The Moser-Veselov Equation

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Abstract

We study the orthogonal solutions of the matrix equation $XJ - JX^T = M$, where J is symmetric positive definite and M is skew-symmetric. This equation arises in the discrete version of the dynamics of a rigid body, investigated by Moser and Veselov in [15]. We show connections between orthogonal solutions of this equation and solutions of a certain algebraic Riccati equation. This will bring out the symplectic geometry of the Moser-Veselov equation and also reduces most computational issues about solutions to finding invariant subspaces of a certain Hamiltonian matrix. Necessary and sufficient conditions for the existence of orthogonal solutions (and methods to compute them) are presented. Our method is contrasted with the Moser-Veselov approach presented in [15]. We also exhibit explicit solutions of a particular case of the Moser-Veselov equation, which appears associated with the continuous version of the dynamics of a rigid body.

Key-words: Algebraic Riccati equation, controllability, stability, primary matrix functions

1 Introduction

In [15] Moser and Veselov investigated a discrete version of the Euler-Arnold equation for the motion of the generalized rigid body. The main feature of this discrete model is the connection between a skew-symmetric matrix M, representing the angular momentum and the orthogonal matrix X representing the angular velocity.

In this paper we present a complete study of the matrix equation

$$XJ - JX^T = M, \tag{1}$$

where J is symmetric positive definite and M is skew-symmetric.

After relating the orthogonal solutions of (1) with the symmetric solutions of a certain algebraic Riccati equation, most results follow from this well known theory, applied to our particular situation. Necessary and sufficient conditions for the existence and uniqueness of orthogonal solutions of equation (1) are presented, followed by computational considerations. Our method is also compared with the approach in [15]. In the last section we derive explicit formulas for solutions of equation (1) when $J > 0$ is a scalar matrix. This particular case is associated with the continuous version of the dynamics of a rigid body.

2 The algebraic Riccati equation revisited

The greatest stimulus for investigation of matrix Riccati equations in differential, difference, and algebraic forms, has been the linear quadratic regulator problem of optimal control. In the next

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section it will become clear the strong connection between orthogonal solutions of the Moser-Veselov equation (1 and symmetric solutions of a particular algebraic Riccati equation, so that a complete study of (1) follows from this well known theory.

We recall in this section some results which will be crucial throughout the paper. There is a vast literature on the subjet, but we refer to Lancaster and Rodman [12], [13] and Kucera [11], for more details about the theory of Riccati equations.

Consider an algebraic matrix Riccati equation of the form

$$SDS + SA + A^T S - C = 0, \tag{2}$$

where D, A and C are $n \times n$ given real matrices with $D \geq 0$ (non-negative definite) and $C = C^T$. As usual, associate to equation (2) the following Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A & D \\ C & -A^T \end{bmatrix}. \tag{3}$$

We are particularly interested in real symmetric solutions of the matrix Riccati equation. The following theorems give necessary and sufficient conditions for the existence and uniqueness of symmetric solutions of (2).

Theorem 2.1 [12]*If (A, D) is controllable, then the statements* (i)–(iv) *are equivalent:*

(i) There exists a solution S of (2) such that $S = S^T$;

(ii) There exists a solution S_1 of (2) such that $S_1 = S_1^T$ and Re $\lambda \leq 0$ for every $\lambda \in \sigma(DS_1 + A)$;

(iii) There exists a solution S_2 of (2) such that $S_2 = S_2^T$ and Re $\lambda \geq 0$ for every $\lambda \in \sigma(DS_2 + A)$;

(iv) The size of the Jordan blocks of $\mathcal{H}$, associated to the pure imaginary eigenvalues is even.

Moreover, if any of the statements (i)–(iv) *holds, then*

(v) The solution S_1 of (2) with the properties from (ii) *is unique;*

(vi) The solution S_2 of (2) with the properties from (iii) *is unique.*

Theorem 2.2 [12] *If (A, D) is controllable then (2) has a unique symmetric solution if and only if all the eigenvalues of $\mathcal{H}$ are pure imaginary and the associated Jordan blocks have even size.*

Now let $\mathcal{C}$ be any subset of eigenvalues of $\mathcal{H}$ with nonzero real parts, which satisfies the following properties:

$$\begin{array}{ll} P1) & \lambda \in \mathcal{C} \;\Rightarrow\; \bar{\lambda} \in \mathcal{C} \text{ and } -\bar{\lambda} \notin \mathcal{C}, \\[1ex] P2) & \mathcal{C} \text{ is maximal with respect to property } P1). \end{array} \tag{4}$$

