ON FUNCTORS WHICH ARE LAX EPIMORPHISMS

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ABSTRACT. We show that lax epimorphisms in the category Cat are precisely the functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ for which the functor $P^* : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ of composition with P is fully faithful. We present two other characterizations. Firstly, they are precisely the "absolutely dense" functors, i.e., functors P such that every object B of $\mathcal{B}$ is an absolute colimit of all arrows $P(E) \longrightarrow B$ for E in $\mathcal{E}$. Secondly, lax epimorphisms are precisely the functors P such that for every morphism f of $\mathcal{B}$ the category of all factorizations through objects of $P[\mathcal{E}]$ is connected.

A relationship between pseudoepimorphisms and lax epimorphisms is discussed.

1. Introduction

What are the epimorphisms of Cat, the category of small categories and functors? No simple answer is known, and the present paper indicates that this may be a "wrong question", disregarding the 2-categorical character of Cat. Anyway, with strong epimorphisms we have more luck: as proved in [2], they are precisely those functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ such that every morphism of $\mathcal{B}$ is a composite of morphisms in $P[\mathcal{E}]$.

Our paper is devoted to lax epimorphisms in the 2-category Cat. We follow the concept of pseudoepimorphism (and pseudomonomorphism) as presented in [3]: a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ is called a *lax epimorphism* provided that for every pair $Q_1, Q_2 : \mathcal{B} \longrightarrow \mathcal{C}$ of functors and every natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ there exists a unique natural transformation $v : Q_1 \longrightarrow Q_2$ with $u = vP$. Briefly, P is a lax epimorphism iff the functor

$$(\_) \cdot P : [\mathcal{B}, \mathcal{C}] \longrightarrow [\mathcal{E}, \mathcal{C}]$$

is fully faithful, for every small category $\mathcal{C}$.

Our first observation is that, instead of all small categories $\mathcal{C}$, one can simply take Set. That is, P is a lax epimorphism iff

$$P^* = (\_) \cdot P : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$$

is fully faithful. This is what Peter Johnstone called "connected functors" in his lecture at the Cambridge PSSL meeting in November 2000. He has asked for a characterization of conected functors, which has inspired the present paper. We provide two characterizations. Recall from [9] that a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ is called dense if every object B of

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$\mathcal{B}$ is a colimit of the diagram of all arrows $P(E) \longrightarrow B$ (more precisely, B is a canonical colimit of the diagram $P/B \longrightarrow \mathcal{B}$ forgetting the codomain). Let us call a functor P *absolutely dense* if every object B of $\mathcal{B}$ is an absolute colimit of the diagram of all arrows $P(E) \longrightarrow B$. The following conditions of a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between small categories will be proved equivalent:

(i) P is a lax epimorphism,

(ii) P is absolutely dense,

(iii) every morphism f of $\mathcal{B}$ has the property that the category $f \mathbin{/\!\!/} P$ of all factorizations of f through objects of $P[\mathcal{E}]$ is connected.

In (iii) above, the objects of $f \mathbin{/\!\!/} P$ are all triples (E, q, m) where E is an object of $\mathcal{E}$ and

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "q"'] & & B' \\ & P(E) \arrow[ur, "m"'] & \end{tikzcd}$$

is a commutative triangle in $\mathcal{B}$. Morphisms of $f \mathbin{/\!\!/} P$ from (E, q, m) to $(\bar{E}, \bar{q}, \bar{m})$ are all morphisms $e : E \longrightarrow \bar{E}$ of $\mathcal{E}$ such that the following diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "q"] \arrow[ddr, "\bar{q}"'] & & B' \\ & P(E) \arrow[ur, "m"] \arrow[d, "P(e)"] & \\ & P(\bar{E}) \arrow[uur, "\bar{m}"'] & \end{tikzcd}$$

commutes.

For the special case of $f = id_B$ we call the category $id_B \mathbin{/\!\!/} P$ a *splitting fibre* of B. (Recall that a fibre of B is the category of all E in $\mathcal{E}$ with $P(E) = B$. Now a splitting fibre is the category of all split subobjects $B \longrightarrow P(E)$ for E in $\mathcal{E}$.) We conclude that every lax epimorphism has all splitting fibres connected. In his PSSL lecture, P. Johnstone announced that every extremal epimorphism with connected splitting fibres is "connected", i.e., is a lax epimorphism. We show a simple example demonstrating that this sufficient condition is not necessary.

