Asymptotic Model of a Nonlinear Adaptive Elastic Rod

Isabel Figueiredo * Luis Trabucho †

Abstract - In this paper we apply the asymptotic expansion method to obtain a nonlinear adaptive elastic rod model. We first consider the model derived in [2, 3] with the modifications proposed in [5], with a remodeling rate equation depending nonlinearly on the strain field and for a thin rod whose cross section is a function of a small parameter. Based on the asymptotic expansion method for the elastic case [6], we prove that, when the small parameter tends to zero the solution of the nonlinear adaptive elastic rod model converges to the leading term of its asymptotic expansion. Moreover, we show that this term is also the solution of a well-known simplified adaptive elastic model, with generalized Bernoulli-Navier equilibrium equations and a remodeling rate equation whose driving mechanism is the strain energy per unit volume, in good agreement with some of the models used in practice.

1 Introduction

In [2, 3] a theory of adaptive elasticity is proposed for the mathematical modeling of the physiological process of bone remodeling. This theory constitutes a generalization of linearized elasticity. The basic system of equations governing the adaptive elasticity model couples the equilibrium and constitutive equations with an ordinary differential equation with respect to time, which is the remodeling rate equation. This latter equation mathematically expresses the process of absorption and deposition of bone material due to an external stimulus and is a consequence of mass conservation. The unknowns of the model are the displacement vector field of the body and the change in volume fraction of the elastic material, which is a scalar vector field. These two unknowns are coupled in the model, because the material coefficients depend on the change in volume fraction and the remodeling rate equation depends on the displacement vector field. In particular, the displacement vector field is the solution of the equilibrium equations and the change in volume fraction is the solution of the remodeling rate equation.

In this paper we begin by considering the model derived in [2, 3], but with the modification proposed in [5], in such a way that the remodeling rate equation is nonlinear on strain, whereas the elastic stress is linear on strain and the body is a thin elastic rod, whose cross section area depends on a small parameter. The justification of these choices regarding the modeling, the type of equations and the geometry of the body are next described. We adopt in our model the modification proposed in [5] (for the case of a remodeling rate equation linear in strain and a generic geometry for the body) in order to be able to prove the existence of solution. The nonlinear and linear dependence on strain, by the remodeling rate equation and by the elastic stress, respectively, is a consequence of experiments and clinical observations as pointed out in [3]. Finally, we choose the specific geometry described before, because one of the main objectives of the paper is to investigate the behaviour of the adaptive elasticity model and the corresponding solution, when the geometric small parameter tends to zero. To this end we apply the asymptotic expansion method described in [6].

Based on the elastic case [6], we assume that the unknown displacement vector field admits an expansion of positive powers of the geometric small parameter. This assumption immediately induces that

*Corresponding author – Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal (e-mail: Isabel.Figueiredo@mat.uc.pt).

†Departamento de Matemática, Universidade de Lisboa, Edifício C1, Piso 3, Campo Grande 1749-016, Lisboa, Portugal (e-mail: trabucho@lmc.fc.ul.pt).

the other unknown of the model, that is the change in volume fraction of elastic material possesses an asymptotic expansion of the same type. This fact is a straightforward consequence of the dependence of the remodeling rate equation on the displacement vector field and also a consequence of an additional condition (physically justified) imposed on the type of dependence of the material coefficients on the change in volume fraction.

The main results of the paper show that the displacement vector field and the change in volume fraction converge to the first terms of the corresponding asymptotic expansions. Moreover, the limit model is a simplified adaptive elasticity model, with generalized Bernoulli-Navier equilibrium equations and a remodeling rate equation whose driving mechanism is the strain energy per unit volume, in good agreement with some of the most popular models used in practice.

The structure of the paper is organized as follows. In section 2 we describe the model for the thin elastic rod and state an existence and uniqueness result. In section 3 we describe the main results of the paper. We introduce some hypotheses and redefine the problem in a set independent of the small parameter in subsection 3.1. Next, in subsections 3.2 and 3.3 we identify the first term of the asymptotic expansion of the displacement vector field and the first three terms of the asymptotic expansion of the change in volume fraction. In subsection 3.4 we indicate an existence and uniqueness result for the limit problem. The strong convergences of the two asymptotic expansions to the corresponding leading terms is proved in subsection 3.5. In subsection 3.6 we show that all the previous results can be particularized for the case of an adaptive elasticity model, with a remodeling rate equation depending linearly on the strain field, and we also derive other weak convergence results for this linear case. Finally, some conclusions are presented in the last section of the paper.

2 Description of the Problem

In this section we begin by introducing the principal notations used in the paper. Next we formulate the three-dimensional nonlinear adaptive elastic rod problem. The formulation is based on the model derived in [2, 3] for adaptive elasticity, with the same modifications proposed in [5]. The existence and uniqueness of solution is stated in theorem 2.1.

2.1 Notations

Let ω be an open, bounded and connected subset of R^2, and $0 < \varepsilon \leq 1$ a small parameter. We denote by $\overline{\Omega}^\varepsilon$ the set occupied by a cylindrical adaptive elastic rod, in its reference configuration, with length $L > 0$ and cross section $\omega^\varepsilon = \varepsilon\omega$, that is

$$\overline{\Omega}^\varepsilon = \omega^\varepsilon \times [0, L] \subset R^3. \tag{1}$$

Moreover we suppose that ε is very small with respect to L. We also introduce the following sets, which represent several regions of the boundary of Ω^ε:

$$\Gamma^\varepsilon = \partial\omega^\varepsilon \times (0, L), \quad \Gamma_0^\varepsilon = \overline{\omega}^\varepsilon \times \{0\}, \quad \Gamma_L^\varepsilon = \overline{\omega}^\varepsilon \times \{L\}, \tag{2}$$

where $\partial\omega^\varepsilon$ is the boundary of ω^ε. An arbitrary point of Ω^ε is denoted by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$ and the outward unit normal vector to the boundary by $n^\varepsilon = (n_1^\varepsilon, n_2^\varepsilon, n_3^\varepsilon)$. We assume that the coordinate system $(O, x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$ is a principal system of inertia associated with the rod Ω^ε. Consequently, axis Ox_3^ε passes through the centroid of each section $\omega^\varepsilon \times \{x_3^\varepsilon\}$ and we have

$$\int_{\omega^\varepsilon} x_\alpha^\varepsilon \, d\omega^\varepsilon = \int_{\omega^\varepsilon} x_1^\varepsilon x_2^\varepsilon \, d\omega^\varepsilon = 0. \tag{3}$$

The set $C^m(\overline{\Omega}^\varepsilon)$ stands for the space of real functions m times continuously differentiable in $\overline{\Omega}^\varepsilon$. The spaces $W^{m,q}(\Omega^\varepsilon)$, $W^{0,q}(\Omega^\varepsilon) = L^q(\Omega^\varepsilon)$ are the usual Sobolev spaces, where q is a real number satisfying

$1 \le q \le \infty$ and m is a positive integer. The norms in these Sobolev spaces are denoted by $\|.\|_{W^{m,q}(\Omega^\varepsilon)}$. The set

$$\mathcal{R}^\varepsilon = \{v^\varepsilon \in R^3 : \ v^\varepsilon = a + b \wedge x^\varepsilon, \ \ a, b \in R^3\} \tag{4}$$

where $\wedge$ is the cross product in R^3, is the set of infinitesimal rigid displacements. We denote by $[W^{m,q}(\Omega^\varepsilon)]^3 \setminus \mathcal{R}^\varepsilon$ the quotient space induced by the set $\mathcal{R}^\varepsilon$ in the Sobolev space $[W^{m,q}(\Omega^\varepsilon)]^3$.

Throughout the paper, the latin indices i, j, k, l... belong to the set $\{1,2,3\}$, the greek indices α, β, μ... vary in the set $\{1,2\}$ and the summation convention with respect to repeated indices is employed, that is, $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

Let $T > 0$ be a real parameter and denote by t the time variable in the interval $[0,T]$. If V is a topological vectorial space, the set $C^m([0,T];V)$ is the space of functions $g : t \in [0,T] \to g(t) \in V$, such that g is m times continuously differentiable with respect to t. If V is a Banach space we denote $\|.\|_{C^m([0,T];V)}$ the usual norm in $C^m([0,T];V)$. Moreover, given a function $g(x^\varepsilon, t)$ defined in $\overline{\Omega}^\varepsilon \times [0,T]$ we denote by $\dot{g}$ its partial derivative with respect to time, by $\partial_i^\varepsilon g$ its partial derivative with respect to x_i^ε, that is $\partial_i^\varepsilon g = \frac{\partial g}{\partial x_i^\varepsilon}$, and by $\partial_n^\varepsilon g$ its normal derivative, that is $\partial_n^\varepsilon g = n_i^\varepsilon \partial_i^\varepsilon g$.

2.2 Setting of the Problem

The three-dimensional problem, for the nonlinear adaptive elastic rod under consideration, is a system of coupled equations formed by the equilibrium equations, the constitutive equations and the remodeling rate equation. This system is stated as follows.

Find $\ u^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \times [0,T] \to R^3, \quad d^\varepsilon : \overline{\Omega}^\varepsilon \times [0,T] \to R \ $ such that :

Equilibrium Equations

$$-\partial_j^\varepsilon \sigma_{ij}^\varepsilon = \gamma(\xi_0 + \mathcal{P}_\eta(d^\varepsilon)) f_i^\varepsilon, \qquad \text{in} \quad \Omega^\varepsilon \times (0,T), \tag{5}$$

Constitutive Equations

$$\left\{ \begin{array}{l} \sigma_{ij}^\varepsilon = c_{ijkl}^\varepsilon(d^\varepsilon) e_{kl}^\varepsilon(u^\varepsilon), \\ c_{ijkl}^\varepsilon(d^\varepsilon) = (\xi_0 + M_\rho \circ \mathcal{P}_\eta(d^\varepsilon)) a_{ijkl}^\varepsilon(M_\rho \circ \mathcal{P}_\eta(d^\varepsilon)), \\ e_{ij}^\varepsilon = \frac{1}{2}(\partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_j^\varepsilon), \end{array} \right. \qquad \text{in} \quad \Omega^\varepsilon \times (0,T), \tag{6}$$

Boundary Conditions (Pure Traction)

$$\left\{ \begin{array}{lll} \sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon, & \text{in} & \Gamma^\varepsilon \times (0,T), \\ \sigma_{ij}^\varepsilon n_j^\varepsilon = h_i^\varepsilon, & \text{in} & (\Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon) \times (0,T), \end{array} \right. \tag{7}$$

Remodeling Rate Equation

$$\dot{d}^\varepsilon = a^\varepsilon(d^\varepsilon) + A_{kl}^\varepsilon(d^\varepsilon) e_{kl}^\varepsilon(u^\varepsilon) + B_{ijkl}^\varepsilon(d^\varepsilon) e_{ij}^\varepsilon(u^\varepsilon) e_{kl}^\varepsilon(u^\varepsilon), \qquad \text{in} \quad \Omega^\varepsilon \times (0,T), \tag{8}$$

Initial Condition

$$d^\varepsilon(x^\varepsilon, 0) = \overline{d}(x^\varepsilon), \qquad \text{in} \quad \overline{\Omega}^\varepsilon. \tag{9}$$

The unknowns of the model (5)-(9) are the displacement vector field $u^\varepsilon(x^\varepsilon, t)$, corresponding to the displacement of the point x^ε of the rod at time t and the measure of change in volume fraction of the elastic material $d^\varepsilon(x^\varepsilon, t)$ at (x^ε, t). The stress tensor $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)$ and the linear strain tensor $e^\varepsilon = (e_{ij}^\varepsilon)$ are a function of theses quantities.

