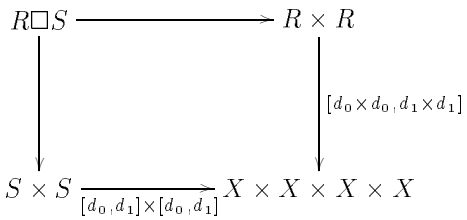
If $\mathcal{C}$ is a variety of universal algebras, we denote by $Con(X)$ the lattice of congruence relations on a given algebra $X \in \mathcal{C}$; if R is a congruence on X we write $q_R \colon X \to \frac{X}{R}$ for the canonical quotient. A variety is congruence modular when the lattice of congruences on any algebra X is modular: this means that, for any $R, S, T \in Con(X)$, $T \leq R$ implies $R \wedge (S \vee T) = (R \wedge S) \vee T$. Among congruence modular varieties there are Maltsev varieties, which have the property that $R \circ S = S \circ R$, and distributive varieties, which have the property that $Con(X)$ is a distributive lattice. During the last twenty years, one of the most interesting aspects of universal algebra has been the development of the theory of commutators, for which the appropriate level of generality seems to be provided by congruence modular varieties (see [7], [8], [9]). We shall adopt here the new approach to commutators as it was developed in [14], and which is based on internal categorical structures. We slightly modify the presentation of the definition of a pseudogroupoid, by emphasizing the role of the congruence relations R and S on a given algebra X, rather than the role of the span

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle q_R}\swarrow & & \searrow{\scriptstyle q_S} & \\ \frac{X}{R} & & & & \frac{X}{S} \end{array}$$

of the quotients. The definition given here below is clearly equivalent to the one given in [14].
From now on $\mathcal{C}$ will denote a fixed variety of universal algebras defined by a set of operators $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \ldots$ (where Ω_n denotes the set of n-ary operators) and a certain set of identities. If R and S are congruence relations on X, an R-S rectangle consists in four elements x, y, t, z in the algebra X with the property that xRy, tRz, xSt and ySz. We shall represent such an R-S rectangle by the square

$$\begin{array}{ccc} x & \overset{S}{\longrightarrow} & t \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ y & \underset{S}{\longrightarrow} & z, \end{array}$$

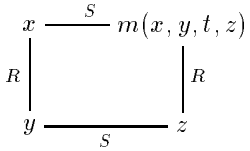
or by a matrix $\begin{pmatrix} x & t \\ y & z \end{pmatrix}$. Let $R\square S$ denote the subset of $X^4 = X \times X \times X \times X$ of all R-S rectangles; it is a subalgebra of X^4, and it can be obtained by the pullback



where d_0 and d_1 represent, respectively, the first and the second projections of the congruence relations R and S. We can then introduce the definition of a pseudogroupoid [14]:

Definition 1.1. *A pseudogroupoid on R and S is a homomorphism $m\colon R\square S \to X$ written as* $m \begin{pmatrix} x & t \\ y & z \end{pmatrix} = m(x,y,t,z)$ *with*

1. $xSm(x,y,t,z)Rz$

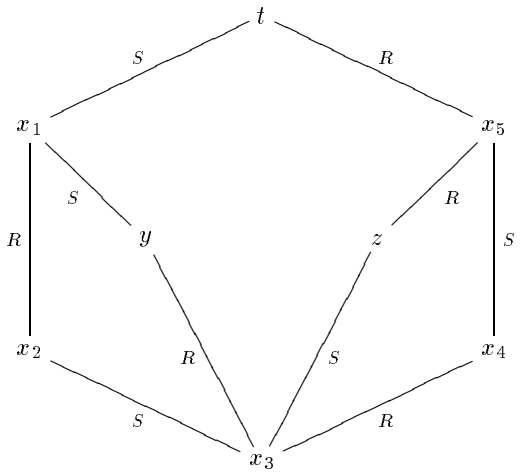


2. *m does not depend on the third variable, i.e. $m(x,y,t,z) = m(x,y,t',z)$ if both sides are defined*