How is our concept related to other epimorphism concepts? We will see easy examples demonstrating that

$$\text{regular epimorphism} \not\Rightarrow \text{lax epimorphism} \not\Rightarrow \text{epimorphism}$$

and so there seems to be no connection to the "strict" concepts. Next, every lax epimorphism is a pseudoepimorphism (defined as above, except that the natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ is supposed to be a natural isomorphism — and then so is v).

1.1. Open Problem. Is every pseudoepimorphism a lax epimorphism?

1.2. Remark. For functors between preorders we prove in Proposition 3.1 that the answer is affirmative.

2. A Characterization of Lax Epimorphisms

2.1. Theorem. *For a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between small categories the following conditions are equivalent:*

1. *P is a lax epimorphism.*
2. *The functor $P^* = (\_) \cdot P : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ is fully faithful.*
3. *All the categories $f \,/\!\!/\, P$ for morphisms f of $\mathcal{B}$ are connected.*
4. *P is absolutely dense.*

2.2. Remark. A category $\mathcal{C}$ is called connected iff the graph whose nodes are the objects and whose arrows are the pairs C, C' of objects with $\mathcal{C}(C, C') \neq \emptyset$ is connected (i.e., has precisely one component — thus, it is nonempty and every pair of nodes can be connected by a non-directed path).

Proof. 1.⇒2. This is trivial: given functors $Q_1, Q_2 : \mathcal{B} \longrightarrow \mathsf{Set}$, let $\mathcal{C}$ be a small full subcategory of Set containing both images so that we have codomain-restrictions $Q_1', Q_2' : \mathcal{B} \longrightarrow \mathcal{C}$. By 1., for every natural transformation $u' : Q_1' \cdot P \longrightarrow Q_2' \cdot P$ there is a unique $v' : Q_1' \longrightarrow Q_2'$ with $u' = v'P$. This is equivalent to having, for every natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ a unique $v : Q_1 \longrightarrow Q_2$ with $u = vP$.

2.⇔3. The functor $P^* = (\_) \cdot P$ has a left adjoint, viz, the functor

$$L : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]$$

of left Kan extension along P. Therefore, P^* is full and faithful iff the counit

$$\varepsilon : L \cdot P^* \longrightarrow Id$$

of the adjunction $L \dashv P^*$ is a natural isomorphism.

Since every object of $[\mathcal{B}, \mathsf{Set}]$ is a colimit of hom-functors, and since $L \cdot P^*$ preserves colimits, it follows that ε is a (pointwise) isomorphism iff the component of ε at every $\mathcal{B}(B, \_)$, for B an object of $\mathcal{B}$, is an isomorphism.

This component can be described as follows: we express

$$P^*(\mathcal{B}(B, \_)) = \mathcal{B}(B, P\_) : \mathcal{E} \longrightarrow \mathsf{Set}$$

as a colimit of representable functors $\mathcal{E}(E, \_)$ indexed by all pairs (E, m) with E in $\mathcal{E}$ and $m : B \longrightarrow P(E)$ in $\mathcal{B}$. Then $G = L \cdot P^*(\mathcal{B}(B, \_))$ is a colimit of the corresponding diagram of all hom-functors $\mathcal{B}(P(E),\_)$ in $[\mathcal{B},\mathsf{Set}]$. We can describe $G:\mathcal{B}\longrightarrow\mathsf{Set}$ as follows: to every object X it assigns the set GX of all triples (E,q,m) where E is an object of $\mathcal{E}$ and

$$B \xrightarrow{m} P(E) \xrightarrow{q} X$$

are morphisms of $\mathcal{B}$, modulo the smallest equivalence $\approx$ merging (E,m,q) and $(\bar{E},\bar{m},\bar{q})$ whenever there exists $e:E\longrightarrow\bar{E}$ such that the following diagram

$$\begin{tikzcd} & P(E) \arrow[dr, "q"] \arrow[dd, "P(e)"] & \\ B \arrow[ur, "m"] \arrow[dr, "\bar{m}"'] & & X \\ & P(\bar{E}) \arrow[ur, "\bar{q}"'] & \end{tikzcd}$$

commutes. With this description of G, the counit

$$\varepsilon_{\mathcal{B}(B,\_)}:G\longrightarrow\mathcal{B}(B,\_)$$

is defined by assigning to the equivalence class of

$$B \xrightarrow{m} P(E) \xrightarrow{q} X$$

the composite $q\cdot m:B\longrightarrow X$. We see that $\varepsilon_{\mathcal{B}(B,\_)}$ is a natural isomorphism iff for every morphism $f:B\longrightarrow X$ there is a triple (E,q,m) in GX with $f=q\cdot m$, unique up to the above equivalence. This is precisely when the category $f\,/\!\!/\,P$ is connected.