On the other hand, the data of the model (5)-(9) are the following: the open set $\Omega^\varepsilon \times (0,T)$, the density γ of the full elastic material, which is supposed to be a constant and independent of ε, the reference volume fraction of the elastic material ξ_0, that belongs to $C^1(\overline{\Omega}^\varepsilon)$ and is independent of ε, the body load $f^\varepsilon = (f_i^\varepsilon)$, such that $f_i^\varepsilon \in C^1[0,T]$ and depends only on t, the normal tractions on the boundary $g^\varepsilon = (g_i^\varepsilon)$ and $h^\varepsilon = (h_i^\varepsilon)$, the initial value of the change in volume fraction $\overline{d}(x^\varepsilon)$, which belongs to $C^0(\overline{\Omega}^\varepsilon)$, the coefficients $a_{ijkl}^\varepsilon(d^\varepsilon)$, $a^\varepsilon(d^\varepsilon)$, $A_{ij}^\varepsilon(d^\varepsilon)$, $B_{ijkl}^\varepsilon(d^\varepsilon)$, which are all material coefficients depending upon the change in volume fraction d^ε, the real numbers $\eta > 0$ and $\rho > 0$, that are small parameters, the truncation operator $\mathcal{P}_\eta(.)$ and the mollification operator M_ρ.

On these data we also suppose further conditions, which we will describe next. We assume that $0 < \xi_0^{min} \leq \xi_0(x^\varepsilon) \leq \xi_0^{max} < 1$ and that the normal tractions verify

$$g_i^\varepsilon \in C^1([0,T]; W^{1-1/p,p}(\Gamma^\varepsilon)), \qquad h_i^\varepsilon \in C^1([0,T]; W^{1-1/p,p}(\Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon)), \qquad \text{with} \quad p > 3. \tag{10}$$

Moreover we assume that the resultant of the system of applied forces is null for rigid displacements, that is, for any $v^\varepsilon = (v_i^\varepsilon)$ in $\mathcal{R}^\varepsilon$

$$\int_{\Omega^\varepsilon} \gamma(\xi_0 + \mathcal{P}_\eta(d^\varepsilon)) f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon + \int_{\Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon} h_i^\varepsilon v_i^\varepsilon d(\Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon) = 0, \quad \text{in} \quad [0,T]. \tag{11}$$

The truncation operator $\mathcal{P}_\eta$, of class C^1, is defined by

$$\mathcal{P}_\eta(d^\varepsilon)(x^\varepsilon) = \begin{cases} -\xi_0(x^\varepsilon) + \frac{\eta}{2}, & \text{if } d^\varepsilon(x^\varepsilon) \leq -\xi_0(x^\varepsilon) + \frac{\eta}{2} \\ d^\varepsilon(x^\varepsilon), & \text{if } \eta - \xi_0(x^\varepsilon) \leq d^\varepsilon(x^\varepsilon) \leq 1 - \xi_0(x^\varepsilon) - \eta \\ 1 - \xi_0(x^\varepsilon), & \text{if } d^\varepsilon(x^\varepsilon) \geq 1 - \xi_0(x^\varepsilon) \end{cases} \tag{12}$$

and satisfies

$$0 < \frac{\eta}{2} \leq (\xi_0 + \mathcal{P}_\eta(d^\varepsilon))(x^\varepsilon) \leq 1, \quad \forall x^\varepsilon \in \overline{\Omega}^\varepsilon. \tag{13}$$

The mollification operator acts from $C^0(\overline{\Omega}^\varepsilon)$ into $C^\infty(R^3)$ and is defined by

$$M_\rho(g(t))(x^\varepsilon) = \omega_\rho * \overline{g}(t)(x^\varepsilon) = \int_{R^3} \omega_\rho(x^\varepsilon - y)\overline{g}(t)(y)dy, \tag{14}$$

for any function $g \in C^0([0,T]; C^0(\overline{\Omega}^\varepsilon))$. The function $\overline{g}(t)$ is the extension of $g(t)$ to R^3, such that $\overline{g}(t) \in C^0(R^3)$, and $\omega_\rho(x^\varepsilon)$ is a mollifier. We choose

$$\omega_\rho(x^\varepsilon) = \Big(\int_{R^3} \omega(y)dy\Big)\frac{1}{\rho^n}\omega(\frac{x^\varepsilon}{\rho}), \tag{15}$$

where ω is the function defined in R^3 by

$$\omega(x) = \begin{cases} \exp(1/(\|x\|^2_{R^3} - 1)), & \text{if } \|x\|_{R^3} < 1 \\ 0, & \text{if } \|x\|_{R^3} \geq 1. \end{cases} \tag{16}$$

The coefficients $a_{ijkl}^\varepsilon(d^\varepsilon)$ are the elasticity coefficients, $c_{ijkl}^\varepsilon(d^\varepsilon)$ are the modified elasticity coefficients, $a^\varepsilon(d^\varepsilon)$ is the constitutive function, $A_{ij}^\varepsilon(d^\varepsilon)$ and $B_{ijkl}^\varepsilon(d^\varepsilon)$ are the remodeling rate coefficients. All these coefficients are continuously differentiable with respect to d^ε. In particular, the elasticity coefficients satisfy the following symmetric and elliptic conditions

$$\begin{cases} a_{ijkl}^\varepsilon(d^\varepsilon) = a_{jikl}^\varepsilon(d^\varepsilon) = a_{klij}^\varepsilon(d^\varepsilon), \\ a_{\alpha\beta\gamma3}^\varepsilon(d^\varepsilon) = a_{\alpha333}^\varepsilon(d^\varepsilon) = 0, \\ \exists c > 0 : \quad (\xi_0 + d^\varepsilon) a_{ijkl}^\varepsilon(d^\varepsilon)\tau_{ij}\tau_{kl} \geq c\,\tau_{ij}\tau_{ij}, \quad \forall \tau_{ij} \in R^{3\times3},\ \tau_{ij} = \tau_{ji}, \end{cases} \tag{17}$$

with c a positive constant independent of ε, x^ε and t. Of course all the properties stated in (17) also apply to the modified elasticity coefficients c_{ijkl}^ε defined in (6) because of the properties of ξ_0 and $\mathcal{P}_\eta$.

There is very few experimental data on the functions $a_{ijkl}^\varepsilon(d^\varepsilon)$, $a^\varepsilon(d^\varepsilon)$, $A_{ij}^\varepsilon(d^\varepsilon)$ and $B_{ijkl}^\varepsilon(d^\varepsilon)$. We define

$$B_{ijkl}^\varepsilon(d^\varepsilon) = (\xi_0 + M_\rho \circ \mathcal{P}_\eta(d^\varepsilon)) a_{ijkl}^\varepsilon(M_\rho \circ \mathcal{P}_\eta(d^\varepsilon)) = c_{ijkl}^\varepsilon(d^\varepsilon). \tag{18}$$

With the previous definition we have

$$B_{ijkl}^\varepsilon(d^\varepsilon) e_{ij}^\varepsilon(u^\varepsilon) e_{kl}^\varepsilon(u^\varepsilon) = c_{ijkl}^\varepsilon(d^\varepsilon) e_{ij}^\varepsilon(u^\varepsilon) e_{kl}^\varepsilon(u^\varepsilon) = \sigma_{ij}^\varepsilon e_{ij}^\varepsilon(u^\varepsilon) \tag{19}$$

that is, the driving mechanism associated to the remodeling rate equation (8) is the strain energy per unit volume $\sigma^{\varepsilon}_{ij} e^{\varepsilon}_{ij}(u^{\varepsilon})$ as some experimental evidence suggests.

The system (5-9) is an adaptive elasticity problem, which models the physiological process of bone remodeling. It generalizes the linear elasticity model and differs from this latter, because of the remodeling rate equation, which mathematically expresses the process of absorption and deposition of bone material due to an external stimulus. Equation (8) specifies the rate of change of the volume fraction of elastic material d^{ε} as a function of the volume fraction itself and the strain $e^{\varepsilon}(u^{\varepsilon})$. If $\dot{d}^{\varepsilon}$ is positive (respectively negative) it means the volume fraction of elastic material is increasing (respectively decreasing). When the volume fraction d^{ε} is zero and the reference volume fraction ξ_0 is equal to 1, we can choose $\mathcal{P}_{\eta}(d^{\varepsilon}) = 0$, and then, the constitutive equation (6) is the generalized Hooke's law, the remodeling rate equation does not exist and problem (5-9) coincides with the classical linearized elasticity model.

2.3 Existence, Regularity and Uniqueness of Solution

In [5] an existence and regularity result is proved, for the model (5-9) but with a remodeling rate equation linear in strain, that is, when (8) is replaced by the following equation

$$\dot{d}^{\varepsilon} = a^{\varepsilon}(d^{\varepsilon}) + A^{\varepsilon}_{kl}(d^{\varepsilon}) e^{\varepsilon}_{kl}(u^{\varepsilon}), \tag{20}$$

which does not contain the nonlinear term on strain $B^{\varepsilon}_{ijkl}(d^{\varepsilon}) e^{\varepsilon}_{ij}(u^{\varepsilon}) e^{\varepsilon}_{kl}(u^{\varepsilon})$. It is possible to prove, that the results of existence and regularity of solution of [5] may be extended to the case where the remodeling rate equation includes the nonlinear term on strain as in (8). Moreover, using arguments similar to those of [4], the uniqueness of solution can also be proven. The next theorem states these existence and uniqueness results for the adaptive elastic rod problem (5-9). A sketch of the proof of existence of solution is given.

