The generalized Lie bialgebroid of a strict Jacobi-Nijenhuis manifold

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Abstract

We associate with each differentiable manifold M equipped with a Jacobi structure and a Nijenhuis operator, a pair of Lie algebroids in duality. We show that this pair constitute a generalized Lie bialgebroid if and only if M is a strict Jacobi-Nijenhuis manifold.

Mathematics Subject Classification (2000): 17B62, 53D05, 53D10, 53D17.

Key words: *Jacobi-Nijenhuis manifold, Poisson-Nijenhuis manifold, Lie algebroid, Lie bialgebroid.*

1 Introduction

The notion of Jacobi-Nijenhuis manifold was introduced in [13] by J. Marrero, J. Monterde and E. Padrón. Their definition is a generalization of the Poisson-Nijenhuis structure presented in [14]. In this paper we introduce the notion of *strict* Jacobi-Nijenhuis manifold, which seems to be the natural generalization of the definition of Poisson-Nijenhuis manifold initially given by F. Magri and C. Morosi in [12], here called strict Poisson-Nijenhuis manifold. Obviously, every strict Jacobi-Nijenhuis manifold is a Jacobi-Nijenhuis manifold in the sense of [13]. An intermediate definition of Jacobi-Nijenhuis structure was used in [16] and [17].

It is well known that when a Poisson manifold (M, Λ) is equipped with a Nijenhuis tensor N, we can associate with this manifold two Lie algebroids structures, one defined on the cotangent bundle T^*M of M and the other on the tangent bundle TM of M. Using the notion of Lie bialgebroid, which was introduced by K. Mackenzie and P. Xu in [11], Y. Kosmann-Schwarzbach established, in [7], a characterization of strict Poisson-Nijenhuis manifolds: (M, Λ, N) is a strict Poisson-Nijenhuis manifold if and only if the two Lie algebroids mentioned above constitute a Lie bialgebroid. The aim of this paper is to show that a similar relation can be obtained when a differentiable manifold is equipped with a Jacobi structure and a Nijenhuis operator. However, the associated structure is no more a Lie bialgebroid, but a generalized Lie bialgebroid. This last notion was introduced by D. Iglesias and J. Marrero in [2] and it admits the Lie bialgebroid as a particular case. Generalized Lie bialgebroids are closely related with Jacobi structures. In fact, it was proved in [2] that with each Jacobi manifold one can associate, in a certain manner, a

*Research partially supported by CMUC-FCT and PRAXIS.

generalized Lie bialgebroid and that the base manifold of a generalized Lie bialgebroid possesses a Jacobi structure.

We must stress that the first relationship between Jacobi manifolds and Lie algebroids was established by Y. Kerbrat and Z. Souici-Benhammadi in [3]. They showed that there exists a Lie algebroid structure on the 1-jet bundle of a Jacobi manifold.

Taking into account that we can associate with each strict Jacobi-Nijenhuis manifold a strict Poisson-Nijenhuis manifold, as it was shown in [17], and using the techniques of [2], we deduce from the result of Y. Kosmann-Scharzbach, a characterization of strict Jacobi-Nijenhuis manifolds by means of generalized Lie bialgebroids.

The paper is divided into 5 sections. In sections 2 and 3 we make a review of the essential results concerning Lie algebroids, Poisson-Nijenhuis manifolds and Jacobi-Nijenhuis manifolds, needed in the sequel. In section 4 we associate with each Jacobi-Nijenhuis manifold $(M,(\Lambda,E),\mathcal{N})$ two Lie algebroids over M, that are in duality. From these Lie algebroids, and using the procedures presented in [2], we deduce two Lie algebroids over $M\times\mathbb{R}$. We show that a manifold endowed with a Jacobi structure and a Nijenhuis operator is a strict Jacobi-Nijenhuis manifold if and only if these two Lie algebroids over $M\times\mathbb{R}$ form a Lie bialgebroid. Finally, in section 5, we rewrite this results in terms of a generalized Lie bialgebroid (over M).

Notation: In the following, we will denote by M a C^∞-differentiable manifold of finite dimension, by $C^\infty(M)$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$, $k\in\mathbb{N}$, the space of k-forms on M, and by $\mathcal{V}^k(M)$, $k\in\mathbb{N}$, the space of skew-symmetric contravariant k-tensors on M.

2 Lie algebroids and Poisson-Nijenhuis manifolds

In this section we review some results on Lie algebroids and Lie bialgebroids and their relation with Poisson-Nijenhuis manifolds.

A *Lie algebroid* $(A,[.,.],\rho)$ over a manifold M is a vector bundle A over M together with a bundle map $\rho : A\to TM$ and a Lie algebra structure $[.,.]$ on the space $\Gamma(A)$ of the global cross sections such that

i) the induced map $\Gamma(\rho):\Gamma(A)\to\mathcal{V}^1(M)$ is a Lie algebra homomorphism;

ii) for any $f\in C^\infty(M)$ and $X,Y\in\Gamma(A)$, then

$$[X,fY]=f[X,Y]+(\rho(X).f)Y.$$

The map ρ is called the *anchor map* and usually the map $\Gamma(\rho)$ is denoted by ρ.

Example 2.1 If M is a differentiable manifold, then the triple $(TM,[.,.],Id_{TM})$ is a Lie algebroid over M.

Example 2.2 Let N be a tensor field of type $(1,1)$ on a manifold M with vanishing Nijenhuis torsion, *i.e.*, for any vector fields X and Y on M,

$$\mathcal{T}(N)=[NX,NY]-N([NX,Y]+[X,NY]-N[X,Y])=0.$$

Then the triple $(TM,[.,.]_N,N)$ is a Lie algebroid over M, where $[.,.]_N$ is the Lie bracket on $\mathcal{V}^1(M)$ given by

$$[X,Y]_N=[NX,Y]+[X,NY]-N[X,Y],\quad X,Y\in\mathcal{V}^1(M). \tag{1}$$

A tensor field N of type $(1,1)$ with vanishing Nijenhuis torsion is called a *Nijenhuis tensor*.

