Superconvergence of the gradient in a fully discrete FEM scheme: one-dimensional case

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Abstract

In this paper we study the convergence of a fully discrete linear finite element solution for a one-dimensional elliptic problem subject to general boundary conditions. We prove for $s \in [1, 2]$ order $O(h^s)$ convergence of solution and gradient if the exact solution is in the Sobolev space $H^{s+1}(0, L)$. The method is equivalent to a finite difference scheme on a nonuniform mesh and the obtained convergence is then a so-called supraconvergence result for solution and gradient.

1 Introduction

We consider the discretization of the differential equation

$$Au := -(au')' + (bu)' + cu = f \quad \text{in } (0, L) \subset \mathbb{R} \tag{1}$$

subject to either Dirichlet boundary conditions

$$u(0) = \alpha_0, \quad u(L) = \alpha_L \tag{2}$$

or third kind boundary conditions

$$-(au)'(0) + \beta_0 u(0) = \gamma_0, \quad (au)'(L) + \beta_L u(L) = \gamma_L. \tag{3}$$

Our scheme can be written as a finite difference approximation on the general non uniform grid

$$\mathbb{I}_h := \{0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = L\},$$

where h is a vector of mesh-sizes $h_j := x_{j+1} - x_j$, $j = 0, \ldots, N-1$. Let $x_{j+1/2} := x_j + h_j/2, j = 0, \ldots, N-1$. By $W_h := \{u_h, v_h, w_h, \ldots\}$ we denote the space of grid functions defined on $\mathbb{I}_h$. We introduce the centered divided finite differences

$$(\delta v_h)_j := \frac{v_{j+1} - v_{j-1}}{x_{j+1} - x_{j-1}}, \quad (\delta^{(1/2)} v_h)_{j+1/2} := \frac{v_{j+1} - v_j}{x_{j+1} - x_j}, \quad (\delta^{(1/2)} v_h)_j := \frac{v_{j+1/2} - v_{j-1/2}}{x_{j+1/2} - x_{j-1/2}}, \tag{4}$$

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where $v_j := v_h(x_j)$, and $v_{j+1/2}$ is used as far as it makes sense. Our scheme has the form

$$A_h u_h := -\delta^{(1/2)}(a\delta^{(1/2)}u_h) + \delta(bu_h) + cu_h = f_h \quad \text{in } \mathbb{I}'_h \tag{5}$$

together with the discretized boundary conditions

$$u_0 = \alpha_0, \quad u_N = \alpha_L \tag{6}$$

or

$$-\delta(au_h)_0 + \beta_0 u_0 = \gamma_0, \quad \delta(au_h)_N + \beta_L u_N = \gamma_L \tag{7}$$

in the case (2) or (3), respectively. Here, for boundary conditions of type (7) $\mathbb{I}'_h := \mathbb{I}_h$ and we have introduced auxiliary variables u_{-1} and u_{N+1} corresponding to auxiliary grid points $x_{-1} := -h_0$ and $x_{N+1} := L + h_{N-1}$. In the case of Dirichlet boundary conditions $\mathbb{I}'_h := \mathbb{I}_h \backslash \{x_0, x_N\}$. In (5) the grid function f_h approximating the right-hand side of (1) is given by

$$f_j := \frac{2}{h_{j-1} + h_j} \int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\, dx\,, \quad j = 0, \ldots, N, \tag{8}$$

where for the sake of a simpler notation we introduced the additional mesh-sizes $h_{-1} := h_N := 0$ and points $x_{-1/2} := 0, x_{N+1/2} := L$.

In Section 2 we show the interesting relation that the scheme (5) with (6) or (7), respectively, is equivalent to a linear finite element method with quadrature. This fact will also be used for studying the convergence properties of the scheme.