Theorem 2.1 *For each $\varepsilon > 0$, and assuming that for each fixed $\hat{d}^{\varepsilon}$ the solution $\hat{u}^{\epsilon}$ of the equilibrium problem (5-7) is regular enough, there exists a unique solution $(u^{\varepsilon}, d^{\varepsilon})$ of problem (5-9), such that $u^{\varepsilon} \in C^1([0,T]; [W^{2,p}(\Omega^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon})$, with $p > 3$, and $d^{\varepsilon} \in C^1([0,T]; C^0(\overline{\Omega}^{\varepsilon}))$.*

Sketch of the proof: The proof of existence relies on Schauder's fixed point theorem, as in [5], theorem 1. That is, there is a fixed point of the composition operator $i \circ R \circ E \circ O \circ e^{\varepsilon}$; e^{ε} is the strain tensor operator, O is a differential operator (solution of problem (8-9) for a fixed u^{ε}), E is the elasticity operator (solution of problem (5-7) for a fixed d^{ε}), R is an injection operator (specifies the regularity of the solution of (5-7) [1, 7]) and i a compact injection operator. These operators are defined in the following spaces

$$\begin{array}{llll}
e^{\varepsilon} : C^0([0,T]; [C^1(\overline{\Omega}^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}) & \longrightarrow & C^0([0,T]; [C^0(\overline{\Omega}^{\varepsilon})]^{3\times 3}), & v^{\varepsilon} \mapsto e^{\varepsilon}(v^{\varepsilon}) = (e^{\varepsilon}_{ij}(v^{\varepsilon})) \\
O : C^0([0,T]; [C^0(\overline{\Omega}^{\varepsilon})]^{3\times 3}) & \longrightarrow & C^1([0,T]; C^0(\overline{\Omega}^{\varepsilon})), & e^{\varepsilon}_{ij}(v^{\varepsilon}) \mapsto O(e^{\varepsilon}_{ij}(v^{\varepsilon})) = d^{\varepsilon} \\
E : C^1([0,T]; C^0(\overline{\Omega}^{\varepsilon})) & \longrightarrow & C^1([0,T]; [W^{1,2}(\Omega^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}), & d^{\varepsilon} \mapsto E(d^{\varepsilon}) = u^{\varepsilon} \\
R : C^1([0,T]; [W^{1,2}(\Omega^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}) & \longrightarrow & C^1([0,T]; [W^{2,p}(\Omega^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}), & u^{\varepsilon} \mapsto R(u^{\varepsilon}) \\
i : C^1([0,T]; [W^{2,p}(\Omega^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}) & \longrightarrow & C^0([0,T]; [C^1(\overline{\Omega}^{\varepsilon})]^3 \setminus \mathcal{R}^{\varepsilon}), & u^{\varepsilon} \mapsto i(u^{\varepsilon}),
\end{array} \tag{21}$$

where $O(e^{\varepsilon}_{ij}(v^{\varepsilon})) = d^{\varepsilon}$ is the solution of the remodeling rate equation (8-9) for a fixed v^{ε}, and $E(d^{\varepsilon}) = u^{\varepsilon}$ is the solution of the elasticity system (5-7) for a fixed d^{ε}. We remark that the assumption of the theorem implies that the injection operator R in the composition $i \circ R \circ E \circ O \circ e^{\varepsilon}$ is well defined. In fact, for a fixed $\hat{d}^{\varepsilon}$ the solution $\hat{u}^{\varepsilon}$ of the elasticity problem (5-7) has the regularity $[W^{2,p}(\Omega^{\varepsilon})]^3$, with $p > 3$, if the domain is of class C^2 (cf. [1, 7]), which is not verified for the domain Ω^{ε}. Nevertheless, this type of regularity on the solution exists, in a great variety of interesting particular cases for a cylindrical rod as Ω^{ε}, which justifies the assumption. •

3 Asymptotic Expansion Method

In this section our purpose is to study the behaviour of the solution $(u^\varepsilon, d^\varepsilon)$, when ε tends to zero. We first reformulate the adaptive elastic model in a set independent of ε, and introduce some appropriate assumptions on the data. Moreover we assume, as in the elastic case [6], that the unknown displacement field admits an expansion of positive powers of ε. This asymptotic expansion induces a similar expansion for the change in volume fraction of elastic material, due to the definition of the remodeling rate equation (8) and the assumptions on the material coefficients. Then, we adapt and extend the results of the elastic displacement case of [6], obtained for displacement-traction boundary conditions, to the adaptive elastic displacement case (5-7), with pure traction boundary conditions. We obtain some properties and a characterization of the term of order ε^0 in the asymptotic expansion of the displacement field. We also identify the three first terms of the asymptotic expansion of the change in volume fraction. These are the unique solutions of ordinary differential equations, with respect to time. Moreover, we prove the convergence of the displacement vector field and the change in volume fraction to the first terms of the corresponding asymptotic expansions. Finally we particularize the results to the case of an adaptive elastic rod model with a remodeling rate equation linear in strain.

3.1 Fundamental Scalings and Assumptions

In this subsection we redefine problem (5-9) in a fixed domain. This is achieved through appropriate scalings, which is a consequence of the change of coordinates to the fixed domain, and imposing assumptions on the data.

The fixed domain is

$$\Omega = \omega \times (0, L) \tag{22}$$

and the corresponding boundary subsets are

$$\Gamma = \partial\omega \times (0, L), \quad \Gamma_0 = \overline{\omega} \times \{0\}, \quad \Gamma_L = \overline{\omega} \times \{L\}, \tag{23}$$

where $\partial\omega$ is the boundary of ω. To every point $x \in \overline{\Omega}$ we associate the point $x^\varepsilon \in \overline{\Omega}^\varepsilon$ defined by the bijection

$$\begin{array}{rl} \Pi^\varepsilon : & \overline{\Omega} \longrightarrow \overline{\Omega}^\varepsilon \\ & x \longrightarrow x^\varepsilon \end{array} \qquad \text{with} \qquad \begin{array}{l} x = (x_1, x_2, x_3) \\ x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (\varepsilon x_1, \varepsilon x_2, x_3), \end{array} \tag{24}$$

and in particular we have

$$\Gamma^\varepsilon = \Pi^\varepsilon(\Gamma), \quad \Gamma_0^\varepsilon = \Pi^\varepsilon(\Gamma_0), \quad \Gamma_L^\varepsilon = \Pi^\varepsilon(\Gamma_L). \tag{25}$$

As d^ε is a scalar field, u^ε, f^ε, g^ε, h^ε are vectors fields and e^ε, σ^ε are tensor fields of order two, the transformation of coordinates (24) from the system (O, x_1, x_2, x_3) to the system $(O, x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$, with the same origin O, induces a transformation on the components (covariant and contravariant) of these tensor fields of orders zero, one and two. Using tensor calculus laws, the expressions relating the components of d^ε, u^ε, f^ε, g^ε, h^ε, e^ε, σ^ε, defined in Ω^ε, and the components of the corresponding tensor fields $d(\varepsilon)$, $u(\varepsilon)$, $f(\varepsilon)$, $g(\varepsilon)$, $h(\varepsilon)$, $e(\varepsilon)$, $\sigma(\varepsilon)$, defined in Ω, are described below, for any $x \in \Omega$, with $x^\varepsilon = \Pi^\varepsilon(x)$, and any $t \in [0, T]$.

- For the tensor field $d(\varepsilon)$ of order zero

$$d(\varepsilon)(x, t) = d^\varepsilon(x^\varepsilon, t). \tag{26}$$

- For the tensor field $u(\varepsilon)$ of order one, the covariant components verify

$$u_\alpha(\varepsilon)(x, t) = \varepsilon u_\alpha^\varepsilon(x^\varepsilon, t), \quad u_3(\varepsilon)(x, t) = u_3^\varepsilon(x^\varepsilon, t). \tag{27}$$

- For the vector fields $f(\varepsilon)$, $g(\varepsilon)$, $h(\varepsilon)$, the contravariant components verify

$$\begin{array}{lll} f^\varepsilon_\alpha(x^\varepsilon,t) = \varepsilon f_\alpha(\varepsilon)(x,t), & g^\varepsilon_\alpha(x^\varepsilon,t) = \varepsilon^2 g_\alpha(\varepsilon)(x,t), & h^\varepsilon_\alpha(x^\varepsilon,t) = \varepsilon h_\alpha(\varepsilon)(x,t) \\ f^\varepsilon_3(x^\varepsilon,t) = f_3(\varepsilon)(x,t), & g^\varepsilon_3(x^\varepsilon,t) = \varepsilon g_3(\varepsilon)(x,t), & h^\varepsilon_3(x^\varepsilon,t) = h_3(\varepsilon)(x,t). \end{array} \tag{28}$$

- For the linearized strain tensor $e^\varepsilon(u^\varepsilon)$ of order two, the covariant components verify

$$\begin{array}{l} e_{\alpha\beta}(u(\varepsilon))(x,t) = \varepsilon^2 e^\varepsilon_{\alpha\beta}(u^\varepsilon)(x^\varepsilon,t) \\ e_{3\beta}(u(\varepsilon))(x,t) = \varepsilon e^\varepsilon_{3\beta}(u^\varepsilon)(x^\varepsilon,t) \\ e_{33}(u(\varepsilon))(x,t) = e^\varepsilon_{33}(u^\varepsilon)(x^\varepsilon,t) \end{array} \tag{29}$$

- For the stress tensor $\sigma_{ij}(\varepsilon)$ of order two, the contravariant components verify

$$\begin{array}{l} \sigma_{\alpha\beta}(\varepsilon)(x,t) = \varepsilon^{-2}\sigma^\varepsilon_{\alpha\beta}(x^\varepsilon,t) \\ \sigma_{3\beta}(\varepsilon)(x,t) = \varepsilon^{-1}\sigma^\varepsilon_{3\beta}(x^\varepsilon,t) \\ \sigma_{33}(\varepsilon)(x,t) = \sigma^\varepsilon_{33}(x^\varepsilon,t). \end{array} \tag{30}$$