Theorem 2.3 [13] *If (2) has symmetric solutions, then for every set $\mathcal{C}$ given as above, there exists a unique symmetric solution S such that $\mathcal{C}$ is exactly the set of eigenvalues of $A+DS$ having nonzero real parts.*

Recall that a pair (A, B) of $n \times n$ matrices is said to be *stabilizable* if there exists a matrix K such that $A + BK$ is stable, that is, $\sigma(A + BK)$ lies on the open left half plane. (A, B) is said to be *detectable* if (B^T, A^T) is stabilizable.

An important necessary and sufficient condition for existence and uniqueness of a non-negative definite solution of (2) is given in the following theorem.

Theorem 2.4 [11] *Assume that $C \geq 0$, $D \geq 0$, $C = EE^T$ with rank(E) = rank(C) and $D = FF^T$ with rank(F) = rank(D). Then (2) has a unique solution $S \geq 0$ such that the matrix $-(A + DS)$ is stable if and only if (A, F) is stabilizable and (E, A) is detectable.*

Parece-me que este outro teorema é mais importante para o que se segue (ver teorema 8.1.11 no livro do Rodman).

Theorem 2.5 [11] *Assume that $C \geq 0$, $D \geq 0$, (A, F) is stabilizable and (E, A) is detectable. Then $\mathcal{H}$ has no pure imaginary (or zero) eigenvalues and equation (2) has a unique symmetric solution $S > 0$ such that the matrix $(A + DS)$ is stable.*

It is also widely known that, under the assumptions of the previous theorem, if $\lambda_1, \cdots \lambda_n$ are the eigenvalues of $\mathcal{H}$ in the open left half plane, the unique positive definite solution $S_\star$ of (2) is given by $S_\star = ZY^{-1}$, where Y, Z are $n \times n$ real matrices and the columns of $\begin{bmatrix} Y \\ Z \end{bmatrix}$ span the invariant subspace of $\mathcal{H}$ associated with $\lambda_1, \cdots \lambda_n$.

3 Existence and uniqueness theorems

We start with a simple observation about general solutions of (1).

Lemma 3.1 *Every solution of (1) (not necessarily orthogonal) can be written in the form*

$$X = (M/2 + S)J^{-1},$$

where S is symmetric.

Proof. If X is a solution of (1), then there exists a symmetric matrix S such that

$$XJ + (XJ)^T = 2S.$$

Adding this equation with (1) it follows immediatly that $X = (M/2 + S)J^{-1}$, with S symmetric. ∎

The connection between orthogonal solutions of (1) and symmetric solutions of an algebraic matrix Riccati equation is given in the following result.

Lemma 3.2 *Every orthogonal solution of (1) can be written in the form $X = (M/2+S)J^{-1}$, where S is a symmetric matrix satisfying*

$$S^2 + S(M/2) + (M/2)^T S - (M^2/4 + J^2) = 0. \tag{5}$$

Proof. By the previous lemma, $X = (M/2+S)J^{-1}$, with S symmetric.Using the requirement that X is orthogonal, we may write

$$\begin{array}{rrl} & \left((M/2+S)J^{-1}\right)^T (M/2+S)J^{-1} & = I \\ \Leftrightarrow & J^{-1}(S - M/2)(S + M/2)J^{-1} & = I \\ \Leftrightarrow & S^2 + S(M/2) - (M/2)S - M^2/4 & = J^2 \\ \Leftrightarrow & S^2 + S(M/2) + (M/2)^T S - (M^2/4 + J^2) & = 0. \end{array}$$

∎

The next theorem, which is an immediate consequence of the previous lemmas, allows the analysis of (1) using the theory of algebraic Riccati equations.

Theorem 3.3 *$X = (M/2 + S)J^{-1}$ is an orthogonal solution of (1) if and only if S is a symmetric solution of the algebraic Riccati equation (5).*

Remark 3.4 Introducing $W = XJ$, the equation (1) reduces to $W - W^T = M$, with the additional constraint $W^T W = J^2$. Note that when X is orthogonal, $W = XJ$ is the polar decomposition of W. So, solving (1) is equivalent to finding the orthogonal factor in the polar decomposition of a matrix, knowing its skewsymmetric part only. Clearly, in general, there are infinitely many solutions.