3. $m(x,x,t,z) = z$, $m(x,y,t,y) = x$

4. $m(m(x_1,x_2,y,x_3),x_4,t,x_5) = m(x_1,x_2,t,m(x_3,x_4,z,x_5))$

for every diagram



As shown in [14] if there is a pseudogroupoid m between R and S in a congruence modular variety, then it is unique. Accordingly, for two congruence relations R and S to have a pseudogroupoid structure becomes a property. Moreover, a remarkable fact on pseudogroupoids is the following:

Theorem 1.1. *[14] If $\mathcal{C}$ is a congruence modular variety, then $[R,S] = \Delta_X$ if and only if R and S have a (unique) pseudogroupoid structure.*

Example 1.1. If R and S are two equivalence relations on the same object X, we denote by $R \underset{X}{\times} S$ the pullback

$$\begin{array}{ccc} R \underset{X}{\times} S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X. \end{array}$$

A connector between R and S is an arrow $p : R \underset{X}{\times} S \to X$ such that

1. $xSp(x,y,z)$ 1.* $zRp(x,y,z)$
2. $p(x,x,y) = y$ 2.* $p(x,y,y) = x$
3. $p(x,y,p(z,u,v)) = p(p(x,y,z),u,v)$

Clearly, any connector determines a pseudogroupoid: for any $\begin{pmatrix} x & t \\ y & z \end{pmatrix}$ in $R\square S$ one simply defines $m(x,y,t,z) = p(x,y,z)$.

Example 1.2. Let X be an internal groupoid in a variety $\mathcal{C}$, represented by the diagram

$$X_1 \underset{X_0}{\times} X_1 \underset{\pi_2}{\overset{\pi_1}{\underset{\longrightarrow}{\overset{m}{\longrightarrow}}}} X_1 \underset{c}{\overset{d}{\underset{\longrightarrow}{\overset{e}{\longleftarrow}}}} X_0$$

where X_0 is the algebra of objects, X_1 is the algebra of arrows, $X_1 \underset{X_0}{\times} X_1$ is the algebra of composable arrows, d is the domain, c is the codomain and m is the groupoid composition. It determines a connector, and then a pseudogroupoid, between the kernel congruences $Eq[d]$ and $Eq[c]$. Conversely, a connector between $Eq[d]$ and $Eq[c]$ determines a groupoid structure on any reflexive graph

$$X_1 \underset{c}{\overset{d}{\underset{\longrightarrow}{\overset{e}{\longleftarrow}}}} X_0 .$$

Example 1.3. If R and S are two congruences on an algebra X with the property that $R \cap S = \Delta_X$, then R and S have a unique pseudogroupoid structure: it is given by $m(x,y,t,z) = t$.

The precise relationship between pseudogroupoids and connectors is given in the proposition here below: the equivalence between 2. and 3. holds more generally in any category with finite limits [5].

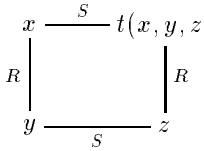
Proposition 1.1. *If $\mathcal{C}$ is a variety and R and S are two congruence relations on an algebra X, then the following conditions are equivalent:*

1. *There is a pseudogroupoid on R and S and $R \circ S = S \circ R$.*
2. *There is a connector between R and S.*
3. *There is a centralizing double relation C on R and S.*

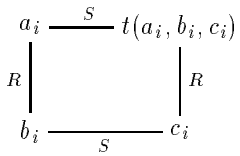
Proof: 1. $\Rightarrow$ 2. Consider the pullback

$$\begin{array}{ccc} R \underset{X}{\times} S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X. \end{array}$$

If $(x,y,z) \in R \underset{X}{\times} S$, then $R \circ S = S \circ R$ implies that there exists $t(x,y,z) \in X$ with