Example 2.3 Let (M, Λ) be a Poisson manifold and

$$\Lambda^\sharp : T^*M \to TM, \quad < \beta, \Lambda^\sharp(\alpha) >= \Lambda(\alpha, \beta), \quad \alpha, \beta \in T^*M,$$

the bundle map associated with the Poisson tensor Λ, which can also be seen as a homomorphism of $C^\infty(M)$-modules, $\Lambda^\sharp : \Omega^1(M) \to \mathcal{V}^1(M)$.

Then the triple $(T^*M, [.,.]_\Lambda, \Lambda^\sharp)$ is a Lie algebroid over M, where $[.,.]_\Lambda$ is the Lie bracket of 1-forms given, for all $\alpha, \beta \in \Omega^1(M)$, by

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\sharp(\alpha)}\beta - \mathcal{L}_{\Lambda^\sharp(\beta)}\alpha - d(\Lambda(\alpha, \beta)). \tag{2}$$

When $(A, [.,.], \rho)$ is a Lie algebroid over a manifold M, we may define a differential d on the space of sections of the algebra bundle $\Lambda A^* = \oplus_{k\in\mathbb{Z}}\Lambda^k A^*$ of the dual bundle of A. Explicitly, if $\theta \in \Gamma(\Lambda^k A^*)$ then $d\theta \in \Gamma(\Lambda^{k+1}A^*)$ and for any $X_1, \dots, X_{k+1} \in \Gamma(A)$,

$$\begin{aligned} d\theta(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k}(-1)^{i+1}\rho(X_i).\,\theta(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\theta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}). \end{aligned} \tag{3}$$

The triple $(\Gamma(\Lambda A^*), \wedge, d)$ is a differential graded algebra.

For the given examples of Lie algebroids, we have:

- the differential of $(TM, [.,.], Id_{TM})$ is the de Rham differential;
- the differential of $(TM, [.,.]_N, N)$ is $d_N = [i_N, d]$, where $[.,.]$ is the graded commutator, d is the de Rham differential and i_N is the derivation of degree zero defined, for any $\beta \in \Omega^k(M)$, by
$$i_N\beta(X_1, \cdots, X_k) = \sum_{i=1}^{k}\beta(X_1, \cdots, NX_i, \cdots, X_k), \quad X_1, \cdots, X_k \in \mathcal{V}^1(M);$$
(If $\beta \in \Omega^1(M)$, $d_N(\beta) = i_N d\beta - d({}^tN\beta)$.)
- the differential of $(T^*M, [.,.]_\Lambda, \Lambda^\sharp)$, (M, Λ) being a Poisson manifold, is the Lichnerowicz-Poisson differential $d_\Lambda = [\Lambda, .]$.

Furthermore, when $(A, [.,.], \rho)$ is a Lie algebroid, the Lie bracket on $\Gamma(A)$ can be extended to the algebra of sections of ΛA, $\Gamma(\Lambda A) = \oplus_{k\in\mathbb{Z}}\Gamma(\Lambda^k A)$. The result is a graded Lie bracket defined on the multivectors of the Lie algebroid, which is called the Schouten bracket of the Lie algebroid. We will also denote the Schouten bracket of the Lie algebroid by $[.,.]$. The triple $(\Gamma(\Lambda A), \wedge, [.,.])$ is a Gerstenhaber algebra. For more details see [6].

The notion of *Lie bialgebroid* was introduced by K. Mackenzie and P. Xu in [11].

Suppose that the vector bundle $(A, [.,.], \rho)$ and its dual vector bundle $(A^*, [.,.]_*, \rho_*)$ are both Lie algebroids over a manifold M. Let d (resp. d_*) denote the differential (3) associated with A (resp. A^*). The pair (A, A^*) is a *Lie bialgebroid* if for all $X, Y \in \Gamma(A)$,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y] \tag{4}$$

or, equivalently, if for all $\alpha, \beta \in \Gamma(A^*)$,

$$d[\alpha, \beta]_* = [d\alpha, \beta]_* + [\alpha, d\beta]_*. \tag{5}$$

Example 2.4 Given a Poisson manifold (M,Λ), let us consider the two Lie algebroids $(TM,[.,.],Id_{TM})$ and $(T^*M,[.,.]_\Lambda,\Lambda^\sharp)$, presented in examples 2.1 and 2.3, respectively. Then, the pair (TM,T^*M) is a Lie bialgebroid, [11].

A *Poisson-Nijenhuis* manifold (M,Λ,N) is a Poisson manifold (M,Λ) equipped with a Nijenhuis tensor N verifying the following compatibility conditions:

i) $N\Lambda^\sharp = \Lambda^\sharp . {}^tN$,

ii) the map $\Lambda^\sharp \circ C(\Lambda,N) : \Omega^1(M)\times\Omega^1(M) \to \Omega^1(M)$ identically vanishes on M,

where ${}^tN$ stands for the transpose of N and $C(\Lambda,N)$ is the Magri-Morosi concomitant which is given, for all $\alpha,\beta\in\Omega^1(M)$, by

$$C(\Lambda,N)(\alpha,\beta) = [\alpha,\beta]_{N\Lambda} - [{}^tN\alpha,\beta]_\Lambda - [\alpha,{}^tN\beta]_\Lambda + {}^tN[\alpha,\beta], \tag{6}$$

where $[.,.]_\Lambda$ (resp. $[.,.]_{N\Lambda}$) is the bracket (2) associated with Λ (resp. $N\Lambda$).

If condition ii) is replaced by

ii') $C(\Lambda,N)=0$,

we say that (M,Λ,N) is a *strict Poisson-Nijenhuis* manifold. This last one is the definition introduced by F. Magri and C. Morosi in [12], while the first version is closer to the one used in [14].

Given a Poisson-Nijenhuis manifold (M,Λ,N), we have both a Nijenhuis tensor and a Poisson structure on M. So, we may associate with it two Lie algebroids whose structures are given respectively by examples 2.2 and 2.3. The next theorem, due to Y. Kosmann-Schwarzbach, shows how the compatibility conditions of a strict Poisson-Nijenhuis structure are related with the notion of Lie bialgebroid.