Remark 3.5 The equation (5) may be written in the form

$$(S + M/2)^T (S + M/2) = J^2,$$

which, in particular, implies that $0 \notin \sigma(S + M/2)$. This agrees with the form of the orthogonal solutions in lemma 3.1.

Remark 3.6 Equation (5) is a matrix Riccati equation of the form (2), with $D = I$ (non-negative definite), $C = M^2/4 + J^2$ (symmetric), and $A = M/2$. In this case the pair (A, D) is always controllable and the associated Hamiltonian matrix is now

$$\overline{\mathcal{H}} = \begin{bmatrix} M/2 & I \\ M^2/4 + J^2 & M/2 \end{bmatrix}.$$

Remark 3.7 A simple calculation shows that if $P = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix}$ and S is a solution of (1), then

$$P^{-1}\overline{\mathcal{H}}P = \begin{bmatrix} M/2 + S & I \\ 0 & -(S + M/2)^T \end{bmatrix}, \tag{6}$$

which implies that

$$\sigma(\overline{\mathcal{H}}) = \sigma(S + M/2) \cup \sigma(-(S + M/2)^T) \tag{7}$$

and consequently,

$$\sigma(\overline{\mathcal{H}}) \cap i\mathbb{R} = \emptyset \Longleftrightarrow \sigma(S + M/2) \cap i\mathbb{R} = \emptyset$$

$$\sigma(\overline{\mathcal{H}}) \subset i\mathbb{R} \Longleftrightarrow \sigma(S + M/2) \subset i\mathbb{R}.$$

Theorem 3.8 *(Existence) There exists a solution $X \in SO(n)$ for (1) if and only if the size of the Jordan blocks associated to the pure imaginary eigenvalues of $\overline{\mathcal{H}}$ (if any) is even.*

Proof. ($\Rightarrow$) Since the pair $(M/2, I)$ is controllable, the result follows by direct application of the previous theorem and the equivalence between the statements (i) and (iv) in theorem 2.1.
($\Leftarrow$) If the size of the Jordan blocks associated to the pure imaginary eigenvalues of $\overline{\mathcal{H}}$ is even, then, according to theorem 2.1, there exists a symmetric solution S of (5) such that Re $\lambda \geq 0$, for every $\lambda \in \sigma(M/2 + S)$. But this implies that $\det(S + M/2) > 0$ and, consequently, the orthogonal solution of (1), $X = (M/2 + S)J^{-1}$, has determinant equal to 1.

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An immediate consequence of the results in this section and theorem 2.2 is the following.

Corollary 3.9 *(Uniqueness) The matrix equation (1) has a unique solution $X \in SO(n)$ if and only if the spectrum of $\overline{\mathcal{H}}$ is pure imaginary and the size of the Jordan blocks associated to each (nonzero) eigenvalue is even.*

Since there is a bijection between orthogonal solutions of (1) and symmetric solutions of (5), we may use theorem 8.4.3 in [13] to decide whether (1) has a finite or infinite number of orthogonal solutions. For the sake of completeness, we summarize that result below, after introducing some notation. For every $\lambda \in \sigma(\overline{\mathcal{H}})$, let $R_\lambda(\overline{\mathcal{H}})$ denote the corresponding generalized eigenspace and $m_g(\lambda)$ the geometric multiplicity of λ.

Theorem 3.10 *The number of orthogonal solutions of (1) is finite if and only if $m_g(\lambda) = 1$, $\forall \lambda \in \sigma(\overline{\mathcal{H}})$. In this case, the number of orthogonal solutions is given by*

$$\left(\prod_{j=1}^{p} \left(dim\, R_{\lambda_j}(\overline{\mathcal{H}}) + 1\right)\right) \cdot \left(\prod_{j=1}^{q} \left(\frac{1}{2} dim\, R_{\alpha_j \pm i\beta_j}(\overline{\mathcal{H}}) + 1\right)\right),$$

where $\lambda_1, \cdots, \lambda_p$ are all the distint real positive eigenvalues of $\overline{\mathcal{H}}$, and $\alpha_1 \pm i\beta_1, \cdots, \alpha_q \pm i\beta_q$ are all the distincts pairs of complex conjugate eigenvalues of $\overline{\mathcal{H}}$ having positive real parts.