3.$\Rightarrow$4. Firstly note that the assumption in 3. is "self-dual", i.e., we deduce from 2.$\Leftrightarrow$3. that the functor

$$(\_)\cdot P^{op}:[\mathcal{B}^{op},\mathsf{Set}]\longrightarrow[\mathcal{E}^{op},\mathsf{Set}]$$

is full and faithful. Thus, the composite

$$\mathcal{B} \xrightarrow{Y} [\mathcal{B}^{op},\mathsf{Set}] \xrightarrow{(\_)\cdot P^{op}} [\mathcal{E}^{op},\mathsf{Set}]$$

of $(\_)\cdot P^{op}$ with the Yoneda embedding Y is full and faithful. This means that P is dense, and this implies that every object B of $\mathcal{B}$ can be expressed as a colimit of a diagram $P:\mathcal{E}\longrightarrow\mathcal{B}$ weighted by $\mathcal{B}(P\_,B):\mathcal{E}^{op}\longrightarrow\mathsf{Set}$ (see Theorem 5.1 of [8]). To prove that P is absolutely dense, we verify that every functor $F:\mathcal{B}\longrightarrow\mathcal{X}$ (with $\mathcal{X}$ small) preserves the weighted colimits $B\cong\mathcal{B}(P\_,B)*P$. In fact, recall that $(\_)\cdot P^{op}$ is full and faithful and use $\mathcal{B}(P\_,B)=\mathcal{B}(\_,B)\cdot P^{op}$ to deduce the following isomorphisms

$$\begin{aligned} \mathcal{X}\Big(\mathcal{B}(P\_,B)*F\cdot P,X\Big) &\cong [\mathcal{E}^{op},\mathsf{Set}]\Big(\mathcal{B}(P\_,B),\mathcal{X}(F\cdot P\_,X)\Big) \\ &\cong [\mathcal{E}^{op},\mathsf{Set}]\Big(\mathcal{B}(\_,B)\cdot P^{op},\mathcal{X}(F\_,X)\cdot P^{op}\Big) \\ &\cong [\mathcal{B}^{op},\mathsf{Set}]\Big(\mathcal{B}(\_,B),\mathcal{X}(F\_,X)\Big) \\ &\cong \mathcal{X}(FB,X) \end{aligned}$$

natural in every object X in $\mathcal{X}$.

4.$\Rightarrow$1. Since we assume that P is absolutely dense, this means that a left Kan extension $\mathrm{Lan}_P P \cong Id_{\mathcal{B}}$ of P along itself is preserved by any functor $F : \mathcal{B} \longrightarrow \mathcal{X}$. Thus, for every pair $F, G : \mathcal{B} \longrightarrow \mathcal{X}$ we have isomorphisms

$$\begin{aligned} [\mathcal{B}, \mathcal{X}](F, G) &\cong [\mathcal{B}, \mathcal{X}](\mathrm{Lan}_P(F \cdot P), G) \\ &\cong [\mathcal{E}, \mathcal{X}](F \cdot P, G \cdot P) \end{aligned}$$

where the last isomorphism is induced by precomposing with P. But this means precisely that P is a lax epimorphism. ■

2.3. Examples. The following are examples of lax epimorphisms:

1. Coinserters: recall that a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$, together with a natural transformation $\alpha : P \cdot F \longrightarrow P \cdot G$ is called a *coinserter* in Cat of the pair $F, G : \mathcal{C} \longrightarrow \mathcal{E}$ iff the following two conditions are satisfied:

   (a) For every natural transformation $\beta : Q \cdot F \longrightarrow Q \cdot G$ with $Q : \mathcal{E} \longrightarrow \mathcal{D}$ there is a unique functor $H : \mathcal{B} \longrightarrow \mathcal{D}$ such that $H \cdot P = Q$ and $H\alpha = \beta$.

   (b) For every pair $H_1, H_2 : \mathcal{B} \longrightarrow \mathcal{D}$ of functors and every natural transformation $\gamma : H_1 \cdot P \longrightarrow H_2 \cdot P$ satisfying $(H_2\alpha)(\gamma F) = (\gamma G)(H_1\alpha)$ there is a unique natural transformation $\delta : H_1 \longrightarrow H_2$ with $\delta P = \gamma$.