In the sequel we refer to $d(\varepsilon)$, $u(\varepsilon)$, $f(\varepsilon)$, $g(\varepsilon)$, $h(\varepsilon)$, $e(\varepsilon)$, $\sigma(\varepsilon)$ as scaled tensor fields. Moreover we assume that the scaled vector fields $f(\varepsilon)$, $g(\varepsilon)$, $h(\varepsilon)$ associated to the forces are independent of ε and verify

$$\begin{array}{ll} f(\varepsilon) = (f_i), & f_i \in C^1([0,T]), \\ g(\varepsilon) = (g_i), & g_i \in C^1([0,T]; W^{1-1/p,p}(\Gamma^)), \\ h(\varepsilon) = (h_i), & h_i \in C^1([0,T]; W^{1-1/p,p}(\Gamma_0 \cup \Gamma_L)). \end{array} \tag{31}$$

Based on the asymptotic expansion method for an elastic rod (cf. [6]), we suppose that for the adaptive elastic rod model (5-9), the unknown scaled displacement $u(\varepsilon)$ is of the form

$$u(\varepsilon) = u^0 + \varepsilon^1 u^1 + \varepsilon^2 u^2 + ... \tag{32}$$

where the terms u^i depend on x and t and are independent of ε. Because of the definition of the remodeling rate equation (8) we further assume that this expansion induces the following expansion for the scaled change in volume fraction $d(\varepsilon)$

$$d(\varepsilon) = d^0 + \varepsilon d^1 + \varepsilon^2 d^2 + ... \tag{33}$$

where $d^i = d^i(x,t)$. Of course (33) is perfectly reasonable if the material coefficients do not depend on negative powers of ε. In fact, this can not occur since it would imply that for very small ε, $d(\varepsilon) > 1$ which is impossible, from the physical view-point. Therefore we suppose that the truncation operator, the elasticity coefficients, the remodeling coefficients and the constitutive function do not depend on negative powers of ε. Moreover we assume that these quantities are all independent of ε, which is acceptable from the mechanical view-point, and depend only on d^0 the first term of the asymptotic expansion of $d(\varepsilon)$, that is

$$\begin{array}{ll} \mathcal{P}_\eta(d^\varepsilon)(x^\varepsilon,t) = \mathcal{P}_\eta(d^0)(x,t), & \\ a^\varepsilon_{ijkl}(d^\varepsilon)(x^\varepsilon,t) = a_{ijkl}(d^0)(x,t), & c^\varepsilon_{ijkl}(d^\varepsilon)(x^\varepsilon,t) = c_{ijkl}(d^0)(x,t) \\ A^\varepsilon_{ij}(d^\varepsilon)(x^\varepsilon,t) = A_{ij}(d^0)(x,t), & a^\varepsilon(d^\varepsilon)(x^\varepsilon,t) = a(d^0)(x,t) \end{array} \tag{34}$$

where $\mathcal{P}_\eta$, a_{ijkl}, c_{ijkl}, a_0 and A_{ij} are independent of ε.

With all these assumptions and scalings the problem (5-9) can be formulated in the fixed domain $\overline{\Omega} \times [0,T]$, as stated in the next theorem.

Theorem 3.1 *The scaled displacement field $u(\varepsilon)$ and the scaled change in volume fraction $d(\varepsilon)$ constitute the unique solution pair of the following scaled problem.*

Find $u(\varepsilon) \in C^1([0,T]; [W^{2,p}(\Omega)]^3 \setminus \mathcal{R})$ *and* $d(\varepsilon) \in C^1([0,T]; C^0(\overline{\Omega}))$ *such that :*

$$
\left[\begin{array}{lll}
-\partial_j \sigma_{ij}(\varepsilon) = \gamma(\xi_0 + \mathcal{P}_\eta(d^0)) f_i, & in & \Omega \times (0,T), \\
\sigma_{ij}(\varepsilon) n_j = g_i, & in & \Gamma \times (0,T), \\
\sigma_{ij}(\varepsilon) n_j = h_i, & in & (\Gamma_0 \cup \Gamma_L) \times (0,T)
\end{array}\right. \tag{35}
$$

$$
\left[\begin{array}{ll}
\dot{d}(\varepsilon) = & \varepsilon^{-4} c_{\alpha\beta\gamma\mu}(d^0) e_{\alpha\beta}(\varepsilon) e_{\mu\gamma}(\varepsilon) + \\
& +\varepsilon^{-2}\big(A_{\alpha\beta}(d^0) e_{\alpha\beta}(\varepsilon) + 2c_{\alpha\beta 33}(d^0) e_{\alpha\beta}(\varepsilon) e_{33}(\varepsilon) + 4c_{3\alpha 3\beta}(d^0) e_{3\alpha}(\varepsilon) e_{3\beta}(\varepsilon)\big) + \\
& +\varepsilon^{-1}(A_{3\alpha}(d^0) + A_{\alpha 3}(d^0)) e_{3\alpha}(\varepsilon) + \\
& +a(d^0) + A_{33}(d^0) e_{33}(\varepsilon) + c_{3333}(d^0) e_{33}(\varepsilon) e_{33}(\varepsilon), \qquad in \quad \overline{\Omega} \times [0,T]
\end{array}\right. \tag{36}
$$

$$
d(\varepsilon)(x,0) = \overline{d}(\varepsilon)(x), \qquad in \quad \overline{\Omega}^\varepsilon, \tag{37}
$$

where $\overline{d}(\varepsilon)$ *is the scaled initial condition (9), and*

$$
\mathcal{R} = \{v \in R^3 : \;\; v = a + b \wedge x, \;\; a, b \in R^3\} \tag{38}
$$

is the scaled set of infinitesimal rigid displacements.

Proof: For each t, we obtain (35) as in [6], theorem 22.2. To derive the expression of the scaled remodeling rate equation (36) we use (19) and the following expressions for the scaled stress tensor components $\sigma_{ij}(\varepsilon)$

$$
\begin{array}{l}
\sigma_{\alpha\beta}(\varepsilon) = \varepsilon^{-4} c_{\alpha\beta\gamma\mu} e_{\gamma\mu}(u(\varepsilon)) + \varepsilon^{-2} c_{\alpha\beta 33} e_{33}(u(\varepsilon)) \\
\sigma_{3\beta}(\varepsilon) = 2\varepsilon^{-2} c_{3\alpha 3\beta} e_{3\alpha}(u(\varepsilon)) \\
\sigma_{33}(\varepsilon) = c_{3333} e_{33}(u(\varepsilon)) + \varepsilon^{-2} c_{\alpha\beta 33} e_{\alpha\beta}(u(\varepsilon)),
\end{array} \tag{39}
$$

which are deduced directly from (29-30) and the constitutive equation (6). •

The next theorem will be important in order to deduce the convergence results in subsection 3.5.

Theorem 3.2 *There exists a constant* $C > 0$*, independent of* ε*,* x *and* t*, such that the pair* $(u(\varepsilon), \sigma(\varepsilon))$*, solution of the scaled equilibrium equation (35), verifies*

$$
\begin{array}{c}
\|u(\varepsilon)\|_{C^0([0,T];[W^{1,2}(\Omega)]^3 \backslash \mathcal{R})} \leq C, \\
\|\sigma_{33}(\varepsilon)\|_{C^0([0,T];L^2(\Omega))} \leq C, \quad \|\varepsilon \sigma_{3\beta}(\varepsilon)\|_{C^0([0,T];L^2(\Omega))} \leq C, \qquad \|\varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)\|_{C^0([0,T];L^2(\Omega))} \leq C,
\end{array} \tag{40}
$$

for all $0 < \varepsilon \leq 1$ *and all* $t \in [0,T]$.

Proof: The arguments are the same as in [6], theorem 10.1, with the difference that in problem (35) there is another constitutive equation and the fields $u(\varepsilon)$ and $\sigma(\varepsilon)$ depend implicitly on the variable t. •

3.2 Characterization of the Term $\mathbf{u^0}$ and Auxiliary Results

In this subsection we extend the results of [6] to problem (35), in order to derive some properties and a characterization of the first term u^0 of the asymptotic expansion (32). We remark that the differences, between the equilibrium equation of the elasticity model considered in [6], chap. IV, for a nonhomogeneous anisotropic linear elastic rod, and the equilibrium equation (35) of the adaptive elasticity model considered in this paper are the boundary conditions and the fact that the displacement vector field depends implicitly on time in the adaptive elastic model. In [6] displacement-traction boundary conditions were considered, whereas in (35) we have pure traction boundary conditions. Neverthless it is possible to prove that the results of [6] can be extended to the elasticity model with pure traction boundary conditions, and therefore can be extended also to the equilibrium problem (35). In this subsection we summarize these extended results for problem (35). These results will be used in subsection 3.3 to deduce the properties of the terms of the asymptotic expansion of $d(\varepsilon)$.

Setting equal to zero the different factors of the successive odd powers of ε in (35), we can prove as in [6] that the asymptotic expansion (32) does not contain odd powers of ε, that is

$$u(\varepsilon) = u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + ... \tag{41}$$

Then, this expansion induces the following expansion on the components of the scaled strain tensor

$$\begin{aligned} &e_{ij}(u(\varepsilon)) = e_{ij}(u^0) + \varepsilon^2 e_{ij}(u^2) + \varepsilon^4 e_{ij}(u^4) + ... = e^0_{ij} + \varepsilon^2 e^2_{ij} + \varepsilon^4 e^4_{ij} + ... \\ &e^p_{ij} = e_{ij}(u^p), \quad p = 0, 2, 4, ... \end{aligned} \tag{42}$$