There are no restrictions made on the nonuniformity of the grid $\mathbb{I}_h$. We consider the behaviour of the scheme for a sequence of grids $\mathbb{I}_h$, $h \in H$, with maximal mesh-size $h_{max} := \max\{h_j, j = 0, \ldots, N-1\}$ tending to zero. The scheme (5) is in general first order accurate only (it is of second order on uniform grids). One purpose of the present paper is to show that nevertheless the solutions u_h are second order accurate. This fact is usually expressed by saying that the scheme is supraconvergent. But the scheme exhibits an interesting additional feature in that the first order derivatives are also second order accurately approximated. This latter fact is a superconvergent behaviour of the fully discrete linear finite element method that is already nonstandard for the simpler situation of uniform partitions. The regularity assumption $u \in H^3(0, L)$ for the continuous solution u to obtain that result, which is contained in the main theorem of this note (Theorem 2 of Section 3), is minimal. The present kind of superconvergence is different from the one obtained by postprocessing (see e.g. [1] and the literature cited there).

Supraconvergence of finite difference schemes for BVODE's is well established in the literature (see [6], [7], [9], [10], [14], [19] and [22]). Different methods of proof have been used by the various authors. The present paper still presents an other kind of proof following the reasoning in [13] for the equidistant case which has two advantages: it allows to obtain the error estimates under optimal smoothness assumptions for the exact solution and it may be extended to the two-dimensional case. The latter will be done in a forthcoming paper by the authors. In this note we also consider general two-point boundary conditions (2) and (3), which was not done so far in the literature. Naturally, the phenomenon of supraconvergence in more than one dimension has already been studied in the literature (see e.g. [3] - [5], [17], [20]). Also superconvergence results for the gradient have been obtained (see e.g. [2], [13], [16], [18], [21], [24], [25]). Another direction of interest lies in setting up discretisation schemes that work also for coefficients with low smoothness in the differential equation (1). The assumptions in [8], [11], [12], [15] and [23] are weaker than ours in this paper as stated at the end of Section 2 .

2 The variational formulation

The linear finite element scheme that is equivalent to (5) and (6) or (7), respectively, can be written with the aid of the piecewise linear interpolation of a grid function v_h with break points in $\mathbb{I}_h$ which we denote by $P_h v_h$. We restrict ourselves to the case of boundary conditions (3), the Dirichlet case (2) is similar but slightly simpler. Let a_h denote the sesquilinear form

$$\begin{aligned} a_h(v_h, w_h) \;&:=\; \sum_{j=0}^{N-1} h_j[a_{j+1/2}(P_h v_h)'_j - (P_h(bv_h))_{j+1/2}](P_h\overline{w}_h)'_j \\ &+\; (c_h v_h, w_h)_h + (\beta_0 - b_0)v_0\overline{w}_0 + (\beta_L + b_N)v_N\overline{w}_N. \end{aligned} \tag{9}$$

Here $c_h := R_h c \in W_h$, where R_h denotes the pointwise restriction to the grid $\mathbb{I}_h$, and

$$(v_h, w_h)_h := \sum_{j=0}^{N} \frac{h_{j-1} + h_j}{2} v_j \overline{w}_j, \quad v_h, w_h \in W_h. \tag{10}$$

The finite difference scheme (5) and (7) is equivalent to the variational problem to find $u_h \in W_h$ such that

$$a_h(u_h, v_h) = (f_h, v_h)_h + \gamma_0\overline{v}_0 + \gamma_L\overline{v}_N \quad \forall v_h \in W_h. \tag{11}$$

The formulation (11) corresponds to the variational formulation of the continuous problem (1) and (3) to find $u \in H^1(0, L)$ such that

$$a(u, v) = (f, v)_0 + \gamma_0\overline{v}(0) + \gamma_L\overline{v}(L) \quad \forall v \in H^1(0, L), \tag{12}$$

where $(\cdot, \cdot)_0$ denotes the standard inner product in $L^2(0, L)$ and for $v, w \in H^1(0, L)$

$$a(v, w) := (av', w')_0 - (bv, w')_0 + (cv, w)_0 + (\beta_0 - b(0))v(0)\overline{w}(0) + (\beta_L + b(L))v(L)\overline{w}(L).$$

The coefficients in the differential equation are assumed to be smooth enough, e.g. $a \in C[0, L]$ and $b, c \in W^{2,\infty}(0, L)$ is sufficient.

3 The main result

The main result of this paper in Theorem 2 relies on the following inverse stability result.