Let us then define the map $p\colon R \underset{X}{\times} S \to X$ as $p(x,y,z) = m(x,y,t(x,y,z),z)$ where $t(x,y,z)$ is any element with $xSt(x,y,z)Rz$. This map is well defined, since, by definition, the pseudogroupoid m does not depend on the third variable. Let us then show that $p\colon R \underset{X}{\times} S \to X$ is a homomorphism: let $\omega \in \Omega_n$ be any n-ary operation and let $(a_1,b_1,c_1),(a_2,b_2,c_2),\ldots,(a_n,b_n,c_n)$ be n elements in $R \underset{X}{\times} S$. Now, for any (a_i,b_i,c_i) there exists $t(a_i,b_i,c_i)$ with



where $i = 1,2,\ldots,n$. It follows that

$$\omega(a_1,a_2,\ldots,a_n)S\omega(t(a_1,b_1,c_1),t(a_2,b_2,c_2),\ldots,t(a_n,b_n,c_n))$$

and

$$\omega(c_1,c_2,\ldots,c_n)R\omega(t(a_1,b_1,c_1),t(a_2,b_2,c_2),\ldots,t(a_n,b_n,c_n)).$$

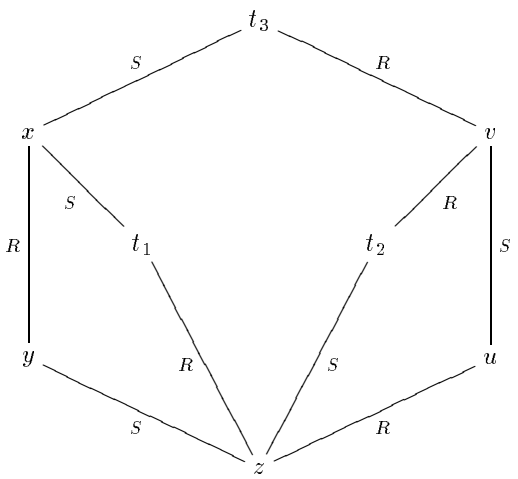
Then

$$\begin{aligned}
& p(\omega(a_1,a_2,\ldots,a_n),\omega(b_1,b_2,\ldots,b_n),\omega(c_1,c_2,\ldots,c_n)) \\
= \;& m(\omega(a_1,a_2,\ldots,a_n),\omega(b_1,b_2,\ldots,b_n),\omega(t(a_1,b_1,c_1),t(a_2,b_2,c_2),\ldots,t(a_n,b_n,c_n)),\omega(c_1,c_2,\ldots,c_n)) \\
= \;& \omega(m(a_1,b_1,t(a_1,b_1,c_1),c_1),m(a_2,b_2,t(a_2,b_2,c_2),c_2),\ldots,m(a_n,b_n,t(a_n,b_n,c_n),c_n) \\
= \;& \omega(p(a_1,b_1,c_1),\ldots,p(a_n,b_n,c_n)).
\end{aligned}$$

The axioms 1., 1.*, 2. and 2.* in the definition of connector easily follow from axioms 1. and 3. in the definition of pseudogroupoid. To check the associativity axiom 3. in the definition of connector, let us consider x,y,z,u,v with $xRySzRuSv$. Since $x(R \circ S)z$, $z(R \circ S)v$ and then $x(R \circ S)v$ by our assumption, there exist t_1,t_2 and t_3 in X with the properties

$$\begin{aligned}
& xSt_1Rz, \\
& zSt_2Rv, \\
& xSt_3Rv.
\end{aligned}$$

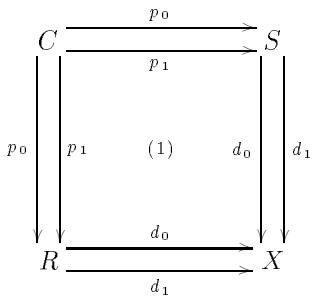
We can then form the diagram



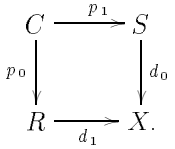
and the associativity of the pseudogroupoid gives

$$\begin{aligned} p(x,y,p(z,u,v)) &= m(x,y,t_3,m(z,u,t_2,v)) \\ &= m(m(x,y,t_1,z),u,t_3,v) \\ &= m(p(x,y,z),u,t_3,v) \\ &= p(p(x,y,z),u,v). \end{aligned}$$