   Thus, every coinserter is a lax epimorphism, since the second condition above is satisfied by every natural transformation $\gamma : H_1 \cdot P \longrightarrow H_2 \cdot P$.

2. The folllowing functor between preordered sets is a lax epimorphism which is not a coinserter:

$$\boxed{\bullet} \longrightarrow \boxed{\bullet \cong \bullet}$$

3. Categories of fractions: given a set Σ of morphisms in a small category $\mathcal{E}$, then the canonical functor

$$P_\Sigma : \mathcal{E} \longrightarrow \mathcal{E}[\Sigma^{-1}]$$

   into the category of fractions is a lax epimorphism (see, e.g., Lemma 1.2 of [4]).

4. Epimorphisms of small categories, which are one-to-one on objects, are lax epimorphisms. This is due to the second equivalent condition of Theorem 2.1 and Corollary 2.2 in [5].

A regular epimorphism in Cat need not be a lax epimorphism:

$$\boxed{\begin{matrix}\bullet \\ \bullet\end{matrix}} \longrightarrow \boxed{\bullet}$$

(See [2] for characterization of regular epimorphisms as precisely those functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ which are surjective on objects and such that every morphism in $\mathcal{B}$ is a composite of morphisms in $P[\mathcal{E}]$.)

3. Pseudoepimorphisms

3.1. Proposition. *For functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is a preordered set we have:*

P is a pseudoepimorphism iff it is a lax epimorphism.

Proof. Let P be a pseudoepimorphism. For every $x \leq y$ in $\mathcal{B}$ we prove that the category $\mathcal{C} = (x \leq y) /\!\!/ P$, which is the full subcategory of $\mathcal{E}$ on all E with $x \leq P(E) \leq y$, is connected.

Define a functor $F : \mathcal{B} \longrightarrow \mathsf{Set}$ on objects by

$$Fb = \begin{cases} 1+1, & \text{if } x \leq b \leq y \\ 1, & \text{if } x \leq b \not\leq y \\ \emptyset, & \text{otherwise} \end{cases}$$

and on morphisms by setting $Ff = id$ for every morphism $f : b \leq b'$ with $Fb = Fb'$.

The category $\mathcal{C}$ is nonempty because otherwise we would have two natural isomorphisms $\beta_1, \beta_2 : F \longrightarrow F$ with $\beta_1 \neq \beta_2$ and $\beta_1 P = id = \beta_2 P$ ($\beta_1 = id$ and β_2 is the transposition of $1+1$), in contradiction to P being a pseudoepimorphism.

Let $\mathcal{C}_0$ be a connected component of $\mathcal{C}$, we will prove that $\mathcal{C}_0 = \mathcal{C}$. We have a natural isomorphism

$$\alpha : F \cdot P \longrightarrow F \cdot P$$

whose components are

$$\alpha_E = \begin{cases} id, & \text{if } E \text{ is not in } \mathcal{C}_0 \\ t, & \text{if } E \text{ is in } \mathcal{C}_0 \end{cases}$$

where $t : 1+1 \longrightarrow 1+1$ swaps the two copies of 1. The naturality squares

$$\begin{tikzcd} FP(E) \arrow[r, "\alpha_E"] \arrow[d, "FPh"'] & FP(E) \arrow[d, "FPh"] \\ FP(E') \arrow[r, "\alpha_{E'}"'] & FP(E') \end{tikzcd}$$

commute for all $h : E \longrightarrow E'$: this is obvious except for the case that E is in $\mathcal{C}_0$ and E' is in $\mathcal{C} \setminus \mathcal{C}_0$, or vice versa, but that case does not happen because $\mathcal{C}_0$ is a connected component of $\mathcal{C}$.

There exists a natural isomorphism $\beta : F \longrightarrow F$ with $\alpha = \beta P$. The component β_x is t because choosing any E in $\mathcal{C}_0$, the following square

$$\begin{tikzcd} 1+1 \arrow[r, "\beta_x"] \arrow[d, "id=F(x\longrightarrow P(E))"'] & 1+1 \arrow[d, "id=F(x\longrightarrow P(E))"] \\ 1+1 \arrow[r, "t=(\beta P)_E"'] & 1+1 \end{tikzcd}$$

commutes. This proves that $\mathcal{C} = \mathcal{C}_0$: by choosing any $E \in \mathcal{C} \setminus \mathcal{C}_0$ we would obtain, analogously, $\beta_x = id$, which is impossible. ∎