Theorem 1 *Assume that the variational problem belonging to (1) and (2) or (3), respectively, is uniquely solvable. Then there exists a positive constant C such that for $h \in H$ with h_{max} small enough*

$$\|P_h v_h\|_1 \leq C \sup_{0 \neq w_h \in W_h} \frac{|a_h(v_h, w_h)|}{\|P_h w_h\|_1} \quad \forall v_h \in W_h. \tag{13}$$

The proof is similar to the one of Theorem 2 in [4] and we do not reproduce it here again.

Theorem 2 *Assume that the variational problem belonging to (1) and (2) or (3), respectively, is uniquely solvable. Then the discretised problem (5) and (6) or (7), respectively, has a unique solution u_h for $h \in H$ with h_{max} sufficiently small. Assume that for some $s \in [1, 2]$ the solution u of (1) and (2) or (3), respectively, lies in $H^{s+1}(0, L)$. Then there holds the error estimate*

$$\|P_h R_h u - P_h u_h\|_1 \leq C \left(\sum_{j=0}^{N-1} h_j^{2s} \|u\|^2_{H^{s+1}(x_j, x_{j+1})} \right)^{1/2} \leq C h^s_{max} \|u\|_{H^{s+1}(0,L)}. \tag{14}$$

Proof: Let u_h be the solution of (5) and (7). Then u_h is solution of the variational problem (11). By means of the stability inequality (13) an estimate to $\|P_h R_h u - P_h u_h\|_1$ will be obtained by bounding

$$a_h(R_h u - u_h, v_h) = a_h(R_h u, v_h) - (f_h, v_h)_h - \gamma_0 \overline{v}_0 - \gamma_L \overline{v}_N. \tag{15}$$

By the definition of f_h in (8) we obtain after an integration and a summation by parts (recall that $x_{-1/2} = 0$ and $x_{N+1/2} = L$)

$$\begin{aligned} (f_h, v_h)_h \;=\; & -\sum_{j=0}^{N-1} [(au')(x_{j+1/2}) - (bu)(x_{j+1/2})](\overline{v}_{j+1} - \overline{v}_j) + \sum_{j=0}^{N} (\int_{x_{j-1/2}}^{x_{j+1/2}} cu\, dx) \overline{v}_j \\ & + [(au')(0) - (bu)(0)]\overline{v}_0 - [(au')(L) - (bu)(L)]\overline{v}_N. \end{aligned}$$

Taking the definitions of a_h and the boundary conditions (3) into account an estimate of (15) will be obtained by the sum of bounds for the quantities

$$\begin{aligned} I_a : &= \sum_{j=0}^{N-1} h_j a_{j+1/2} [(P_h R_h u)'_j - u'(x_{j+1/2})](P_h \overline{v}_h)'_j \\ I_b : &= \sum_{j=0}^{N-1} h_j [(bu)(x_{j+1/2}) - \frac{(bu)(x_j) + (bu)(x_{j+1})}{2}](P_h \overline{v}_h)'_j \\ I_c : &= \sum_{j=0}^{N} [\frac{h_{j-1} + h_j}{2}(cu)(x_j) - \int_{x_{j-1/2}}^{x_{j+1/2}} (cu)(x)\, dx] \overline{v}_j. \end{aligned}$$

Estimate for I_a: Let the function w be defined by $w(\xi) := u(x_j + \xi h_j)$ for $\xi \in [0,1]$. We have

$$u'(x_{j+1/2}) - (P_h u)'_j := \frac{1}{h_j}[w'(\frac{1}{2}) - w(1) + w(0)]. \tag{16}$$

The functional

$$\lambda(g) = g'(\frac{1}{2}) - g(1) + g(0)$$

is bounded on $W^{2,1}(0,1)$ and vanishes for $g = 1, \xi$ and ξ^2. Thus the Bramble-Hilbert Lemma gives the existence of a positive constant C such that

$$|\lambda(g)| \leq C\|g^{(r)}\|_{L_1(0,1)}$$

for $r \in \{2,3\}$. The last estimate applied to $g := w$ and (16) yield

$$|u'(x_{j+1/2}) - (P_h R_h u)'_j| \leq C h_j^{s-1} \|u^{(s+1)}\|_{L^1(x_j,x_{j+1})} \leq C h_j^{s-1/2} \|u\|_{H^{s+1}(x_j,x_{j+1})} \tag{17}$$

for $u \in H^{s+1}(x_j, x_{j+1}), s \in 1, 2$. By interpolation the outer inequality (17) holds for all $s \in [1,2]$ and we obtain the bound

$$|I_a| \leq C\|a\|_\infty \left(\sum_{j=0}^{N-1} h_j^{2s} \|u\|^2_{H^{s+1}(x_j,x_{j+1})} \right)^{1/2} \|P_h v_h\|_1 \,\text{for}\, u \in H^{s+1}(0,L) \,\text{and}\, s \in [1,2]. \tag{18}$$