Theorem 2.5 ([7]) *Let (M,Λ) be a Poisson manifold and N a Nijenhuis tensor on M. Then, (M,Λ,N) is a strict Poisson-Nijenhuis manifold* if and only if *the pair* $((TM,[.,.]_N,N),(T^*M,[.,.]_\Lambda,\Lambda^\sharp))$ *is a Lie bialgebroid.*

3 Jacobi-Nijenhuis manifolds

The notion of Jacobi-Nijenhuis manifold was introduced by J. Marrero and *al.* in [13]. In this paper we will consider a more strict definition, and we will call it *strict* Jacobi-Nijenhuis manifold. This section is dedicated to a review of the essential results concerning the (strict) Jacobi-Nijenhuis manifolds. For more details about this structure, see [17].

Let M be a C^∞-differentiable manifold and $\mathcal{N} : \mathcal{V}^1(M)\times C^\infty(M) \to \mathcal{V}^1(M)\times C^\infty(M)$ a $C^\infty(M)$-linear map defined, for all $(X,f)\in\mathcal{V}^1(M)\times C^\infty(M)$, by

$$\mathcal{N}(X,f) = (NX + fY, \langle\gamma,X\rangle + gf), \tag{7}$$

where N is a tensor field of type (1,1) on M, $Y\in\mathcal{V}^1(M)$, $\gamma\in\Omega^1(M)$ and $g\in C^\infty(M)$. $\mathcal{N} := (N,Y,\gamma,g)$ can be considered as a vector bundle map, $\mathcal{N} : TM\times\mathbb{R} \to TM\times\mathbb{R}$. Since the space $\mathcal{V}^1(M)\times C^\infty(M)$ endowed with the bracket

$$[(X,f),(Z,h)] = ([X,Z], X\cdot h - Z\cdot f), \tag{8}$$

$(X,f),(Z,h)\in\mathcal{V}^1(M)\times C^\infty(M)$, is a real Lie algebra, we may define the *Nijenhuis torsion* $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$,

$$
\begin{aligned}
\mathcal{T}(\mathcal{N})\,((X,f),(Z,h)) &= [\mathcal{N}(X,f),\mathcal{N}(Z,h)] - \mathcal{N}[\mathcal{N}(X,f),(Z,h)] \\
&- \mathcal{N}[(X,f),\mathcal{N}(Z,h)] + \mathcal{N}^2[(X,f),(Z,h)], \qquad (9)
\end{aligned}
$$

$(X,f),(Z,h) \in \mathcal{V}^1(M)\times C^\infty(M)$. When $\mathcal{T}(\mathcal{N})$ identically vanishes, we say that $\mathcal{N}$ is a *Nijenhuis operator* on M.

Now let (M,Λ,E) be a Jacobi manifold. We denote by $(\Lambda,E)^\# : T^*M\times \mathbb{R} \to TM\times\mathbb{R}$, the vector bundle map associated with (Λ,E), i.e., for any section α of T^*M and $f\in C^\infty(M)$,

$$
(\Lambda,E)^\#(\alpha,f) = (\Lambda^\#(\alpha) + fE, -\langle\alpha,E\rangle). \qquad (10)
$$

This vector bundle map can be considered as a homomorphism of $C^\infty(M)$-modules, $(\Lambda,E)^\# : \Omega^1(M)\times C^\infty(M) \to \mathcal{V}^1(M)\times C^\infty(M)$.

The space $\Omega^1(M)\times C^\infty(M)$ possesses a real Lie algebra structure with a bracket $[.,.]_{(\Lambda,E)}$ defined as follows, (cf. [3]): for all $(\alpha,f),(\beta,h)\in\Omega^1(M)\times C^\infty(M)$,

$$
[(\alpha,f),(\beta,h)]_{(\Lambda,E)} := (\gamma,r), \qquad (11)
$$

where

$$
\begin{aligned}
\gamma &:= \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - h\mathcal{L}_E\alpha - i_E(\alpha\wedge\beta), \\
r &:= -\Lambda(\alpha,\beta) + \Lambda(\alpha,dh) - \Lambda(\beta,df) + \langle f dh - h df, E\rangle.
\end{aligned}
$$

Suppose now that M is equipped with a Jacobi structure (Λ,E) and a Nijenhuis operator $\mathcal{N}$ and consider a tensor field Λ_1 of type $(2,0)$ and a vector field E_1 on M, defined by

$$
(\Lambda_1,E_1)^\# = \mathcal{N}\circ(\Lambda,E)^\#. \qquad (12)
$$

Definition 3.1 *A* Jacobi-Nijenhuis manifold $(M,(\Lambda,E),\mathcal{N})$ *is a Jacobi manifold* (M,Λ,E) *with a Nijenhuis operator* $\mathcal{N}$ *verifying the following compatibility conditions:*

i) $\mathcal{N}\circ(\Lambda,E)^\# = (\Lambda,E)^\#\circ {}^t\mathcal{N}$,

ii) the map $(\Lambda,E)^\#\circ\mathcal{C}((\Lambda,E),\mathcal{N}) : (\Omega^1(M)\times C^\infty(M))^2 \to \mathcal{V}^1(M)\times C^\infty(M)$ *identically vanishes on M,*

where ${}^t\mathcal{N}$ *is the transpose of* $\mathcal{N}$ *and* $\mathcal{C}((\Lambda,E),\mathcal{N})$ *is the concomitant of* (Λ,E) *and* $\mathcal{N}$ *which is given, for all* $(\alpha,f),(\beta,h)\in\Omega^1(M)\times C^\infty(M)$, *by*

$$
\begin{aligned}
\mathcal{C}((\Lambda_0,E_0),\mathcal{N})((\alpha,f),(\beta,h)) &= [(\alpha,f),(\beta,h)]_{(\Lambda_1,E_1)} - [{}^t\mathcal{N}(\alpha,f),(\beta,h)]_{(\Lambda,E)} \\
&- [(\alpha,f),{}^t\mathcal{N}(\beta,h)]_{(\Lambda,E)} + {}^t\mathcal{N}[(\alpha,f),(\beta,h)]_{(\Lambda,E)}, \qquad (13)
\end{aligned}
$$

where $[.,.]_{(\Lambda,E)}$ *(resp.* $[.,.]_{(\Lambda_1,E_1)}$*) is the bracket (11) associated with* (Λ,E)*(resp.* (Λ_1,E_1)*).*

If condition ii) is replaced by

ii') $\mathcal{C}((\Lambda,E),\mathcal{N}) = 0$

we say that $(M,(\Lambda,E),\mathcal{N})$ *is a* strict Jacobi-Nijenhuis manifold.