Remark 3.11 For the particular case when $\overline{\mathcal{H}}$ has distinct eigenvalues, (1) has a finite (even) number of orthogonal solutions, half of which are special orthogonal. This follows from (7).

Definition 3.12 *We say that a subset $\Sigma \subset \mathbb{C} \setminus i\mathbb{R}$ admits a* good splitting *if $\Sigma = \Sigma_+ \cup \Sigma_-$, where $\Sigma_+ \cap \Sigma_- = \emptyset$ and both Σ_+ and Σ_- satisfy the properties $P1)$ and $P2)$ in (4).*

Now assume that $\sigma(\overline{\mathcal{H}}) \cap i\mathbb{R} = \emptyset$. Under this assumption, the spectrum of $\overline{\mathcal{H}}$ always admits a good splitting. An example of such a splitting consists in considering Σ_- the set of all eigenvalues of $\overline{\mathcal{H}}$ in the open left half plane and Σ_+ the set of all eigenvalues in the open right half plane.

Theorem 3.13 *If $\overline{\mathcal{H}}$ has no pure imaginary eigenvalues, then for any good splitting of the spectrum of $\overline{\mathcal{H}}$, $\sigma(\overline{\mathcal{H}}) = \Sigma_+ \cup \Sigma_-$, there exists a unique orthogonal solution X of (1) with $\sigma(XJ) = \Sigma_+$.*

Proof. Under this spectral assumption, the equivalence between (i) and (iv) in theorem 2.1 guarantees the existence of symmetric solutions of (5). And theorem 2.3 ensures the existence of a unique symmetric solution S of (5) such that $\sigma(M/2+S) = \Sigma_+$. Since $M/2+S = XJ$, the result follows.

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4 Comparison with the Moser-Veselov approach

In order to contrast our method with the Moser-Veselov approach in [15], we outline here their main steps. The first observation is that if X is orthogonal, equation (1) is equivalent to the quadratic matrix equation

$$W^2 - MW - J^2, \tag{8}$$

with the additional condition $W^T W = J^2$. Clearly, the relationship between solutions of equations (1) and (8) is $X = WJ^{-1}$. Now, if λ is an eigenvalue of W, it follows from (8) that

$$\det(\lambda^2 I - \lambda M - J^2) = 0. \tag{9}$$

The main result in Moser and Veselov [15], concerning solutions of equation (1), is theorem $1'$ in page 228, which we may state in the following way.

Theorem $1'$ - *If the set Σ of the roots of equation (9) does not intersect the imaginary axis $i\mathbb{R}$, then for any good splitting $\Sigma = \Sigma_+ \cup \Sigma_-$, there exists a unique solution W of (8)(and therefore a unique solution of (1)) with $\sigma W = \Sigma_+$.*

Two observations have to be made in order to see the connection between this theorem and theorem 3.13. First note that the characteristic polynomial of the matrix

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ J^2 & M \end{bmatrix},$$

used in [15] to prove theorem $1'$, is equal to $\det(\lambda^2 I - \lambda M - J^2)$. Secondly, the matrix $\mathcal{A}$ is similar to the Hamiltonian matrix $\overline{\mathcal{H}}$. Indeed, the matrix $P = \begin{bmatrix} I & 0 \\ M/2 & I \end{bmatrix}$ is such that $P\overline{\mathcal{H}}P^{-1} = \mathcal{A}$. So, theorem $1'$ in [15] is just a restatement of theorem 3.13.

5 Computation of special orthogonal solutions

Based on our analysis, the computation of orthogonal or special orthogonal solutions of (1) depends on the computation of symmetric solutions of the associated algebraic Riccati equation. There are several numerical methods for this purpose, one of the most reliable being the Schur method [14], [9], which is stable. Other reliable methods include the Newton method [10], the method of the matrix sign function [1] and symplectic QR-like methods [1],[3], [17], which take into account the particular structure of Hamiltonian matrices and are consequently less costly.

In general, these methods compute the unique non-negative definite solution (provided it exists) for the algebraic Riccati equation and require that $\mathcal{M}$ has no imaginary eigenvalues. Under some assumptions, there are several sufficient and necessary/sufficient conditions that guarantee the existence of such a solution [11]. In our case, it is enough to assume that $M^2/4 + J^2 > 0$ to conclude that (5) has a unique non-negative definite solution that corresponds to a special orthogonal solution of (1).