3.2. Remark. Given a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ and an object B of $\mathcal{B}$ we can form a *splitting fibre* of B: it is the category whose objects are all pairs of morphisms

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E) \qquad \text{with } q \cdot m = id$$

where E is an object of $\mathcal{E}$. And morphisms into

$$B \underset{\bar{q}}{\overset{\bar{m}}{\rightleftarrows}} p(\bar{E})$$

are those morphisms $e : E \longrightarrow \bar{E}$ in $\mathcal{E}$ for which the following diagram

$$\begin{tikzcd}
& P(E) \arrow[dd, "P(e)"] \arrow[dr, "q"] & \\
B \arrow[ur, "m"] \arrow[dr, "\bar{m}"'] & & B \\
& P(\bar{E}) \arrow[ur, "\bar{q}"'] &
\end{tikzcd}$$

commutes.

Every lax epimorphism has all splitting fibres connected. In fact, these are just the categories $id_B \mathbin{/\!\!/} P$.

3.3. Proposition. *Let $P : \mathcal{E} \longrightarrow \mathcal{B}$ have connected splitting fibres and let*

$(*)$ *all morphisms in $\mathcal{B}$ be composites of isomorphisms and morphisms in $P[\mathcal{E}]$.*

Then P is a lax epimorphism.

Proof. The functor $P^* : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ is faithful. In fact, for distinct natural transformations $\alpha, \beta : F \longrightarrow G$ (where F, G are functors in $[\mathcal{B}, \mathsf{Set}]$) we find an object B with $\alpha_B \neq \beta_B$, and we choose an object

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre of B. Then $(\alpha P)_E \neq (\beta P)_E$. In fact: assuming the contrary, we get a contradiction:

$$\alpha_B = \alpha_B \cdot Fq \cdot Fm = Gq \cdot \alpha_E \cdot Fm = Gq \cdot \beta_E \cdot Fm = \beta_B.$$

The functor P^* is full: consider functors $F, G : \mathcal{B} \longrightarrow \mathsf{Set}$ and a natural transformation $\alpha = FP \longrightarrow GP$. Define, for every object B of $\mathcal{B}$, the morphism $\beta_B : FB \longrightarrow GB$ as follows: choose

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre and put

$$\beta_B = Gq \cdot \alpha_E \cdot Fm.$$

This is independent of the choice: to show this, we use the connectedness of the splitting fibre and verify only that given a morphism as in 3.2, then $Gq \cdot \alpha_E \cdot Fm = G\bar{q} \cdot \alpha_{\bar{E}} \cdot F\bar{m}$:

$$\begin{tikzcd}
 & FP(E) \arrow[r, "\alpha_E"] \arrow[dd, "FP(e)"'] & GP(E) \arrow[dr, "Gq"] \arrow[dd, "GP(e)"] & \\
FB \arrow[ur, "Fm"] \arrow[dr, "F\bar{m}"'] & & & GB \\
 & FP(\bar{E}) \arrow[r, "\alpha_{\bar{E}}"'] & GP(E) \arrow[ur, "G\bar{q}"'] &
\end{tikzcd}$$

Consequently,

$$\beta_{P(E)} = \alpha_E \quad \text{for each } E \text{ in } \mathcal{E}$$

because we choose $q = m = id$. To show that β_B is natural in B, it is sufficient — due to $(*)$ — to consider all isomorphisms and all morphisms in $P[\mathcal{E}]$.

Let $h : B \longrightarrow B'$ be an isomorphism. Given

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre of B, then

$$B' \underset{h \cdot q}{\overset{m \cdot h^{-1}}{\rightleftarrows}} P(E)$$

lies in the splitting fibre of B', thus,

$$\beta_{B'} = G(h \cdot q) \cdot \alpha_E \cdot F(m \cdot h^{-1})$$

which implies

$$Gh \cdot \beta_B = Gh \cdot Gq \cdot \alpha_E \cdot Fm = \beta_{B'} \cdot Fh.$$

Let $h : B \longrightarrow B'$ have the form $h = P(k)$ for $k : E \longrightarrow E'$ in $\mathcal{E}$. Since $P(E) = B$ and $P(E') = B'$, we conclude

$$Gh \cdot \beta_B = GP(k) \cdot \alpha_E = \alpha_{E'} \cdot FP(k) = \beta_{B'} \cdot Fh.$$

Thus, β is natural. ∎

3.4. PROPOSITION. *For a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between finite preorders the following conditions are equivalent:*

1. *P is a lax epimorphism.*
2. *P has connected splitting fibres and satisfies condition $(*)$ of Proposition 3.3.*

Proof. 1.⇒2. Every morphism in $\mathcal{B}$ is, since $\mathcal{B}$ is a finite preorder, a composite of isomorphism and coverings $x_0 \longrightarrow y_0$ (i.e., $x_0 < y_0$ and no x in $\mathcal{B}$ fulfils $x_0 < x < y_0$). Thus, it is sufficient to prove condition $(*)$ of Proposition 3.3 for every covering $x_0 \longrightarrow y_0$.