Remarks 3.2 **A)** The bivector Λ_1 given by (12) is skew-symmetric if and only if $\mathcal{N}\circ(\Lambda,E)^\# = (\Lambda,E)^\#\circ {}^t\mathcal{N}$;

B) When Λ_1 is skew-symmetric, the pair (Λ_1, E_1) given by (12) defines a Jacobi structure on M if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$,

$$\mathcal{T}(\mathcal{N})\left((\Lambda, E)^{\#}(\alpha, f), (\Lambda, E)^{\#}(\beta, h)\right) =$$
$$= \mathcal{N} \circ (\Lambda, E)^{\#}\left(\mathcal{C}\left((\Lambda, E), \mathcal{N}\right)\left((\alpha, f), (\beta, h)\right)\right);$$

C) In the case where (Λ_1, E_1) is a Jacobi structure, it is compatible with (Λ, E) (*i.e.*, the sum $(\Lambda + \Lambda_1, E + E_1)$ is again a Jacobi structure, see [15]) if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$,

$$(\Lambda, E)^{\#}\left(\mathcal{C}\left((\Lambda, E), \mathcal{N}\right)\left((\alpha, f), (\beta, h)\right)\right) = 0.$$

Theorem 3.3 ([13]) *Let $((\Lambda_0, E_0), \mathcal{N})$ be a Jacobi-Nijenhuis structure on a differentiable manifold M. Then, there exists a hierarchy $((\Lambda_k, E_k), k \in \mathbb{N})$ of Jacobi structures on M, which are pairwise compatible. For all $k \in \mathbb{N}$, (Λ_k, E_k) is the Jacobi structure associated with the vector bundle map $(\Lambda_k, E_k)^{\#}$ given by $(\Lambda_k, E_k)^{\#} = \mathcal{N}^k \circ (\Lambda_0, E_0)^{\#}$. Moreover, for all $k, l \in \mathbb{N}$, the pair $((\Lambda_k, E_k), \mathcal{N}^l)$ defines a Jacobi-Nijenhuis structure on M.*

In order to show the relation between Jacobi-Nijenhuis manifolds and Poisson-Nijenhuis structures, we recall that with each Jacobi manifold (M, Λ, E) we may associate a Poisson manifold $(\tilde{M}, \tilde{\Lambda})$, with

$$\tilde{M} = M \times \mathbb{R}, \quad \tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E) \tag{14}$$

where t is the usual coordinate on $\mathbb{R}$, [9]. The manifold $(\tilde{M}, \tilde{\Lambda})$ is called the *Poissonization* of (M, Λ, E).

Proposition 3.4 ([17]) *With each (strict) Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, a (strict) Poisson-Nijenhuis manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{N})$ can be associated, where $(\tilde{M}, \tilde{\Lambda})$ is the Poissonization of (M, Λ, E) and $\tilde{N}$ is the Nijenhuis tensor field on $\tilde{M}$, given by*

$$\tilde{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt, \tag{15}$$

and reciprocally.

4 Lie algebroids associated with a Jacobi-Nijenhuis manifold

In this section we start by associating with each Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$, two Lie algebroids over M that are in duality. Then, we apply the techniques used in [2] to these Lie algebroids, in order to obtain a Lie bialgebroid over $M \times \mathbb{R}$ associated with a Jacobi-Nijenhuis manifold.

Let (M, Λ, E) be a Jacobi manifold. In opposition to the case of a Poisson manifold, in general one cannot define a Lie algebroid structure on the cotangent bundle of a Jacobi manifold. However, it is possible to associate a Lie algebroid with a Jacobi manifold if one considers the 1-jet bundle $T^*M \times \mathbb{R} \to M$. In fact, if (M, Λ, E) is a Jacobi manifold,

$$(T^*M \times \mathbb{R}, [.,.]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^{\sharp}) \tag{16}$$

is a Lie algebroid over M, where $[.,.]_{(\Lambda,E)}$ is the bracket (11), $(\Lambda, E)^{\sharp} : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ is the bundle map given by (10) and $\pi : TM \times \mathbb{R} \to TM$ is the projection over the first factor (see [3]).

Remark 4.1 In (16), if $E = 0$, by projection, we obtain the Lie algebroid structure on the cotangent of a Poisson manifold, presented in example 2.3.

The differential d_* of the Lie algebroid $(T^*M \times \mathbb{R}, [.,.]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\sharp)$, defined by expression (3), is given for all $(P,Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$, by

$$d_*(P,Q) = ([\Lambda, P] + kE \wedge P + \Lambda \wedge Q, -[\Lambda, Q] + (1-k)E \wedge Q + [E, P]). \tag{17}$$

Now we consider a differentiable manifold equipped with a Nijenhuis operator $\mathcal{N} := (N, Y, \gamma, g)$, given by (7). Using the operator $\mathcal{N}$, we may define a new bracket on $\mathcal{V}^1(M) \times C^\infty(M)$, which is a deformation of the bracket (8), by setting, for all $(X,f),(Z,h) \in \mathcal{V}^1(M) \times C^\infty(M)$,

$$\begin{aligned} [(X,f),(Z,h)]_{\mathcal{N}} &= [\mathcal{N}(X,f),(Z,h)] + [(X,f),\mathcal{N}(Z,h)] - \mathcal{N}[(X,f),(Z,h)] \\ &= ([X,Z]_N + f[Y,Z] - h[Y,X],\ h(X.g - Y.f) - f(Z.g - Y.h) \\ &\quad +(NX).h - (NZ).f + d\gamma(X,Z)), \end{aligned} \tag{18}$$

where $[.,.]_N$ is the bracket on $\mathcal{V}^1(M)$ given by (1). Since the Nijenhuis torsion $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ given by (9) vanishes, the bracket $[.,.]_{\mathcal{N}}$ is a Lie bracket on $\mathcal{V}^1(M) \times C^\infty(M)$. Moreover, we have the following.