Estimate for I_b: Let w be defined as before but with u replaced by bu. Then

$$\frac{(bu)(x_j) + (bu)(x_{j+1})}{2} - (bu)(x_{j+1/2}) = \frac{w(0) + w(1)}{2} - w(\frac{1}{2}).$$

The functional

$$\lambda(g) := \frac{g(0) + g(1)}{2} - g(\frac{1}{2})$$

is bounded on $W^{2,1}(0,1)$ and vanishes for $g = 1$ and ξ. Again by the Bramble-Hilbert Lemma the estimate

$$|\lambda(g)| \leq C\|g''\|_{L^1(0,1)}, \quad g \in W^{2,1}(0,1),$$

holds and we obtain the bound

$$|I_b| \leq C\|b\|_{\infty,2} \left(\sum_{j=0}^{N-1} h_j^4 \|u\|^2_{H^2(x_j,x_{j+1})} \right)^{1/2} \|P_h v_h\|_1. \tag{19}$$

Estimate for I_c: Recall that $x_{-1/2} = 0, x_{N+1/2} = L, h_{-1} = h_N = 0$. Thus I_c may be written as the sum

$$I_c = 2I_1 + 2I_2 \tag{20}$$

with
$$I_1 := \sum_{j=0}^{N-1} [\frac{h_j}{2}((cu)_j + (cu)_{j+1}) - \int_{x_j}^{x_{j+1}} cu\, dx](\overline{v}_j + \overline{v}_{j+1})$$
and
$$I_2 := \sum_{j=0}^{N-1} [\frac{h_j}{2}((cu)_{j+1} - (cu)_j) + \int_{x_j}^{x_{j+1/2}} cu\, dx - \int_{x_{j+1/2}}^{x_{j+1}} cu\, dx](\overline{v}_{j+1} - \overline{v}_j).$$
The sum in I_1 contains the errors of the trapezoidal rule that can be bounded with the aid of the Bramble-Hilbert Lemma by
$$|\frac{h_j}{2}((cu)_j + (cu)_{j+1}) - \int_{x_j}^{x_{j+1}} cu\, dx| \leq C h_j^2 \|(cu)''\|_{L^1(x_j, x_{j+1})}$$
for $u \in W^{2,1}(x_j, x_{j+1})$. Since $P_h v_h$ is piecewise linear the estimate
$$|I_1| \leq C\|c\|_{2,\infty} \left(\sum_{j=0}^{N-1} h_j^4 \|u\|^2_{H^2(x_j,x_{j+1})} \right)^{1/2} \|P_h v_h\|_0 \tag{21}$$
follows. In I_2 we have only the first order bound
$$|\frac{h_j}{2}((cu)_{j+1} - (cu)_j) + \int_{x_j}^{x_{j+1/2}} cu\, dx - \int_{x_{j+1/2}}^{x_{j+1}} cu\, dx| \leq C h_j \|(cu)'\|_{L^1(x_j,x_{j+1})}$$
for $u \in W^{1,1}(x_j, x_{j+1})$. But the factor $(\overline{v}_{j+1} - \overline{v}_j)$ allows us to estimate I_2 with the same order as I_1 by
$$|I_2| \leq C\|c\|_{1,\infty} \left(\sum_{j=0}^{N-1} h_j^4 \|u\|^2_{H^1(x_j,x_{j+1})} \right)^{1/2} \|P_h v_h\|_1. \tag{22}$$
The first inequality in (14) now follows from the bounds (18), (19), (21) and (22). For the proof of the second one first use the first one with $s = 1$ and $s = 2$ and estimate each h_j by h_{max}. The proof for the remaining $s \in (1, 2)$ is then obtained by interpolation. ■

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