Consequently, and because of (39), the previous expansion of the strain tensor induces the following expansion in the scaled stress tensor

$$\begin{aligned} &\sigma_{\alpha\beta}(\varepsilon) = \varepsilon^{-4}\sigma^{-4}_{\alpha\beta} + \varepsilon^{-2}\sigma^{-2}_{\alpha\beta} + \sigma^0_{\alpha\beta} + ... \\ &\sigma_{3\beta}(\varepsilon) = \varepsilon^{-2}\sigma^{-2}_{3\beta} + \sigma^0_{3\beta} + \varepsilon^2\sigma^2_{3\beta} + ... \\ &\sigma_{33}(\varepsilon) = \varepsilon^{-2}\sigma^{-2}_{33} + \sigma^0_{33} + \varepsilon^2\sigma^2_{33} + ... \end{aligned} \tag{43}$$

where

$$\begin{aligned} &\sigma^{-4}_{\alpha\beta} = c_{\alpha\beta\gamma\mu}e_{\gamma\mu}(u^0) \\ &\sigma^p_{\alpha\beta} = c_{\alpha\beta\gamma\mu}e_{\gamma\mu}(u^{p+4}) + c_{\alpha\beta 33}e_{33}(u^{p+2}), \qquad p \geq -2 \\ &\sigma^p_{3\beta} = 2c_{3\alpha3\beta}e_{3\alpha}(u^{p+2}), \qquad p \geq -2 \\ &\sigma^{-2}_{33} = c_{\alpha\beta33}e_{\alpha\beta}(u^0) \\ &\sigma^p_{33} = c_{3333}e_{33}(u^p) + c_{\alpha\beta33}e_{\alpha\beta}(u^{p+2}), \qquad p \geq 0. \end{aligned} \tag{44}$$

Using again the same techniques as in [6] we can prove that

$$\sigma^p_{ij} = 0, \qquad \text{for} \quad p < 0 \tag{45}$$

and so, the asymptotic expansion for the scaled stress tensor is

$$\sigma(\varepsilon) = \sigma^0 + \varepsilon^2\sigma^2 + \varepsilon^4\sigma^4 + ... \tag{46}$$

We introduce now the coefficients $b_{ijkl}(d^0)$ of the inverse matrix defined by the elastic coefficients $c_{ijkl}(d^0)$, that is

$$e_{ij}(u(\varepsilon)) = b_{ijkl}(d^0)\sigma_{kl}(\varepsilon), \qquad \sigma_{ij}(\varepsilon) = c_{ijkl}(d^0)e_{kl}(u(\varepsilon)). \tag{47}$$

We remark that because of the properties of $c_{ijkl}(d^0)$

$$b_{ijkl}(d^0) = b_{jikl}(d^0) = b_{klij}(d^0) \qquad \text{and} \qquad b_{\alpha\beta\gamma 3}(d^0) = b_{\alpha 333}(d^0) = 0. \tag{48}$$

Moreover we define the following functions of x_3, that depend also on d^0 and, consequently, on t

$$l(x_3,t) = \int_\omega \frac{1}{b_{3333}(d^0)}d\omega, \quad e_\alpha(x_3,t) = \int_\omega \frac{x_\alpha}{b_{3333}(d^0)}d\omega, \quad h_{\alpha\beta}(x_3,t) = \int_\omega \frac{x_\alpha x_\beta}{b_{3333}(d^0)}d\omega. \tag{49}$$

We can now characterize the first term u^0 in the asymptotic expansion (41) of $u(\varepsilon)$. The results and properties concerning u^0 are summarized in the next theorem.

Theorem 3.3 *For each $t \in (0,T)$, the first term $u^0(.,t) = (u^0_1, u^0_2, u^0_3)(.,t)$ of the expansion (41) of $u(\varepsilon)(.,t)$ is in the space $[W^{1,2}(\Omega)]^3 \setminus \mathcal{R}$ and verifies*

$$\begin{aligned} &u^0(.,t) \in V(\Omega) \setminus \mathcal{R}, && V(\Omega) = \{v : \Omega \to R^3, \quad e_{\alpha\beta}(v) = e_{3\beta}(v) = 0\} \\ &u^0_\alpha = u^0_\alpha(x_3,t), && u^0_\alpha(.,t) \in W^{2,2}(0,L) \\ &u^0_3 = \underline{u}^0_3(x_3,t) - x_\alpha\partial_3 u^0_\alpha(x_3,t), && \underline{u}^0_3(.,t) \in W^{1,2}(0,L) \end{aligned} \tag{50}$$

where u_α^0 *and* $\underline{u}_3^0$ *depend only on* x_3 *and* t. *Furthermore, for each* t, $u^0(.,t) \in V(\Omega) \setminus R$ *is the unique solution of the following generalized Bernoulli-Navier model*

Equilibrium Equations in $(0,L) \times (0,T)$:

$$\left[\begin{array}{l} -\partial_3(l\partial_3\underline{u}_3^0 - e_\alpha\partial_{33}u_\alpha^0) = \int_\omega \gamma(\xi_0 + P_\eta(d^0))f_3 d\omega + \int_{\partial\omega} g_3 d\partial\omega \\ \partial_{33}(-e_\beta\partial_3\underline{u}_3^0 + h_{\alpha\beta}\partial_{33}u_\alpha^0) = \int_\omega \gamma(\xi_0 + P_\eta(d^0))f_\beta d\omega + \int_{\partial\omega} g_\beta d\partial\omega \\ \qquad\qquad + \int_\omega x_\beta\partial_3[\gamma(\xi_0 + P_\eta(d^0))f_3]d\omega + \int_{\partial\omega} x_\beta\partial_3 g_3 d\partial\omega \end{array}\right. \tag{51}$$

Boundary Conditions for $\{\overline{x}_3\} \times (0,T)$, *with* $\overline{x}_3 = 0, L$

$$\left[\begin{array}{l} (l\partial_3\underline{u}_3^0 - e_\alpha\partial_{33}u_\alpha^0)(\overline{x}_3) = \int_\omega h_3(\overline{x}_3)d\omega \\ (e_\beta\partial_3\underline{u}_3^0 - h_{\alpha\beta}\partial_{33}u_\alpha^0)(\overline{x}_3) = \int_\omega x_\beta h_3(\overline{x}_3)d\omega \\ \partial_3(e_\beta\partial_3\underline{u}_3^0 - h_{\alpha\beta}\partial_{33}u_\alpha^0)(\overline{x}_3) = \int_\omega h_\beta(\overline{x}_3)d\omega - \int_\omega x_\beta\gamma(\xi_0 + P_\eta(d^0))f_3(\overline{x}_3)d\omega - \int_{\partial\omega} x_\beta g_3(\overline{x}_3)d\partial\omega \end{array}\right. \tag{52}$$

where $h_i(\overline{x}_3)$ *represents the function* h_i *in* $\overline{\omega} \times \{\overline{x}_3\}$.
Moreover the terms σ_{33}^0 *and* u^2 *verify*

$$\begin{array}{l} \sigma_{33}^0 = \frac{1}{b_{3333}(d^0)}(\partial_3\underline{u}_3^0 - x_\alpha\partial_{33}u_\alpha^0) = \frac{1}{b_{3333}(d^0)}\partial_3 u_3^0 = \frac{1}{b_{3333}(d^0)}e_{33}(u^0) = \frac{1}{b_{3333}(d^0)}e_{33}^0 \\ e_{\alpha\beta}^2 = e_{\alpha\beta}(u^2) = b_{\alpha\beta 33}(d^0)\sigma_{33}^0 = \frac{b_{\alpha\beta 33}(d^0)}{b_{3333}(d^0)}e_{33}^0. \end{array} \tag{53}$$

Proof: For each t we obtain the characterization (50) and the equations (51-52) and (53) using the same reasoning of theorem 23.1 of [6]. The proof of existence and uniqueness of solution is based on Lax-Milgram theorem (cf. [1]). In fact, in the quotient space $V(\Omega) \setminus \mathcal{R}$ the following Korn's type inequality is verified (cf. [1, 7]) :

$$\exists c > 0 : \quad \|v\|^2_{[W^{1,2}(\Omega)]^3} \leq c\|e_{33}(v)\|^2_{L^2(\Omega)}, \tag{54}$$

where

$$\begin{array}{l} v = (v_1, v_2, v_3), \quad v_\alpha \in W^{2,2}(0,L), \quad v_3 = \underline{v}_3 - x_\alpha\partial_3 v_\alpha, \quad \underline{v}_3 \in W^{1,2}(0,L), \\ \|e_{33}(v)\|^2_{L^2(\Omega)} = \|\partial_3\underline{v}_3\|^2_{L^2(0,L)} + \left(\int_\omega x_\alpha^2 d\omega\right)\|\partial_{33}v_\alpha\|^2_{L^2(0,L)}. \end{array} \tag{55}$$

In addition, the variational formulation of (51-52) is

$$\left\{\begin{array}{l} u^0 \in V(\Omega) \setminus \mathcal{R} \\ a(u^0, v) = L(v), \qquad \forall v \in V(\Omega) \setminus \mathcal{R} \end{array}\right. \tag{56}$$

with the bilinear form $a(.,.)$ and the linear form $L(.)$ defined by

$$\begin{array}{l} a(u,v) = \displaystyle\int_\Omega \frac{1}{b_{3333}(d^0)}e_{33}(u)e_{33}(v)d\Omega, \qquad \forall u, v \in V(\Omega) \setminus \mathcal{R} \\ L(v) = \displaystyle\int_\Omega \gamma(\xi_0 + P_\eta(d^0))f_i v_i d\Omega + \int_\Gamma g_i v_i d\Gamma + \int_{\Gamma_0\cup\Gamma_L} h_i v_i d(\Gamma_0 \cup \Gamma_L), \qquad \forall v \in V(\Omega) \setminus \mathcal{R}. \end{array} \tag{57}$$

Using the inequality (54) it is possible to show that $a(.,.)$ is elliptic in $V(\Omega) \setminus R$, in the sense that there exists a constant c, such that

$$c > 0, \qquad a(v,v) \geq c\|v\|_{[W^{1,2}(\Omega)]^3}, \quad \forall v \in V(\Omega) \setminus R. \tag{58}$$

Since $L(.)$ is also continuous in $V(\Omega) \setminus \mathcal{R}$ equipped with the norm $\|.\|_{[W^{1,2}(\Omega)]^3}$, we conclude by the Lax-Milgram theorem that for each t, $u^0(.,t)$ is the unique solution of (51-52). •

3.3 Characterization of the Terms of the Asymptotic Expansion of $\mathbf{d}(\varepsilon)$

Using the results of the two previous subsections we can now identify and derive some conclusions about the terms of the asymptotic expansion (33) of $d(\varepsilon)$. To this end we also suppose that the scaled initial condition $\overline{d}(\varepsilon)$ defined in (37) is of the following form

$$\overline{d}(\varepsilon) = \overline{d}^0 + \varepsilon\overline{d}^1 + \varepsilon^2\overline{d}^2 + ..., \tag{59}$$

where $\overline{d}^i$, for $i = 1, 2, ...$ are functions defined in $\overline{\Omega}$ but independent of ε. The next theorem characterizes the three first terms of $d(\varepsilon)$.