2. $\Rightarrow$ 3. A double equivalence relation C on R and S

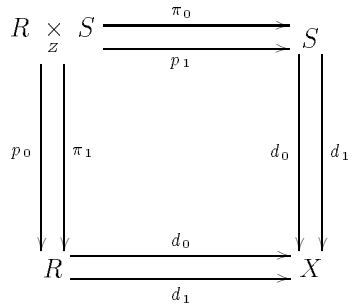


is called a centralizing relation [6] when the following square is a pullback:



If $p\colon R \underset{z}{\times} S \to X$ is a connector between R and S, then by defining $\pi_0(x,y,z) = (x, p(x,y,z))$

and $\pi_1(x,y,z) = (p(x,y,z),z)$ one gets a centralizing relation on R and S:



3. $\Rightarrow$ 1. With the same notations as in diagram (1) above, if C is a centralizing relation on R and S, then the arrow $d_1 \circ p_0 : C = R \underset{X}{\times} S \to S \to X$ defines a connector p between R and S, so in particular a pseudogroupoid. Moreover, for any $xRySz$, the element $p(x,y,z)$ is such that $xSp(x,y,z)Rz$, proving that $R \circ S \leq S \circ R$, and then $R \circ S = S \circ R$. □

Corollary 1.1. *Let*

$$X_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} X_0$$

be a reflexive graph X in a variety $\mathcal{C}$. Then the following conditions are equivalent:

1. *there is a pseudogroupoid on $Eq(d)$ and $Eq(c)$ and $Eq(d) \circ Eq(c) = Eq(c) \circ Eq(d)$*
2. *X has an internal groupoid structure*

Corollary 1.2. *Let $\mathcal{C}$ be a variety. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is Maltsev*
2. *if there is a pseudogroupoid on R and S, then $R \circ S = S \circ R$*

Proof: 1. $\Rightarrow$ 2. is trivial, so let us prove that 2. $\Rightarrow$ 1. Consider a reflexive relation R in $\mathcal{C}$:

$$R \underset{c}{\overset{d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} X.$$

The kernel congruences $Eq(d)$ and $Eq(c)$ have the property that $Eq(d) \wedge Eq(c) = \Delta_R$, so that there is a pseudogroupoid structure on $Eq(d)$ and $Eq(c)$. It follows that $Eq(d) \circ Eq(c) = Eq(c) \circ Eq(d)$ and then there is a connector between $Eq(d)$ and $Eq(c)$. Accordingly, the reflexive relation R is an internal groupoid, and then an equivalence relation. □

2 The unification theorem

In this section we prove that any algebraically central extension in a congruence modular variety is categorically central.

The categorical theory of central extensions developed by Janelidze and Kelly [10] gives a classification therorem describing the category of central extensions in an exact category $\mathcal{C}$, centrality being defined relatively to any "admissible" subcategory $\mathcal{X}$. When $\mathcal{C}$ is a congruence modular variety, the admissibility condition on the subcategory $\mathcal{X}$ was shown to assert that $\mathcal{X}$ is a subvariety of $\mathcal{C}$

$$\mathcal{X} \underset{H}{\overset{I}{\rightleftarrows}} \mathcal{C}.$$

If $H : \mathcal{X} \to \mathcal{C}$ denotes the inclusion functor and $I : \mathcal{C} \to \mathcal{X}$ its left adjoint, we write $\eta_A : A \to HIA$ for the A-component of the unit of this adjunction. An extension of B, i.e. a surjective homomorphism $f : A \to B$ in $\mathcal{C}$, is called a *trivial* extension if the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HIA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HIA} \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

is a pullback in $\mathcal{C}$. An extension $f : A \to B$ is said to be central if there exists a surjective homomorphism $p : E \to B$ such that, in the pullback

$$\begin{array}{ccc} E \underset{B}{\times} A & \xrightarrow{\pi_A} & A \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

the extension $\pi_E : E \underset{B}{\times} A \to E$ is a trivial extension. Following [12] we shall call *categorically central* an extension $f : A \to B$ in a congruence modular variety $\mathcal{C}$ which is central with respect to the subvariety $\mathcal{C}_{Ab}$ of abelian algebras in $\mathcal{C}$ (an algebra A being abelian if the largest commutator on A is trivial: $[\nabla_A, \nabla_A] = \Delta_A$).
On the other hand, there is a well known notion of "centre" of an algebra A in a congruence modular variety: the centre $Z(A)$ of A is the largest congruence relation on A with the property that $[Z(A), \nabla_A] = \Delta_A$. An extension $f : A \to B$ is *algebraically central* if $[Eq(f), \nabla_A] = \Delta_A$.