Theorem 5.1 *Suppose that $M^2/4 + J^2$ is positive definite. Then:*

(i) The equation (5) has a unique solution $S \geq 0$ such that the eigenvalues of $S + M/2$ have positive real parts.

(ii) $X = (S + M/2)J^{-1}$, where S is the matrix of the previous (i), *is special orthogonal.*

Proof.

(i) Since $M^2/4 + J^2$ is positive definite, there exists an invertible matrix E such that $M^2/4 + J^2 = E^T E$. We will show that $(M/2, I)$ is stabilizable and $(E, M/2)$ is detectable, so that part (i) follows from theorem 2.4. Indeed, since the eigenvalues of $\pm M/2 - I$ are of the form $-1 \pm \alpha i$ $(\alpha \in \mathbb{R})$, they always have negative real part. Therefore both $(M/2, I)$ and $(-M/2, E^T)$ are stabilizable, that is, $(M/2, I)$ is stabilizable and $(E, M/2)$ is detectable.

(ii) Immediate consequence of (i).

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By the previous theorem together with (7) it is enough to assume that $M^2/4 + J^2 > 0$ to ensure that the unique non-negative definite solution of (5) can be computed and that $\mathcal{M}$ has no pure imaginary eigenvalues.

We now suppose that $M^2/4 + J^2$ is non-negative definite instead of positive definite as above. In this case, $\overline{\mathcal{H}}$ may have pure imaginary eigenvalues, which occur, for example, when $J = I$ and $\overline{\mathcal{H}} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ in (1). However, the existence of a special orthogonal solution is guaranteed by the next theorem.

Theorem 5.2 *If $M^2/4 + J^2 \geq 0$, then equation (1) admits a special orthogonal solution.*

Proof. By theorem 1.5, there exists a solution $S \geq 0$ for equation (2). It remains to show that the real eigenvalues of $M/2 + S$ are positive. In fact, if J is a real eigenvalue of such a matrix, then it admits an associated real eigenvector u. Since

$$0 \leq u^T S u = u^T (M/2 + S) u = u^T \lambda u = \lambda u u^T$$

it follows that $\lambda = (u^T S u)/(u^T u) \geq 0$. Since $M/2 + S$ is invertible, we have $\lambda \neq 0$, and therefore $\lambda > 0$.

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For the sake of completeness we outline the main steps of an algorithm to compute the special orthogonal matrix $X = (S + M/2)J^{-1}$ using the Schur method of [14].

Algorithm
Assume that $M^2/4 + J^2$ is positive definite. This algorithm computes a special orthogonal solution X of (1).

1. Find a real Schur form of $\overline{\mathcal{H}}$,

$$R^T \overline{\mathcal{H}} R = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

   such that the real parts of the spectrum of H_{11} are negative and the real parts of the spectrum of H_{22} are positive, and partition R conformably into four blocks

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

2. Compute $S = R_{21} R_{11}^{-1}$. (R_{11} is invertible and $S^T = S \geq 0$ is a solution of (5)).

3. Compute $X = (S + M/2)J^{-1}$.

6 A particular case of the Moser-Veselov equation

While equation (1) is associated to the discrete model of the dynamics of a generalized rigid body, the continuous model is associated to the following algebraic matrix equation

$$YQ^T - QY^T = M, \tag{10}$$

where Q orthogonal and M skewsymmetric are given. The objective now is to find orthogonal solutions of (10). Introducing $X = YQ^T$, this equation reduces to

$$X - X^T = M, \tag{11}$$

which is a particular case of equation (1), with $J = I$. So, the previous analysis may be used. However, in this case important simplification occur which leads to explicit formulas for the corresponding solutions. Also in this case, there exists a necessary and sufficient condition for the existence of orthogonal solutions of (11), in terms of the spectrum of M.

Theorem 6.1 *Every orthogonal solution of (11) can be written in the form*

$$X = \frac{M}{2} + \left(\frac{M^2}{4} + I\right)^{1/2},$$

where $\left(\frac{M^2}{4} + I\right)^{1/2}$ *is a symmetric square root that commutes with M.*

Proof. We first note that, if X is an orthogonal solution of (11), then X commutes with M. Indeed, $MX = (X - X^T)X = X^2 - I$ and $XM = X(X - X^T) = X^2 - I$. Using theorem 3.3 we know that if X is an orthogonal solution of (11), then $X = S + M/2$, where S is a symmetric solutions of the Riccati equation

$$S^2 + S(M/2) + (M/2)^T - (M^2/4 + I) = 0.$$

But since X commutes with M, also S commutes with M and the previous equation reduces to

$$S^2 - (M^2/4 + I) = 0.$$

This implies the result.