Assuming the contrary, we extend $\mathcal{B}$ to a category $\mathcal{C}$ by adding, for every pair of objects $x \cong x_0$ and $y \cong y_0$ a new morphism

$$r_{xy} : x \longrightarrow y.$$

The composition in $\mathcal{C}$ extends that of $\mathcal{B}$ by the following rules: given $x \cong x_0$ and $y \cong y_0$ then for every $z \leq x$ put

$$r_{xy} \cdot (z \longrightarrow x) = \begin{cases} r_{zy}, & \text{if } z \cong x \\ z \longrightarrow y, & \text{otherwise} \end{cases}$$

and for every $y \geq z$ put

$$(y \longrightarrow z) \cdot r_{xy} = \begin{cases} r_{xz}, & \text{if } y \cong z \\ x \longrightarrow z, & \text{otherwise} \end{cases}$$

Since $x_0 \longrightarrow y_0$ (thus, $x \longrightarrow y$) is not a composite of isomorphisms and morphisms in $P[\mathcal{E}]$, we have a well-defined functor $Q_1 : \mathcal{B} \longrightarrow \mathcal{C}$ with $Q_1(x \longrightarrow y) = r_{xy}$ for all $x \cong x_0$, $y \cong y_0$ and otherwise Q_1 is the identity function. Let $Q_2 : \mathcal{B} \longrightarrow \mathcal{C}$ denote the inclusion functor. Then $Q_1 \cdot P = Q_2 \cdot P$. But there exists no natural transformation $v : Q_1 \longrightarrow Q_2$ because the square

$$\begin{tikzcd} x_0 \arrow[r, equal, "v_{x_0}"] \arrow[d, "x_0 \longrightarrow y_0"'] & x_0 \arrow[d, "r_{x_0 y_0}"] \\ y_0 \arrow[r, equal, "v_{y_0}"'] & y_0 \end{tikzcd}$$

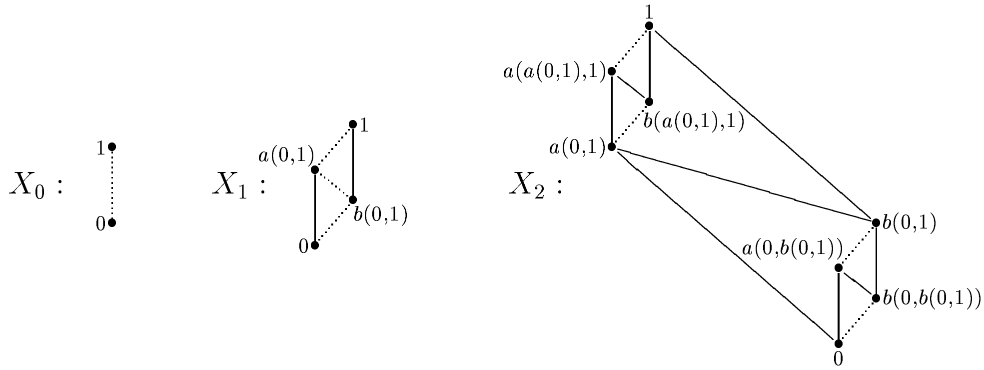
does not commute.

2.⇒1. This follows from Proposition 3.3. ■

3.5. REMARK. The above condition $(*)$ together with surjectivity characterizes finite quotients (i.e., regular epimorphisms) in the category Top_0 of topological T_0 spaces. More precisely, if we identify a finite topological space with the induced order ($x \leq y$ iff x lies in the closure of y), then continuous functions are precisely the functors. And quotients are precisely the surjective functors satisfying $(*)$, as proved in [6] and [7].

3.6. EXAMPLE. We show a lax epimorphism $P : \mathcal{E} \longrightarrow \mathcal{B}$ between posets which does not satisfy condition $(*)$ of Proposition 3.3.