Theorem 3.4 *The terms d^0, d^1 and d^2 of the asymptotic expansion (33) are the solutions of the following independent ordinary differential equations:*

$$\left[\begin{array}{lll} \dot{d}^0 = \frac{1}{b_{3333}(d^0)} e^0_{33} e^0_{33} + \left[A_{\alpha\beta}(d^0)\frac{b_{\alpha\beta 33}(d^0)}{b_{3333}(d^0)} + A_{33}(d^0)\right] e^0_{33} + a(d^0), & in & \Omega\times(0,T) \\ d^0(x,0) = \overline{d}^0(x), & in & \Omega \end{array}\right. \tag{60}$$

$$\left[\begin{array}{lll} \dot{d}^1 = \left(A_{3\beta}(d^0) + A_{\beta 3}(d^0)\right) e^2_{3\beta}, & in & \Omega\times(0,T) \\ d^1(x,0) = \overline{d}^1(x), & in & \Omega \end{array}\right. \tag{61}$$

$$\left[\begin{array}{lll} \dot{d}^2 = 2\sigma^0_{33} e^2_{33} + 2\sigma^0_{3\beta} e^2_{3\beta} + A_{\alpha\beta}(d^0) e^4_{\alpha\beta} + A_{33}(d^0) e^2_{33}, & in & \Omega\times(0,T) \\ d^2(x,0) = \overline{d}^2(x), & in & \Omega. \end{array}\right. \tag{62}$$

Proof: From the expansions (33) and (59) and the scaled initial condition (37) we have for $t = 0$

$$d^0(x,0) + \varepsilon d^1(x,0) + \varepsilon^2 d^2(x,0) + ... = \overline{d}^0(x) + \varepsilon\overline{d}^1(x) + \varepsilon^2\overline{d}^2(x) + ... \tag{63}$$

which gives the initial conditions of (60-62). Introducing (42) in (36) we have

$$\left\{\begin{array}{l} \varepsilon^4\dot{d}^0 + \varepsilon^5\dot{d}^1 + \varepsilon^6\dot{d}^2 + ... = c_{\alpha\beta\gamma\mu}(e^0_{\alpha\beta} + \varepsilon^2 e^2_{\alpha\beta} + \varepsilon^4 e^4_{\alpha\beta} + ...)(e^0_{\mu\gamma} + \varepsilon^2 e^2_{\mu\gamma} + \varepsilon^4 e^4_{\mu\gamma} + ...) + \\ \quad +\varepsilon^2\Big[A_{\alpha\beta}(e^0_{\alpha\beta} + \varepsilon^2 e^2_{\alpha\beta} + \varepsilon^4 e^4_{\alpha\beta} + ...) + \\ \quad\quad 2c_{\alpha\beta 33}(e^0_{\alpha\beta} + \varepsilon^2 e^2_{\alpha\beta} + \varepsilon^4 e^4_{\alpha\beta} + ...)(e^0_{33} + \varepsilon^2 e^2_{33} + \varepsilon^4 e^4_{33} + ...) \\ \quad\quad +4c_{3\alpha 3\beta}(e^0_{3\alpha} + \varepsilon^2 e^2_{3\alpha} + \varepsilon^4 e^4_{3\alpha} + ...)(e^0_{3\beta} + \varepsilon^2 e^2_{3\beta} + \varepsilon^4 e^4_{3\beta} + ...)\Big] + \\ \quad +\varepsilon^3(A_{3\alpha} + A_{\alpha 3})(e^0_{3\alpha} + \varepsilon^2 e^2_{3\alpha} + \varepsilon^4 e^4_{3\alpha} + ...) + \\ \quad +\varepsilon^4\Big[a + A_{33}(e^0_{33} + \varepsilon^2 e^2_{33} + \varepsilon^4 e^4_{33} + ...) + \\ \quad\quad c_{3333}(e^0_{33} + \varepsilon^2 e^2_{33} + \varepsilon^4 e^4_{33} + ...)(e^0_{33} + \varepsilon^2 e^2_{33} + \varepsilon^4 e^4_{33} + ...)\Big]. \end{array}\right. \tag{64}$$

In order to obtain the differential equations (60-62) it is enough to identify the coefficients of the same power of ε in (64). That is, we must identify the coefficients of ε^0, ε^2, ε^4, ε^5 and ε^6.

- For the coefficients of ε^0 we obtain that

$$0 = c_{\alpha\beta\gamma\mu} e^0_{\alpha\beta} e^0_{\mu\gamma} \tag{65}$$

  which is true, since $e^0_{\alpha\beta} = e_{\alpha\beta}(u^0) = 0$ (cf. theorem 3.3).

- For the coefficients of ε^2 one has

$$0 = 4c_{3\alpha 3\beta} e^0_{3\alpha} e^0_{3\beta} \tag{66}$$

  which is correct, since $e^0_{3\alpha} = e_{3\alpha}(u^0) = 0$ (cf. theorem 3.3).

- For the coefficients of ε^4 we have the equation

$$\dot{d}^0 = c_{\alpha\beta\gamma\mu} e^2_{\alpha\beta} e^2_{\mu\gamma} + A_{\alpha\beta} e^2_{\alpha\beta} + 2c_{\alpha\beta 33} e^2_{\alpha\beta} e^0_{33} + A_{33} e^0_{33} + c_{3333} e^0_{33} e^0_{33} + a. \tag{67}$$

  But, we can group the terms, such that

$$\dot{d}^0 = [c_{\alpha\beta\gamma\mu} e^2_{\mu\gamma} + c_{\alpha\beta 33} e^0_{33}] e^2_{\alpha\beta} + [c_{3333} e^0_{33} + c_{\alpha\beta 33} e^2_{\alpha\beta}] e^0_{33} + A_{\alpha\beta} e^2_{\alpha\beta} + A_{33} e^0_{33} + a. \tag{68}$$

  But using (53), (44) and the fact that $\sigma^p_{ij} = 0$, for $p < 0$, we have that

$$\begin{array}{l} e^2_{\alpha\beta} = e_{\alpha\beta}(u^2) = b_{\alpha\beta 33} \sigma^0_{33} = \frac{b_{\alpha\beta 33}}{b_{3333}} e^0_{33} \\ \sigma^{-2}_{\alpha\beta} = c_{\alpha\beta\gamma\mu} e^2_{\gamma\mu} + c_{\alpha\beta 33} e^0_{33} = 0, \\ \sigma^0_{33} = c_{3333} e^0_{33} + c_{\alpha\beta 33} e^2_{\alpha\beta} = \frac{1}{b_{3333}} e^0_{33}. \end{array} \tag{69}$$

  Then, we obtain the differential equation (60) introducing relations (69) in (68).

- For the coefficients of ε^5 we get directly equation (61).

- For the coefficients of ε^6 we have

$$\left\{ \begin{array}{ll} \dot{d}^1 = & c_{\alpha\beta\gamma\mu}(e^2_{\alpha\beta} e^4_{\mu\gamma} + e^4_{\alpha\beta} e^2_{\mu\gamma}) + A_{\alpha\beta} e^4_{\alpha\beta} + 2c_{\alpha\beta 33}(e^2_{\alpha\beta} e^2_{33} + e^4_{\alpha\beta} e^0_{33}) + \\ & \quad +4c_{3\alpha 3\beta} e^2_{3\alpha} e^2_{3\beta} + A_{33} e^2_{33} + 2c_{3333} e^0_{33} e^2_{33}. \end{array} \right. \tag{70}$$

  Using again (44) and the fact that $\sigma^p_{ij} = 0$, for $p < 0$, we have equation (62).

We remark that continuing this process we could identify the remaining terms in powers of higher order of ε in the expansion (33) of $d(\varepsilon)$. •

3.4 Existence and Uniqueness of Solution of the Asymptotic Model

From theorems 3.3 and 3.4 we conclude that the pair (u^0, d^0), whose components are the first terms of the asymptotic expansions of $u(\varepsilon)$ and $d(\varepsilon)$ respectively, is a solution of the following nonlinear adaptive elasticity model.

$$\begin{array}{l} \text{Find} \quad u^0 = (u^0_i) : \overline{\Omega} \times [0,T] \to R^3, \quad d^0 : \overline{\Omega} \times [0,T] \to R \quad \text{with,} \\ u^0_\alpha : [0,L] \times [0,T] \to R, \quad u^0_3 = \underline{u}^0_3 - x_\alpha \partial_3 u^0_\alpha \quad \text{and} \quad \underline{u}^0_3 : [0,L] \times [0,T] \to R, \quad \text{such that :} \end{array} \tag{71}$$

Equilibrium Equations in $(0,L) \times (0,T)$

$$\left[\begin{array}{l} -\partial_3(l\partial_3 \underline{u}^0_3 - e_\alpha \partial_{33} u^0_\alpha) = \int_\omega \gamma(\xi_0 + P_\eta(d^0)) f_3 d\omega + \int_{\partial\omega} g_3 d\partial\omega \\ \partial_{33}(-e_\beta \partial_3 \underline{u}^0_3 + h_{\alpha\beta} \partial_{33} u^0_\alpha) = \int_\omega \gamma(\xi_0 + P_\eta(d^0)) f_\beta d\omega + \int_{\partial\omega} g_\beta d\partial\omega \\ \qquad + \int_\omega x_\beta \partial_3 [\gamma(\xi_0 + P_\eta(d^0)) f_3] d\omega + \int_{\partial\omega} x_\beta \partial_3 g_3 d\partial\omega \end{array} \right. \tag{72}$$

Boundary Conditions for $\{\overline{x}_3\} \times (0,T)$, with $\overline{x}_3 = 0, L$

$$\left[\begin{array}{l} (l\partial_3 \underline{u}^0_3 - e_\alpha \partial_{33} u^0_\alpha)(\overline{x}_3) = \int_\omega h_3(\overline{x}_3) d\omega \\ (e_\beta \partial_3 \underline{u}^0_3 - h_{\alpha\beta} \partial_{33} u^0_\alpha)(\overline{x}_3) = \int_\omega x_\beta h_3(\overline{x}_3) d\omega \\ \partial_3(e_\beta \partial_3 \underline{u}^0_3 - h_{\alpha\beta} \partial_{33} u^0_\alpha)(\overline{x}_3) = \int_\omega h_\beta(\overline{x}_3) d\omega - \int_\omega x_\beta \gamma(\xi_0 + P_\eta(d^0)) f_3(\overline{x}_3) d\omega - \int_{\partial\omega} x_\beta g_3(\overline{x}_3) d\partial\omega \end{array} \right. \tag{73}$$

Remodeling Rate Equation

$$\left[\begin{array}{ll} \dot{d}^0 = \frac{1}{b_{3333}(d^0)} e^0_{33} e^0_{33} + \left[A_{\alpha\beta}(d^0) \frac{b_{\alpha\beta 33}(d^0)}{b_{3333}(d^0)} + A_{33}(d^0) \right] e^0_{33} + a(d^0), & \text{in} \quad \Omega \times (0,T) \\ d^0(x,0) = \overline{d}^0(x), & \text{in} \quad \Omega. \end{array} \right. \tag{74}$$

The proof of existence and uniqueness of solution of (71-74) is stated in the next theorem and is analogous to the proof of theorem 2.1 for the problem (5-9).