■

Theorem 6.2 *The matrix equation (11) has orthogonal solutions if and only if* $\sigma(M) \subset [-2i, 2i]$.

Proof: ($\Rightarrow$) This follows from the canonical real Jordan forms of skew-symmetric matrices and orthogonal matrices. (See Horn and Johnson [8] for details). Indeed, given M skew-symmetric, there exists V orthogonal such that

$$M = V \operatorname{diag}(M_1, \cdots, M_k, \mu_1, \cdots, \mu_s)\, V^T,$$

where, $M_j = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}$, $\alpha_j \in \mathbb{R}^+$, $j = 1, \cdots, k$, and $\mu_i = 0$, $i = 1, \cdots, s$. Also, if X is an orthogonal solution of (11), then there exists U orthogonal such that

$$X = U \operatorname{diag}(X_1, \cdots, X_k, \lambda_1, \cdots, \lambda_s)\, U^T,$$

where , $X_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$, $\theta_j \in \mathbb{R}$, $j = 1, \cdots, k$, and $\lambda_i = \pm 1$, $i = 1, \cdots, s$. Replacing this expressions of M and X in (11), a simple calculation shows that $\sigma(M) \subset [-2i, 2i]$.

($\Leftarrow$) We will show that, if $\sigma(M) \subset [-2i, 2i]$, then there exists a square root of $\frac{M^2}{4} + I$ such that $X = \frac{M}{2} + \left(\frac{M^2}{4} + I\right)^{1/2}$ is special orthogonal. Indeed, under this spectral condition on M, the principal matrix square root, i.e. the unique square root with eigenvalues in the open right half plane, here denoted by $\sqrt{\frac{M^2}{4}+I}$, is well defined, and is skew-symmetric. Besides, $Y=\frac{M}{2}+\sqrt{\frac{M^2}{4}+I}$ is special orthogonal since its eigenvalues are of the form $\lambda=\frac{\alpha}{2}i\pm\sqrt{1-\alpha^2/4}$, where $\alpha i\in\sigma(M)$.

■

As far as we know, the matrix equation (10) appeared first in Bloch and Crouch [2], associated to the dynamics of the generalized rigid body. Special orthogonal solutions of (10) were given in terms of the inverse matrix hyperbolic sine. Numerical considerations about computing such solutions were presented in Cardoso and Silva Leite [4]. We now contrast the results presented above with those obtained previously.

Assume that X is a special orthogonal solution of (11). Then, there exists a skew-symmetric matrix A such that $X=e^A$. Replacing in (11) we obtain

$$e^A-(e^A)^T=M\Leftrightarrow\frac{e^A-e^{-A}}{2}=\frac{M}{2}\Leftrightarrow A=\sinh^{-1}(M/2).$$

Now we have to use a result proved in Cardoso and Silva Leite [4] which states the following:

> If C is skew-symmetric, then the matrix equation $\sinh X=C$ has a skew-symmetric solution if and only if $\sigma(C)\subset\{\alpha i:\,-1\leq\alpha\leq1\}$.

Clearly this agrees with the spectral condition in theorem 6.2. We have also proved that, in this case, $Sinh^{-1}(M/2)$, where $Sinh^{-1}$ denotes the principal inverse hyperbolic sine, is a well defined primary matrix function given by

$$Sinh^{-1}(M/2)=Log\left(\frac{M}{2}+\sqrt{\frac{M^2}{4}+I}\right),$$

where Log stands for the principal matrix logarithm. (See Horn and Johnson [7] for more details about primary matrix functions).

So, a special orthogonal solution of (11) may be written as

$$X=e^{Sinh^{-1}(M/2)}=e^{Log\left(\frac{M}{2}+\sqrt{\frac{M^2}{4}+I}\right)}=\frac{M}{2}+\sqrt{\frac{M^2}{4}+I}$$

which agrees with the construction in the proof of the theorem 6.2. However, the approach in [4], which is based on the assumption that $Y=e^A$, for some skew-symmetric matrix A, is less general than the method presented here.

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