Proposition 4.2 *Let $\mathcal{N} := (N, Y, \gamma, g)$ be a Nijenhuis operator on M. Then,*

$$(TM \times \mathbb{R},\ [.,.]_{\mathcal{N}},\ \pi \circ \mathcal{N}) \tag{19}$$

is a Lie algebroid over M, where $[.,.]_{\mathcal{N}}$ is the bracket (18) and $\pi : TM \times \mathbb{R} \to TM$ is the projection over the first factor.

Proof. Take any (X,f), $(Z,h) \in \mathcal{V}^1(M) \times C^\infty(M)$. Using the fact that the Nijenhuis torsion of $\mathcal{N}$ vanishes, *i.e.*

$$[\mathcal{N}(X,f), \mathcal{N}(Z,h)] = \mathcal{N}([(X,f),(Z,h)]_{\mathcal{N}}),$$

we show that the anchor map $\pi \circ \mathcal{N}$ induces a Lie algebra homomorphism from $(\mathcal{V}^1(M) \times C^\infty(M), [.,.]_{\mathcal{N}})$ to $(\mathcal{V}^1(M), [.,.])$, where $[.,.]$ is the usual Lie bracket of vector fields:

$$\begin{aligned} \pi \circ \mathcal{N}([(X,f),(Z,h)]_{\mathcal{N}}) &= \pi([\mathcal{N}(X,f), \mathcal{N}(Z,h)]) \\ &= [NX + fY, NZ + hY] \\ &= [\pi \circ \mathcal{N}(X,f), \pi \circ \mathcal{N}(Z,h)]. \end{aligned}$$

Finally, a straightforward computation shows that for all $s \in C^\infty(M)$,

$$[(X,f), s(Z,h)]_{\mathcal{N}} = s[(X,f),(Z,h)]_{\mathcal{N}} + ((NX + fY).s)(Z,h).$$

$\diamondsuit$

The differential of the Lie algebroid $(TM \times \mathbb{R}, [.,.]_{\mathcal{N}}, \pi \circ \mathcal{N})$ is $d_{\mathcal{N}} = [i_{\mathcal{N}}, \mathbf{d}]$, where $[.,.]$ is the graded commutator, $\mathbf{d} = (d, -d)$ with d the de Rham differential and $i_{\mathcal{N}}$ is the derivation of degree zero defined, for all $(\beta, \alpha) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$, by

$$i_{\mathcal{N}}(\beta,\alpha)((X_1,f_1),\cdots,(X_k,f_k)) = \sum_{i=1}^{k} (\beta,\alpha)((X_1,f_1),\cdots,\mathcal{N}(X_i,f_i),\cdots,(X_k,f_k)),$$

$(X_1, f_1), \cdots, (X_k, f_k) \in \mathcal{V}^1(M) \times C^\infty(M)$. If $(\beta, f) \in \Omega^1(M) \times C^\infty(M)$, $d_{\mathcal{N}}(\beta, f) = (d_N\beta - fd\gamma, \mathcal{L}_Y\beta - {}^tN(df) + fdg)$, where d_N is the differential of the Lie algebroid $(TM, [.,.]_N, N)$ (cf. §1).

Now let us take a Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$. Since we have simultaneously a Jacobi structure and a Nijenhuis operator on M, we can associate with this Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$ the two Lie algebroids over M that are in duality, given by (16) and (19), respectively:

$$(T^*M \times \mathbb{R}, [.,.]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\sharp) \quad \text{and} \quad (TM \times \mathbb{R}, [.,.]_{\mathcal{N}}, \pi \circ \mathcal{N}).$$

But this pair is `not` a Lie bialgebroid, because condition (4) doesn't hold!

Next we briefly recall the techniques introduced in [2], in order to apply them to the Lie algebroids (16) and (19), considered above.

Let $(A, [.,.], \rho)$ be a Lie algebroid over M and consider the Lie algebroid cohomology complex with trivial coefficients (see [10]). A 1-cochain $\theta \in \Gamma(A^*)$ is a 1-cocycle if, for all $X, Z \in \Gamma(A)$,

$$\theta([X, Z]) = \rho(X).(\theta(Z)) - \rho(Z).(\theta(X)). \tag{20}$$

Given a Lie algebroid $(A, [.,.], \rho)$ over M, let us consider the vector bundle $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of $\tilde{A}$ can be identified with the t-dependent sections of A, t being the canonical coordinate on $\mathbb{R}$, i.e., for any $\tilde{X} \in \Gamma(\tilde{A})$ and $(x, t) \in M \times \mathbb{R}$, $\tilde{X}(x, t) = \tilde{X}_t(x)$, where $X_t \in \Gamma(A)$. This identification induces, in a natural way, a Lie bracket on $\Gamma(\tilde{A})$, also denoted by $[.,.]$:

$$[\tilde{X}, \tilde{Z}](x, t) = [\tilde{X}_t, \tilde{Z}_t](x), \quad \tilde{X}, \tilde{Z} \in \Gamma(\tilde{A}), (x, t) \in M \times \mathbb{R},$$

and a bundle map, also denoted by ρ, $\rho : \tilde{A} \to T(M \times \mathbb{R}) \equiv TM \oplus T\mathbb{R}$, in such a way that $(\tilde{A}, [.,.], \rho)$ becomes a Lie algebroid over $M \times \mathbb{R}$.