Theorem 3.5 *Assuming that for each fixed $\hat{d}^0$ the solution $\hat{u}^0$ of the equilibrium problem (72-73) is regular enough, there exists a unique solution (u^0, d^0) of problem (71-74), such that $u^0 \in C^1([0,T]; V(\Omega) \setminus \mathcal{R})$, and $d^0 \in C^1([0,T]; C^0(\overline{\Omega}))$.*$\bullet$

3.5 Convergence Results

In this subsection we state the strong convergence results for the scaled sequences $u(\varepsilon)$ and $d(\varepsilon)$, when ε tends to zero.

Theorem 3.6 *If the material of the rod is such that the coefficients $\frac{b_{\alpha\beta 33}}{b_{3333}}$ do not depend on x_1 and x_2, then the scaled displacement and stress fields $u(\varepsilon)$ and $\sigma(\varepsilon)$ satisfy the following strong convergences when the small parameter ε tends to zero*

$$\begin{array}{ll} u(\varepsilon) \longrightarrow u^0 & \textit{in} \quad C^0([0,T]; [W^{1,2}(\Omega)]^3 \setminus \mathcal{R}) \\ \left[\begin{array}{l} \sigma_{33}(\varepsilon) \longrightarrow \sigma^0_{33} \\ \varepsilon\sigma_{3\beta}(\varepsilon) \longrightarrow 0 \\ \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon) \longrightarrow 0 \end{array} \right. & \textit{in} \quad C^0([0,T]; L^2(\Omega)). \end{array} \tag{75}$$

Proof: It is a consequence of the bound conditions of theorem 3.2 and theorem 23.3 of [6]. $\bullet$

The next theorem states the strong convergence of $d(\varepsilon)$.

Theorem 3.7 *With the same assumptions of theorem 3.6, and if the scaled initial condition verifies $\overline{d}(\varepsilon)(x) = d^0(x,0)$ the sequence $d(\varepsilon)$ converges strongly to d^0 in the space $C^1([0,T]; L^2(\Omega))$, when ε tends to zero.*

Proof: Using the relations (29) and (30) directly in the remodeling rate equation (8) we obtain the scaled equation

$$\left\{ \begin{array}{rl} \dot{d}(\varepsilon) = & a(d^0) + A_{\alpha\beta}(d^0)\varepsilon^{-2} e_{\alpha\beta}(u(\varepsilon)) + \big(A_{3\beta}(d^0) + A_{\beta 3}(d^0)\big)\varepsilon^{-1} e_{3\beta}(u(\varepsilon)) + \\ & +A_{33}(d^0) e_{33}(u(\varepsilon)) + \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)\varepsilon^{-2} e_{\alpha\beta}(u(\varepsilon)) + \\ & +2\varepsilon\sigma_{3\beta}(\varepsilon)\varepsilon^{-1} e_{3\beta}(u(\varepsilon)) + \sigma_{33}(\varepsilon) e_{33}(u(\varepsilon)) \end{array} \right. \tag{76}$$

which is exactly the same as equation (36). Using the definitions of the scaled strain and stress tensors we introduce the tensors $\hat{e}(u(\varepsilon))$ and $\hat{\sigma}(\varepsilon)$ by

$$\begin{array}{ll} \hat{e}_{\alpha\beta}(u(\varepsilon)) = \varepsilon^{-2} e_{\alpha\beta}(u(\varepsilon)) & \hat{\sigma}_{\alpha\beta}(u(\varepsilon)) = \varepsilon^2 \sigma_{\alpha\beta}(u(\varepsilon)) \\ \hat{e}_{3\beta}(u(\varepsilon)) = \varepsilon^{-1} e_{3\beta}(u(\varepsilon)) & \hat{\sigma}_{3\beta}(u(\varepsilon)) = \varepsilon\sigma_{3\beta}(u(\varepsilon)) \\ \hat{e}_{33}(u(\varepsilon)) = e_{33}(u(\varepsilon)) & \hat{\sigma}_{33}(u(\varepsilon)) = \sigma_{33}(u(\varepsilon)). \end{array} \tag{77}$$

Analysing the constitutive equation (6) we immediately conclude that, in the set Ω

$$\hat{\sigma}_{ij}(\varepsilon) = c_{ijkl}(d^0)\hat{e}_{kl}(u(\varepsilon)) \tag{78}$$

and inverting this relation

$$\hat{e}_{ij}(u(\varepsilon)) = b_{ijkl}(d^0)\hat{\sigma}_{kl}(\varepsilon). \tag{79}$$

But, by the theorem 3.6 we have the following strong convergences in the space $C^0([0,T]; L^2(\Omega))$

$$\sigma_{33}(\varepsilon) \longrightarrow \sigma^0_{33}, \qquad \varepsilon\sigma_{3\beta}(\varepsilon) \longrightarrow 0, \qquad \varepsilon^2\sigma_{\alpha\beta}(\varepsilon) \longrightarrow 0. \tag{80}$$

From (79-80) and because $b_{ijkl}(d^0)$ belongs to the space $C^1(\Omega \times (0,T))$ and is independent of ε, we conclude that, when ε tends to zero, the following strong converges in $C^0([0,T]; L^2(\Omega))$ are verified

$$\begin{array}{l} e_{33}(u(\varepsilon)) = \hat{e}_{33}(u(\varepsilon)) \longrightarrow b_{3333}(d^0)\sigma^0_{33} = e^0_{33}, \\ \varepsilon^{-1} e_{3\beta}(u(\varepsilon)) = \hat{e}_{3\beta}(u(\varepsilon)) \longrightarrow b_{3\beta 33}(d^0)\sigma^0_{33} = 0, \qquad \text{since} \quad b_{3\beta 33}(d^0) = 0, \\ \varepsilon^{-2} e_{\alpha\beta}(u(\varepsilon)) = \hat{e}_{\alpha\beta}(u(\varepsilon)) \longrightarrow b_{\alpha\beta 33}(d^0)\sigma^0_{33} = \frac{b_{\alpha\beta 33}(d^0)}{b_{3333}(d^0)} e^0_{33}. \end{array} \tag{81}$$

Then, we conclude that

$$\dot{d}(\varepsilon) \longrightarrow M \qquad \text{in} \quad C^0([0,T];L^2(\Omega)), \qquad \text{when} \quad \varepsilon \to 0, \tag{82}$$

where

$$M = a(d^0) + A_{\alpha\beta}(d^0)\frac{b_{\alpha\beta 33}(d^0)}{b_{3333}(d^0)}e^0_{33} + A_{33}(d^0)e^0_{33} + \sigma^0_{33}e^0_{33}. \tag{83}$$

By introducing the definition of σ^0_{33} in (83) we obtain that $M = \dot{d}^0$. Therefore we conclude that

$$\dot{d}(\varepsilon) - \dot{d}^0 \longrightarrow 0 \qquad \text{in} \quad C^0([0,T];L^2(\Omega)). \tag{84}$$

Using, for each x, the equality

$$(d(\varepsilon) - d^0)(x,t) = \int_0^t (\dot{d}(\varepsilon) - \dot{d}^0)(x,s)ds + (d(\varepsilon) - d^0)(x,0) \tag{85}$$

and assuming $d(\varepsilon)(x,0) = \overline{d}(\varepsilon)(x) = d^0(x,0)$ we have

$$d(\varepsilon) - d^0 \longrightarrow 0 \qquad \text{in} \quad C^1([0,T];L^2(\Omega)).\bullet \tag{86}$$

So from theorems 3.6 and 3.7 we have the following final strong convergence result.

Theorem 3.8 *Let the hypothesis of theorem 3.6 be satisfied and for each ε let $(u(\varepsilon), d(\varepsilon))$ be the unique solution of the scaled adaptive elastic problem (35-37). Then, when ε tends to zero, the scaled sequence $\{(u(\varepsilon), d(\varepsilon))\}$ converges strongly to (u^0, d^0), solution of problem (71-74), in the product space $C^0([0,T];[W^{1,2}(\Omega)]^3 \setminus \mathcal{R}) \times C^1([0,T];L^2(\Omega))$.*$\bullet$

3.6 Linear Remodeling Rate Equation

In this subsection we will particularize the previous results for the case where the remodeling rate equation is linear in strain, in the initial adaptive elastic rod problem.

If in problem (5-9) we consider the linear remodeling rate equation (20) instead of the nonlinear one (8), then the scaled adaptive elastic rod problem (35-37) is replaced by the following one

Find $\quad u(\varepsilon) \in C^1([0,T];[W^{2,p}(\Omega)]^3 \setminus \mathcal{R}) \quad$ and $\quad d(\varepsilon) \in C^1([0,T];C^0(\overline{\Omega})) \quad$ such that :

$$\left[\begin{array}{ll} -\partial_j \sigma_{ij}(\varepsilon) = \gamma(\xi_0 + \mathcal{P}_\eta(d^0))f_i, & \text{in} \quad \Omega \times (0,T), \\ \sigma_{ij}(\varepsilon)n_j = g_i, & \text{in} \quad \Gamma \times (0,T), \\ \sigma_{ij}(\varepsilon)n_j = h_i, & \text{in} \quad (\Gamma_0 \cup \Gamma_L) \times (0,T) \end{array}\right. \tag{87}$$

$$\left[\begin{array}{l} \dot{d}(\varepsilon) = A_{\alpha\beta}(d^0)\varepsilon^{-2}e_{\alpha\beta}(\varepsilon) + (A_{3\alpha}(d^0) + A_{\alpha 3}(d^0))\varepsilon^{-1}e_{3\alpha}(\varepsilon) + a(d^0) + A_{33}(d^0)e_{33}(\varepsilon), \\ \qquad \text{in} \quad \overline{\Omega} \times [0,T] \end{array}\right. \tag{88}$$

$$d(\varepsilon)(x,0) = \overline{d}(\varepsilon)(x), \qquad \text{in} \quad \overline{\Omega}^\varepsilon, \tag{89}$$

with a linear scaled remodeling rate equation (88). As the equilibrium equation (87) is formally equivalent to (35) the results of theorems 3.2, 3.3 and 3.6 still apply to the scaled displacement field $u(\varepsilon)$, solution of (87-89). Moreover the results of theorems 3.4, 3.7 and 3.8 can be particularized for the linear remodeling rate equation. In fact, it is enough to neglect the nonlinear terms in the results of these theorems. The next theorem summarizes for the linear remodeling rate equation, the theorems 3.4, 3.7 and 3.8.