Theorem 2.1. *If $f : A \to B$ is an algebraically central extension in a congruence modular variety, then it is categorically central.*

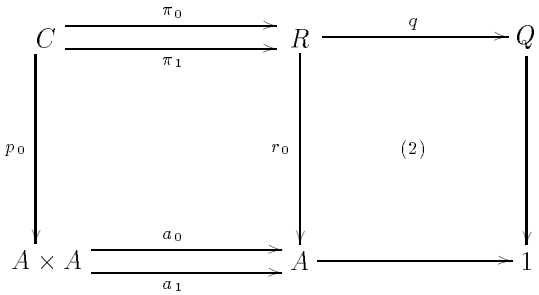
Proof: Let us denote by R the kernel congruence of the algebraically central extension f. First remark that the condition $[R, \nabla_A] = \Delta_A$ implies that there is a centralizing double relation C on R and ∇_A:

$$\begin{array}{ccc} C & \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} & R \\ {\scriptstyle p_0}\downarrow\downarrow{\scriptstyle p_1} & & \downarrow\downarrow{\scriptstyle r_1} \\ A \times A & \overset{a_0}{\underset{a_1}{\rightrightarrows}} & A \end{array}$$

indeed, this follows by Proposition 1.1 and the fact that $R \circ \nabla_A = \nabla_A \circ R$.

Now by taking the coequalizer $q : R \to Q$ of the arrows π_0 and π_1 we obtain the commutative

diagram



By assumption the arrow $(p_0, r_0) : C \to \nabla_A$ is a discrete fibration of internal equivalence relations, so that the square (2) is a pullback (see for instance [2]). The object Q is abelian: indeed

$$\begin{aligned} [Q \times Q, Q \times Q] &= [q(Eq(r_0)), q(Eq(r_1))] \\ &= q[Eq(r_0), Eq(r_1)] \\ &\leq q(Eq(r_0) \wedge Eq(r_1)) \\ &= q(\Delta_R) \\ &= \Delta_Q, \end{aligned}$$

since in modular varieties the commutator preserves regular images (in the categorical sense). This shows that the kernel congruence R of f is isomorphic to

$$Q \times A \underset{r_1}{\overset{\pi_A}{\rightrightarrows}} A \xrightarrow{f} B$$

with Q an abelian algebra in $\mathcal{C}$. There is a factorization $i \circ g : Q \to I \to 1$ of the unique arrow from Q to 1, with g a surjective homomorphism and i a monomorphism. The arrow $g : Q \to I$ clearly lies in $\mathcal{C}_{Ab}$, the algebra I being a subalgebra of 1. There is an induced homomorphism $j : A \to I$ with $\pi_A \circ j = g \circ \pi_Q$. The extension $\pi_A : Q \times A \to A$ appears then as the pullback of the trivial extension g along j, and is then itself trivial. Accordingly, the extension f is categorically central, as desired. □

Let $Ext(B)$ denote the category of extensions of an algebra B and let $Centr(B)$ denote its full subcategory of central extensions of B.

Proposition 2.1. *If $\mathcal{C}$ is a congruence modular variety, then $Centr(B)$ is a reflective subcategory of $Ext(B)$.*

Proof: For any extension $f : A \to B$ there is a canonical quotient $\eta_A : A \to \frac{A}{[Eq(f), \nabla_A]}$. This defines a centralisation functor $J : Ext(B) \to Centr(B)$ associating with $f : A \to B$ the unique extension $f' : \frac{A}{[Eq(f), \nabla_A]} \to B$ with $f' \circ \eta_A = f$. By using the properties of commutators in congruence modular varieties one can see that the functor J is the left adjoint of the inclusion functor $U : Centr(B) \to Ext(B)$. □

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