Now, take a 1-cocycle $\theta \in \Gamma(A^*)$ and consider the following new brackets on $\Gamma(\tilde{A})$:

$$[\tilde{X}, \tilde{Z}]^{\star\theta} = \exp(-t)([\tilde{X}, \tilde{Z}] + \theta(\tilde{X})(\frac{\partial \tilde{Z}}{\partial t} - \tilde{Z}) - \theta(\tilde{Z})(\frac{\partial \tilde{X}}{\partial t} - \tilde{X})) \tag{21}$$

and

$$[\tilde{X}, \tilde{Z}]^{-\theta} = [\tilde{X}, \tilde{Z}] + \theta(\tilde{X})\frac{\partial \tilde{Z}}{\partial t} - \theta(\tilde{Z})\frac{\partial \tilde{X}}{\partial t}, \tag{22}$$

$\tilde{X}, \tilde{Z} \in \Gamma(\tilde{A})$. Also consider the maps $\rho^{\star\theta}$, $\rho^{-\theta} : \Gamma(\tilde{A}) \to \mathcal{V}^1(M \times \mathbb{R})$ given, for any $\tilde{X} \in \Gamma(\tilde{A})$, respectively by

$$\rho^{\star\theta}(\tilde{X}) = \exp(-t)(\rho(\tilde{X}) + \theta(\tilde{X})\frac{\partial}{\partial t}) \tag{23}$$

and

$$\rho^{-\theta}(\tilde{X}) = \rho(\tilde{X}) + \theta(\tilde{X})\frac{\partial}{\partial t}. \tag{24}$$

Proposition 4.3 ([2]) *Let $A \to M$ be a vector bundle over M, $[.,.] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ a bracket on $\Gamma(A)$, $\rho : \Gamma(A) \to \mathcal{V}^1(M)$ a homomorphism of $C^\infty(M)$-modules and θ a section of the dual bundle A^*. Then the following conditions are equivalent:*

i) $(A, [.,.], \rho)$ is a Lie algebroid over M and θ is a 1-cocycle,

ii) $(\tilde{A}, [.,.]^{\star\theta}, \rho^{\star\theta})$ is a Lie algebroid over $M \times \mathbb{R}$,

iii) $(\tilde{A}, [.,.]^{-\theta}, \rho^{-\theta})$ is a Lie algebroid over $M \times \mathbb{R}$, *where* $[.,.]^{\star\,\theta}$ *and* $\rho^{\star\,\theta}$ *(resp.* $[.,.]^{-\theta}$ *and* $\rho^{-\theta}$*) are given by (21) and (23) (resp. (22) and (24)).*

Let us now come back to the Lie algebroids $(T^*M \times \mathbb{R},\, [.,.]_{(\Lambda,E)},\, \pi \circ (\Lambda,E)^\sharp)$ and $(TM \times \mathbb{R},\, [.,.]_{\mathcal{N}},\, \pi \circ \mathcal{N})$ over M, associated with a Jacobi-Nijenhuis manifold $(M,(\Lambda,E),\mathcal{N})$, $\mathcal{N} := (N,Y,\gamma,g)$.

Lemma 4.4 *The pair* $(\gamma,g) \in \Omega^1(M) \times C^\infty(M)$ *is a* 1*-cocycle in the Lie algebroid cohomology with trivial coefficients of* $(TM \times \mathbb{R},\, [.,.]_{\mathcal{N}},\, \pi \circ \mathcal{N})$.

Proof. In order to show that (γ,g) verifies condition (20), we will use the equalities

$$\mathcal{L}_N\gamma = gd\gamma \tag{25}$$

and

$$^tN(dg) = \mathcal{L}_Y\gamma + gdg, \tag{26}$$

which together with $\mathcal{T}(N) = Y \otimes d\gamma$ and $\mathcal{L}_Y N = -Y \otimes dg$ are equivalent to $\mathcal{T}(\mathcal{N}) = 0$ (see [17]). Let (X,f) and (Z,h) be any sections of $\mathcal{V}^1(M) \times C^\infty(M)$. Then,

$$\begin{array}{rcl}
(\gamma,g)([(X,f),(Z,h)]_{\mathcal{N}}) & = & <\gamma,[X,Z]_N + f[Y,Z] - h[Y,X]> + gh(X.g - Y.f) \\
 & & -fg(Z.g - Y.h) + g((NX).h - (NZ).f) + gd\gamma(X,Z) \\
 & \overset{(25)}{=} & <\gamma,[NX,Z]> + <\gamma,[X,NZ]> - <\gamma,N[X,Z]> \\
 & & + <\mathcal{L}_Y\gamma + gdg, hX - fZ> + fY.<\gamma,Z> - hY.<\gamma,X> \\
 & & -gh(Y.f) + fg(Y.h) + g(NX).h - g(NZ).f \\
 & & +d\gamma(NX,Z) + d\gamma(X,NZ) - d(^tN\gamma)(X,Z) \\
 & \overset{(26)}{=} & -f((NZ).g) + ((NX).g) + f(Y.<\gamma,Z>) - h(Y.<\gamma,X>) \\
 & & -gh(Y.f) + fg(Y.h) + g((NX).h) - g((NZ).f) \\
 & & +(NX).<\gamma,Z> - (NZ).<\gamma,X> \\
 & = & (NX + fY).(<\gamma,Z> + gh) - (NZ + hY).(<\gamma,X> + fg).
\end{array}$$

$\diamondsuit$

Taking into account that $(-E,0) \in \mathcal{V}^1(M) \times C^\infty(M)$ is a 1-cocycle in the Lie algebroid cohomology with trivial coefficients of $(T^*M\times\mathbb{R},\, [.,.]_{(\Lambda,E)},\, \pi\circ(\Lambda,E)^\sharp)$, (see [2]), from Proposition 4.3 and Lemma 4.4, we end up with the following.

Proposition 4.5 *If* $(M,(\Lambda,E),\mathcal{N})$, $\mathcal{N} =: (N,Y,\gamma,g)$, *is a Jacobi-Nijenhuis manifold, then the triples* $((TM \times \mathbb{R}) \times \mathbb{R}, [.,.]_{\mathcal{N}}^{-(\gamma,g)}, (\pi \circ \mathcal{N})^{-(\gamma,g)})$ *and* $((T^*M \times \mathbb{R}) \times \mathbb{R},\, [.,.]_{(\Lambda,E)}^{\star\,(-E,0)},\, (\pi \circ (\Lambda,E)^\sharp)^{\star\,(-E,0)})$ *are Lie algebroids over* $M \times \mathbb{R}$.

Let $(\tilde{M},\tilde{\Lambda},\tilde{N})$ be the strict Poisson-Nijenhuis manifold associated with the strict Jacobi-Nijenhuis manifold $(M,(\Lambda,E),\mathcal{N})$, $\mathcal{N} =: (N,Y,\gamma,g)$, in the sense of Proposition 3.4 and take the two Lie algebroids $(T\tilde{M},[.,.]_{\tilde{N}},\tilde{N})$ and $(T^*\tilde{M},[.,.]_{\tilde{\Lambda}},\pi\circ\tilde{\Lambda}^\sharp)$ over $\tilde{M} = M\times\mathbb{R}$ which constitute a Lie bialgebroid (cf. Theorem 2.5).