Theorem 3.9 *For the scaled problem (87-89) the terms d^0, d^1 and d^2 of the asymptotic expansion of $d(\varepsilon)$ are the solutions of the following independent ordinary differential equations :*

$$\left[\begin{array}{ll} \dot{d^0} = \left[A_{\alpha\beta}(d^0)\frac{b_{\alpha\beta33}(d^0)}{b_{3333}(d^0)} + A_{33}(d^0)\right]e^0_{33} + a(d^0), & in \quad \Omega\times(0,T) \\ d^0(x,0) = \overline{d}^0(x), & in \quad \Omega \end{array}\right. \tag{90}$$

$$\left[\begin{array}{ll} \dot{d^1} = \big(A_{3\beta}(d^0) + A_{\beta3}(d^0)\big)e^2_{3\beta}, & in \quad \Omega\times(0,T) \\ d^1(x,0) = \overline{d}^1(x), & in \quad \Omega \end{array}\right. \tag{91}$$

$$\left[\begin{array}{ll} \dot{d^2} = A_{\alpha\beta}(d^0)e^4_{\alpha\beta} + A_{33}(d^0)e^2_{33}, & in \quad \Omega\times(0,T) \\ d^2(x,0) = \overline{d}^2(x), & in \quad \Omega. \end{array}\right. \tag{92}$$

Moreover if the assumptions of theorem 3.6 are verified and if the scaled initial condition is $\overline{d}(\varepsilon)(x) = d^0(x,0)$, with d^0 the solution of (90), then, when ε tends to zero, the solution $(u(\varepsilon), d(\varepsilon))$ of (87-89) converges strongly to (u^0, d^0) in the space $C^0([0,T];[W^{1,2}(\Omega)]^3\setminus\mathcal{R})\times C^1([0,T];L^2(\Omega))$, where u^0 is the solution of (51-52).•

If the coefficients $\frac{b_{\alpha\beta33}}{b_{3333}}$ do not satisfy the hypothesis of theorem 3.6, then, instead of the strong convergences (75) for $u(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ we have weak convergences. In fact, using theorem 3.2 and theorem 23.2 of [6], there exists at least a subsequence of $(u(\varepsilon), \sigma_{33}(\varepsilon), \varepsilon\sigma_{3\beta}(\varepsilon), \varepsilon^2\sigma_{\alpha\beta}(\varepsilon))$, denoted in the same way, such that the following weak convergences are verified, when ε tends to zero

$$\begin{array}{ll} u(\varepsilon) \rightharpoonup u^0 + w & \text{in} \quad C^0([0,T];[W^{1,2}(\Omega)]^3\setminus\mathcal{R}) \\ \left[\begin{array}{l} \sigma_{33}(\varepsilon) \rightharpoonup \psi_{33} \\ \varepsilon\sigma_{3\beta}(\varepsilon) \rightharpoonup \psi_{3\beta} \\ \varepsilon^2\sigma_{\alpha\beta}(\varepsilon) \rightharpoonup \psi_{\alpha\beta} \end{array}\right. & \text{in} \quad C^0([0,T];L^2(\Omega)) \end{array} \tag{93}$$

where w and ψ_{ij} are defined in theorem 23.2 of [6]. Consequently because of (79) the convergences (93) imply the following weak convergences in $C^0([0,T];L^2(\Omega))$

$$\begin{array}{ll} e_{33}(u(\varepsilon)) = \hat{e}_{33}(u(\varepsilon)) \rightharpoonup & b_{33kl}(d^0)\psi_{kl} = e^0_{33} + e_{33}(w), \\ \varepsilon^{-1}e_{3\beta}(u(\varepsilon)) = \hat{e}_{3\beta}(u(\varepsilon)) \rightharpoonup & b_{3\beta kl}(d^0)\psi_{kl} = 2b_{3\beta3\mu}(d^0)\psi_{3\mu}, \\ \varepsilon^{-2}e_{\alpha\beta}(u(\varepsilon)) = \hat{e}_{\alpha\beta}(u(\varepsilon)) \rightharpoonup & \left[\begin{array}{l} b_{\alpha\beta kl}(d^0)\psi_{kl} = b_{\alpha\beta\gamma\mu}(d^0)\psi_{\gamma\mu} + b_{\alpha\beta33}(d^0)\psi_{33} = \\ = b_{\alpha\beta\gamma\mu}(d^0)\psi_{\gamma\mu} + \frac{b_{\alpha\beta33}(d^0)}{b_{3333}(d^0)}\left[e^0_{33} + e_{33}(w) - b_{\alpha\beta33}(d^0)\psi_{\alpha\beta}\right]. \end{array}\right. \end{array} \tag{94}$$

Thus using arguments similar to those of theorem 3.7 we have the following result of weak convergence for in the linear case.

Theorem 3.10 *There exists at least a subsequence of the solution $d(\varepsilon)$ of the scaled linear remodeling rate equation (88), that converges weakly to $\hat{d}$ in $C^1([0,T];L^2(\Omega))$ when ε tends to zero, where $\hat{d}$ is the solution of the following ordinary differential equation*

$$\left[\begin{array}{ll} \dot{\hat{d}} = & A_{\alpha\beta}(d^0)\Big(b_{\alpha\beta\gamma\mu}(d^0)\psi_{\gamma\mu} + \frac{b_{\alpha\beta33}(d^0)}{b_{3333}(d^0)}\left[e^0_{33} + e_{33}(w) - b_{\alpha\beta33}(d^0)\psi_{\alpha\beta}\right]\Big) + \\ & +A_{33}(d^0)\big(e^0_{33} + e_{33}(w)\big) + a(d^0) + (A_{3\beta}(d^0) + A_{\beta3}(d^0))2b_{3\beta3\mu}\psi_{3\mu} \end{array}\right. \tag{95}$$

with the initial condition $\hat{d}(x,0) = d^0(x,0)$. Moreover, $u(\varepsilon)$ converges weakly to $u^0 + w$ in the space $C^0([0,T];[W^{1,2}(\Omega)]^3\setminus\mathcal{R})$, where u^0 is the solution of (51-52). •

If the assumptions of theorem 3.6 are verified it is clear that the weak limit $\hat{d}$ coincides with the strong limit d^0 of theorem 3.9, since in that case (cf. [6]) $w = 0$, $\psi_{33} = \sigma^0_{33}$, $\psi_{3\beta} = 0$, $\psi_{\alpha\beta} = 0$ and so the differential equations (90) and (95) are equal.

We remark that for the case of the nonlinear remodeling rate equation (36) or equivalently (76) we can not apply the weak convergence results of (94) to deduce a weak convergence result for the scaled change in volume fraction $d(\varepsilon)$. In fact in (76) the nonlinear terms are the products $\varepsilon^2\sigma_{\alpha\beta}(\varepsilon)\varepsilon^{-2}e_{\alpha\beta}(u(\varepsilon))$, $2\varepsilon\sigma_{3\beta}(\varepsilon)\varepsilon^{-1}e_{3\beta}(u(\varepsilon))$ and $\sigma_{33}(\varepsilon)e_{33}(u(\varepsilon))$ where each factor is weakly convergent.

4 Conclusion

In this paper we considered the Adaptive Elasticity model derived in [2, 3], with the modification proposed in [5] and applied to a thin rod characterized by the dependence of the cross section on a small parameter. We applied the asymptotic expansion techniques described in [6], by assuming that the unknown displacement vector field admits an expansion of positive powers of the geometric small parameter. This assumption immediately induces an asymptotic expansion of the same type on the other unknown of the model, that is, the change in volume fraction of elastic material.

We then proved that the displacement vector field and the change in volume fraction strongly converge to the first terms of the corresponding asymptotic expansions. Moreover, the limit model, for which we proved existence and uniqueness, is a simplified adaptive elasticity model, with generalized Bernoulli-Navier equilibrium equations and a remodeling rate equation whose driving mechanism is the strain energy per unit volume, in good agreement with some of the most popular models used in practice.

Finally, in subsection 3.6 we showed that all the previous results can be particularized for the case of an adaptive elasticity model, with a remodeling rate equation depending linearly on the strain field, and we also derived additional weak convergence results for this linear case.

References

[1] P.G. Ciarlet, *Mathematical Elasticity, Vol.1: Three-Dimensional Elasticity*, Studies in Mathematics and its Applications, Vol.20, North-Holland, Amsterdam, 1988.

[2] S.C. Cowin and D.H. Hegedus, *Bone remodeling I: theory of adaptive elasticity*, Journal of Elasticity, 6, 3, 313-326 (1976).

[3] S.C. Cowin and D.H. Hegedus, *Bone remodeling II: small strain adaptive elasticity*, Journal of Elasticity, 6, 4, 337-352 (1976).

[4] S.C. Cowin and R.R. Nachlinger, *Bone remodeling III: uniqueness and stability in adaptive elasticity theory*, Journal of Elasticity, 8, 3, 285-295 (1978).

[5] J. Monnier and L. Trabucho, *An existence and uniqueness result in bone remodeling theory*, Computer Methods in Applied Mechanics and Engineering, 151, 539-544 (1998).

[6] L. Trabucho and J.M. Viaño, *Mathematical Modeling of Rods*, in Handbook of Numerical Analysis, pp.487-974, ed., P.G. Ciarlet & J.L Lions, North-Holland, Amsterdam, 1996.

[7] T. Valent, *Boundary Value Problems of Finite Elasticity*, Springer Tracts in Natural Philosophy, Vol. 31, Springer-Verlag, New York, 1988.