We define a set $X = \bigcup_{n\in\omega} X_n$ with two partial orderings $(\leq) \subseteq (\sqsubseteq)$ on it, where $(\leq) = \bigcup_{n\in\omega}(\leq_n)$ and $(\sqsubseteq) = \bigcup_{n\in\omega}(\sqsubseteq_n)$, by induction. The first three steps are illustrated below (where $\leq$ is indicated by full lines and $\sqsubseteq$ by dotted lines).



FIRST STEP. $X_0 = \{0,1\}$, $\leq_0$ is discrete, $\sqsubseteq_0$ is the chain $0 \sqsubseteq_0 1$.
INDUCTION STEP. Let Q_n be the set of all $\sqsubseteq_n$-coverings (i.e., pairs $x \sqsubseteq_n y$ with no element z satisfying $x \sqsubset_n z \sqsubset_n y$) which are not related by $\leq_n$. Let X_{n+1} be obtained by adding to X_n elements $a(x,y)$ and $b(x,y)$ for each $(x,y) \in Q_n$. Then $\leq_{n+1}$ is the reflective and transitive closure of $\leq_n$ extended by

$$x \leq_{n+1} a(x,y) \quad \text{and} \quad b(x,y) \leq_{n+1} y \quad \text{and} \quad b(x,y) \leq_{n+1} a(x,y)$$

for all $(x,y) \in Q_n$. And $\sqsubseteq_{n+1}$ is the transitive closure of $(\sqsubseteq_n) \cup (\leq_{n+1})$ extended by

$$a(x,y) \leq_{n+1} y \quad \text{and} \quad x \leq_{n+1} b(x,y)$$

for all $(x,y) \in Q_n$.

CLAIM. $id : \langle X, \leq\rangle \longrightarrow \langle X, \sqsubseteq\rangle$ is a lax epimorphism.

That is, given a pair $x \sqsubseteq y$ in X, then the subposet V of $\langle X, \leq\rangle$ of all z with $x \sqsubseteq z \sqsubseteq y$ is connected. We prove this by induction on n with $x, y \in X_n$. If $n = 0$, the only interesting case is $x = 0$, $y = 1$ and $V = X$. The poset $\langle X, \leq\rangle$ is indeed connected, see Theorem 2.1 because $\langle X_1, \leq_1\rangle$ is connected, and every new element added to X_{k+1}, $k > 0$, is connected by $\leq_k$ to some element of X_k. For the induction case, observe that since $\langle X_n, \sqsubseteq_n\rangle$ is a finite poset, every morphism is a composite of coverings. And the elements we add at any stage later are only added in between coverings. (More precisely, given $z \in X_k$, for $k > n$, with $x \sqsubseteq_k z \sqsubseteq_k y$ there exists a covering $x' \sqsubseteq_n y'$ with $x \sqsubseteq_n x' \sqsubseteq_k z \sqsubseteq_k y' \sqsubseteq_n y$.) Thus, it is sufficient to prove that $\langle V, \leq\rangle$ is connected assuming that $x \sqsubseteq_n y$ is a covering. If $x \leq_n y$ then $V = \{x, y\}$ is connected. If $(x,y) \in Q_n$, the argument is as for 0, 1 at the beginning: $V \cap X_{n+1} = \{x, y, a(x,y), b(x,y)\}$ is connected, and every new element added to $V \cap X_{k+1}$, $k > n$, is connected by $\leq_k$ to some element of $V \cap X_k$.

CLAIM. The morphism $0 \longrightarrow 1$ of $\langle X, \sqsubseteq\rangle$ is not a composite of isomorphisms and morphisms in $P[\mathcal{E}]$ — in other words, $0 \not\lesssim 1$. This is clear.

4. Faithfulness of P^*

4.1. Proposition. *For every functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between small categories the following conditions are equivalent:*

1. *P^* is faithful.*
2. *P^* is conservative (i.e., reflects isomorphisms).*
3. *P^* is monadic.*
4. *Every object of $\mathcal{B}$ is a retract of an object in $P[\mathcal{E}]$.*

Proof. 1.⇒2. Since P^* is a right adjoint of $L : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]$ (the functor of left Kan extension), faithfulness means that the counit is an epimorphism in $[\mathcal{B}, \mathsf{Set}]$; and since epimorphisms in $[\mathcal{B}, \mathsf{Set}]$ are regular, we conclude that the comparison functor $K : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]^T$ of the monad T of that adjunction

$$\begin{tikzcd} {[\mathcal{E}, \mathsf{Set}]} \arrow[rr, "K"] \arrow[dr, "P^*"'] & & {[\mathcal{B}, \mathsf{Set}]^T} \arrow[dl, "U"] \\ & {[\mathcal{B}, \mathsf{Set}]} & \end{tikzcd}$$

is full and faithful, thus, conservative. Since the forgetful functor $U : [\mathcal{B}, \mathsf{Set}]^T \longrightarrow [\mathcal{B}, \mathsf{Set}]$ is conservative, it follows that so is $P^* = U \cdot K$.