Lemma 4.6 *The map*

$$\psi : (T\tilde{M},[.,.]_{\tilde{N}},\tilde{N}) \to ((TM \times \mathbb{R}) \times \mathbb{R}, [.,.]_{\mathcal{N}}^{-(\gamma,g)}, (\pi \circ \mathcal{N})^{-(\gamma,g)}), \quad \psi(\tilde{X} + \tilde{f}\frac{\partial}{\partial t}) = (\tilde{X},\tilde{f}),$$

is a Lie algebroid isomorphism, where $\tilde{X}$ *is a vector field on* M *depending on* t *and* $\tilde{f} \in C^\infty(M \times \mathbb{R})$.

Proof. For any section $\tilde{X}+\tilde{f}\frac{\partial}{\partial t}$ of $T\tilde{M}=TM\oplus T\mathbb{R}$ we have, using (24) and (15),

$$\begin{aligned}(\pi\circ\mathcal{N})^{-(\gamma,g)}\circ\psi(\tilde{X}+\tilde{f}\frac{\partial}{\partial t}) &= (\pi\circ\mathcal{N})(\tilde{X},\tilde{f})+<(\gamma,g),(\tilde{X},\tilde{f})>\frac{\partial}{\partial t}\\ &= (N\tilde{X}+\tilde{f}Y)+(<\gamma,\tilde{X}>+\tilde{f}g)\frac{\partial}{\partial t}\\ &\overset{(15)}{=} \tilde{N}(\tilde{X}+\tilde{f}\frac{\partial}{\partial t}).\end{aligned}$$

On the other hand, a long but straightforward computation shows that for all sections $\tilde{X}+\tilde{f}\frac{\partial}{\partial t}$ and $\tilde{Z}+\tilde{h}\frac{\partial}{\partial t}$ of $T\tilde{M}=TM\oplus T\mathbb{R}$, the following equality holds:

$$[\psi(\tilde{X}+\tilde{f}\frac{\partial}{\partial t}),\,\psi(\tilde{Z}+\tilde{h}\frac{\partial}{\partial t})]_{\mathcal{N}}^{-(\gamma,g)}=\psi([(\tilde{X}+\tilde{f}\frac{\partial}{\partial t}),\,(\tilde{Z}+\tilde{h}\frac{\partial}{\partial t})]_{\tilde{N}}),$$

where $[.,.]_{\tilde{N}}$ is the bracket (1). $\diamondsuit$

In [2] it was proved that the adjoint ismorphism of ψ,

$$\psi^*:((T^*M\times\mathbb{R})\times\mathbb{R},\,[.,.]^{\star(-E,0)}_{(\Lambda,E)},\,(\pi\circ(\Lambda,E)^\sharp)^{\star(-E,0)})\to(T^*\tilde{M},[.,.]_{\tilde{\Lambda}},\pi\circ\tilde{\Lambda}^\sharp),$$

$\psi^*(\tilde{\alpha},\tilde{f})=\tilde{\alpha}+\tilde{f}dt$, is also a Lie algebroid isomorphism, i.e., for all sections $(\tilde{\alpha},\tilde{f})$ and $(\tilde{\beta},\tilde{h})$ of $T^*\tilde{M}=T^*M\oplus T^*\mathbb{R}$, one has

$$(\pi\circ\tilde{\Lambda}^\sharp)(\psi^*(\tilde{\alpha},\tilde{f}))=(\pi\circ(\Lambda,E)^\sharp)^{\star(-E,0)}(\tilde{\alpha},\tilde{f}) \tag{27}$$

and

$$[\psi^*(\tilde{\alpha},\tilde{f}),\,\psi^*(\tilde{\beta},\tilde{h})]_{\tilde{\Lambda}}=\psi^*([(\tilde{\alpha},\tilde{f}),\,(\tilde{\beta},\tilde{h})]^{\star(-E,0)}_{(\Lambda,E)}). \tag{28}$$

Suppose that A_1 and A_2 are Lie algebroids over M such that their duals A_1^* and A_2^* are also Lie algebroids over M. Next result from [2] will be useful in the sequel.

Lemma 4.7 ([2]) *Let $\psi:A_1\to A_2$ be a Lie algebroid isomorphism such that its adjoint homomorphism $\psi^*:A_2^*\to A_1^*$ is also a Lie algebroid isomorphism. Then, if (A_1,A_1^*) is a Lie bialgebroid, so is (A_2,A_2^*).*

Taking into account the previous comments and using Lemma 4.7 and Theorem 2.5, we can state the following.

Proposition 4.8 *Let (M,Λ,E) be a Jacobi manifold and $\mathcal{N}=:(N,Y,\gamma,g)$ a Nijenhuis operator on M. Then, $((M,\Lambda,E),\mathcal{N})$ is a strict Jacobi-Nijenhuis manifold if and only if the pair*

$$(((TM\times\mathbb{R})\times\mathbb{R},[.,.]_{\mathcal{N}}^{-(\gamma,g)},(\pi\circ\mathcal{N})^{-(\gamma,g)}),\,((T^*M\times\mathbb{R})\times\mathbb{R},\,[.,.]^{\star(-E,0)}_{(\Lambda,E)},\,(\pi\circ(\Lambda,E)^\sharp)^{\star(-E,0)})) \tag{29}$$

is a Lie bialgebroid over $M\times\mathbb{R}$.

5 The generalized Lie bialgebroid of a strict Jacobi-Nijenhuis manifold

Using the notion of generalized Lie bialgebroid, introduced by D. Iglesias and J. Marrero in [2], we are going to give a characterization of a strict Jacobi-Nijenhuis manifold.

Let us recall how we can "add" a 1-cocycle in the differential calculus of a Lie algebroids. Fore more details, see [2].

Let $(A,[.,.],\rho)$ be a Lie algebroid over M and $\theta\in\Gamma(A^*)$ a 1-cocycle (cf. (20)). Using the 1-cocycle θ, we can define a new representation ρ^θ of the Lie algebra $(\Gamma(A),[.,.])$ on $C^\infty(M)$, by setting

$$\rho^\theta:\Gamma(A)\times C^\infty(M)\to C^\infty(M),\quad (X,f)\mapsto\rho^\theta(X,f)=\rho(X).f+\theta(X)f. \tag{30}$$

Therefore, we obtain a new cohomology complex, whose differential cohomology operator is given by

$$d^\theta:\Gamma(\Lambda^kA^*)\to\Gamma(\Lambda^{k+1}A^*),\quad \beta\mapsto d^\theta(\beta)=d\beta+\theta\wedge\beta. \tag{31}$$

Also, for any $X\in\Gamma(A)$, the Lie derivative operator with respect to X is given by

$$\mathcal{L}^\theta_X:\Gamma(\Lambda^kA^*)\to\Gamma(\Lambda^kA^*),\quad \beta\mapsto\mathcal{L}^\theta_X(\beta)=\mathcal{L}_X\beta+\theta(X)\beta. \tag{32}$$

It is also possible to consider a θ-Schouten bracket on the graded algebra $\Gamma(\Lambda A)$, denoted by $[.,.]^\theta$, which is defined as follows:

$$\begin{aligned}[.,.]^\theta\ :\ &\Gamma(\Lambda^pA)\times\Gamma(\Lambda^qA)\to\Gamma(\Lambda^{p+q-1}A)\\ &(P,Q)\mapsto[P,Q]^\theta=[P,Q]+(p-1)P\wedge(i_\theta Q)+(-1)^p(q-1)(i_\theta P)\wedge Q.\end{aligned} \tag{33}$$

Suppose that $(A,[.,.],\rho)$ is a Lie algebroid over M such that in the dual bundle A^* of A also exists a Lie algebroid structure over M, $([.,.]_*,\rho_*)$. Let $\theta\in\Gamma(A^*)$ (resp. $W\in\Gamma(A)$) be a 1-cocycle in the Lie algebroid cohomology complex of $(A,[.,.],\rho)$ (resp. $(A^*,[.,.]_*,\rho_*)$).

Definition 5.1 ([2]) *The pair* $((A,\theta),(A^*,W))$ *is a* generalized Lie bialgebroid *if for all* $X,Z\in\Gamma(A)$ *and* $P\in\Gamma(\Lambda^pA)$,

1. $d^W_*[X,Z]=[d^W_*X,Z]^\theta+[X,d^W_*Z]^\theta$
2. $(\mathcal{L}^W_*)_\theta(P)+\mathcal{L}^\theta_W(P)=0$,

where d^W_* *and* $\mathcal{L}^W_*$ *are, repectively, the* W*-differential and the* W*-Lie derivative on* A^*.

Note that when $\theta=0$ and $W=0$, the generalized Lie bialgebroid is a Lie bialgebroid.

Let the vector bundles $\tilde{A}=A\times\mathbb{R}$ and $\tilde{A^*}=A^*\times\mathbb{R}$ be equipped with the Lie algebroid structures over $\tilde{M}=M\times\mathbb{R}$, $([.,.]^{-\theta},\rho^{-\theta})$ and $([.,.]^{\star W}_*,\rho^{\star W}_*)$ given by (22) and (24), and (21) and (23), respectively.

Proposition 5.2 ([2]) *If* $((A\times\mathbb{R},[.,.]^{-\theta},\rho^{-\theta}),\ (A^*\times\mathbb{R},[.,.]^{\star W}_*,\rho^{\star W}_*))$ *is a Lie bialgebroid (over* $\tilde{M}=M\times\mathbb{R}$*), then* $((A,\theta),(A^*,W))$ *is a generalized Lie bialgebroid (over* M*), and reciprocally.*

Using this result, we may rewrite Proposition 4.8, to obtain a characterization of a strict Jacobi-Nijenhuis structure on a manifold by means of a generalized Lie bialgebroid.

Theorem 5.3 *Let* (M,Λ,E) *be a Jacobi manifold and* $\mathcal{N}=:(N,Y,\gamma,g)$ *a Nijenhuis operator on* M*. Then,* $((M,\Lambda,E),\mathcal{N})$ *is a strict Jacobi-Nijenhuis manifold* if and only if *the pair*

$$(((TM\times\mathbb{R},[.,.]_{\mathcal{N}},\ \pi\circ\mathcal{N}),(\gamma,g)),\ ((T^*M\times\mathbb{R},[.,.]_{(\Lambda,E)},\ \pi\circ(\Lambda,E)^\sharp),(-E,0))) \tag{34}$$

is a generalized Lie bialgebroid.

Remark 5.4 In [2] it was proved that on the base manifold M of a generalized Lie bialgebroid $((A,\theta),(A^*,W))$, there exists a Jacobi structure whose bracket is given by

$$\{f,h\} = < d^\theta f,\, d_*^W h >, \quad f,h\in C^\infty(M).$$

In the case of the generalized Lie bialgebroid (34) associated with a strict Jacobi-Nijenhuis manifold we have, for all $f,h\in C^\infty(M)$,

$$\begin{aligned}
\{f,h\} &= < d_{\mathcal{N}}^{(\gamma,g)} f,\, d_*^{(-E,0)} h > \\
&= < ({}^tN(df) + f\gamma,\, Y.f + fg), (-\Lambda^\sharp(dh) - hE, E.h) > \\
&= < df, (-N\Lambda^\sharp + Y\oplus E)dh > -h < df, NE > +f < dh, \Lambda^\sharp(\gamma) + gE > .
\end{aligned}$$

Taking into account that

$$N\Lambda^\sharp - Y\oplus E = \Lambda_1^\sharp \quad \text{and} \quad \Lambda^\sharp(\gamma) + gE = NE = E_1,$$

where Λ_1 and E_1 are given by (12), see [17], we obtain

$$\{f,h\} = \{f,h\}_1,$$

for all $f,h\in C^\infty(M)$, where $\{.,.\}_1$ is the Jacobi bracket associated with (Λ_1, E_1).

Acknowledgments. The author wishes to thank Fani Petalidou for helpful discussions.

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