2.⇒3. This is clear from Beck's Theorem: P^* preserves coequalizers (in fact, colimits) and has a left adjoint.

3.⇒4. This is analogous to the proof of 2.⇔3. in Theorem 2.1: here ε is an epitransformation (because P^* is faithful) and we conclude that $id_B : B \longrightarrow B$ has a preimage under $\varepsilon_{\mathcal{B}(B,\_)}$, i.e., there are

$$B \xrightarrow{m} P(E) \xrightarrow{q} B$$

with $q \cdot m = id_B$.

4.⇒1. Let $\alpha, \beta : F \longrightarrow G$ be different morphisms of $[\mathcal{B}, \mathsf{Set}]$. We are to prove $\alpha P \neq \beta P$. Given an object B with $\alpha_B \neq \beta_B$, find

$$B \xrightarrow{m} P(E) \xrightarrow{q} B$$

with $q \cdot m = id_B$. Since Fq is a split epimorphism, we conclude that $\alpha_B \cdot Fq \neq \beta_B \cdot Fq$, or, equivalently, $Gq \cdot \alpha_{P(E)} \neq Gq \cdot \beta_{P(E)}$, thus $(\alpha P)_E \neq (\beta P)_E$. ■

4.2. Remark. If P is a pseudoepimorphism, then P^* is obviously faithful. But the converse does not hold, e.g., for the embedding

$$P : \boxed{\begin{array}{c}\bullet 1\\ \\ \bullet 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\bullet 1\\ \uparrow\\ \bullet 0\end{array}}$$

which is certainly no pseudoepimorphism, P^* is faithful.

4.3. Remark.

(a) Recall that P^* is fully faithful iff each of the categories $f \mathbin{/\!\!/} P$ is connected, i.e., iff every morphism f of $\mathcal{B}$ has a factorization through some object of $P[\mathcal{E}]$ unique up to the equivalence $\approx$ of Theorem 2.1:

$$(E, m, q) \approx (\bar{E}, \bar{m}, \bar{q}) \quad \text{iff} \quad \bar{m} = P(e) \cdot m, \; q = \bar{q} \cdot P(e) \text{ for some morphism } e \text{ in } \mathcal{E}.$$

Now P^* is faithful iff each of the categories $f \mathbin{/\!\!/} P$ is nonempty, i.e., iff every morphism of $\mathcal{B}$ has a factorization through some object of $P[\mathcal{E}]$. In fact, given $f : B \longrightarrow B'$, choose a retraction $q : P(E) \longrightarrow B$ then, given $m : B \longrightarrow P(E)$ with $q \cdot m = id$, we factorize

$$f \equiv B \xrightarrow{m} P(E) \xrightarrow{f \cdot q} B'$$

(b) We can also characterize functors such that P^* is full. These are precisely the functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ such that for every object B in $\mathcal{B}$ there exists an object E_0 in $\mathcal{E}$ and morphisms

$$B \xrightarrow{m_0} P(E_0) \xrightarrow{q_0} B$$

with the following property: given morphisms

$$B \xrightarrow{m} P(E) \xrightarrow{q} X$$

in $\mathcal{B}$ then

$$(E, m, q) \approx (E_0, m_0, q \cdot m \cdot q_0).$$

In fact, in the adjunction $L \dashv P^*$ we have P^* full iff ε is componentwise a split monomorphism (see 19.4 in [1]). This is the case iff the components

$$\varepsilon_{\mathcal{B}(B, \_)} : G \longrightarrow \mathcal{B}(B, \_)$$

(see the proof of Theorem 2.1) are, for all objects B in $\mathcal{B}$, split monomorphisms. To give a natural transformation $\alpha : \mathcal{B}(B, \_) \longrightarrow G$ with $\alpha \cdot \varepsilon_{\mathcal{B}(B, \_)} = id$ means precisely to give $(E_0, m_0, q_0) = \alpha_B(id_B)$ as above.

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