A Line Thicker than the Michael Line

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The Michael line was introduced in [3] in connection with the proof that the product of a normal space and a metric space need not be normal. Topological structures of the same type were considered in [4]. Our interest here is on some coarser topologies on the line, their connected components and the topology of their orbit space under their group of self-homeomorphisms.

1. The Finer Topology $\mathcal{T}_Y$

Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{T}|Y)$ be a subspace of X. We denote by $\mathcal{T}_Y$ the topology on X obtained by the union of $\mathcal{T}$ with $\mathcal{T}|Y$, i.e., a basis of $\mathcal{T}_Y$ is given by the open sets in $\mathcal{T}$ and their restrictions to Y.

Then $\mathcal{T}_Y = \mathcal{T}$ if and only if Y is open with respect to $\mathcal{T}$. Also the topological structure of Y as a subspace of $(X, \mathcal{T})$ is the same as the topological structure of Y as a subspace of $(X, \mathcal{T}_Y)$. Moreover if $f : X \to X$ is a homeomorphism with respect to $\mathcal{T}$ such that $f(Y) = Y$ then f is also a homeomorphism with repect to $\mathcal{T}_Y$.

In what follows X will be the standard real line R and for the final conclusions Y will consist of the irrational numbers or the rational numbers Q. Reference [5] is meant for the topological background.

2. Connectedness

Definition: $Y \subset R$, $Y \neq \emptyset$, is called an *NLE-set* (set without local extrema) if there is no open interval $(a, b) \subset R$ such that $Y \cap (a, b)$ has a minimum or a maximum.

Remark: Examples for NLE-sets are intersections of the irrationals or rationals with open intervals or more generally dense subsets of open subsets (in $\mathcal{T}$). But NLE-sets non-necessarily have to be of that type.

Proposition 1: Let $Y \subset R$ be a non-empty set. Then we have: The connected sets with respect to $\mathcal{T}_Y$ are exactly the intervals $\Leftrightarrow$ Y is an NLE-set.

Proof: $\Rightarrow$: Assume that Y is not an NLE-set. Then there is an open interval (a,b) such that (w.l.o.g.) Y has a maximum m in (a,b). Hence $(a,b) = (a,m] \cup (m,b)$, the union being disjoint, $(a,m] = (a,m) \cup (Y \cap (a,b))$ is nonempty and in $\mathcal{T}_Y$, and (m,b) also is nonempty and in $\mathcal{T}_Y$. Hence (a,b) is not connected with respect to $\mathcal{T}_Y$.

$\Leftarrow$: $\mathcal{T}_Y \supset \mathcal{T}$ implies that a connected set with respect to $\mathcal{T}_Y$ necessarily has to be in interval. We first show that the bounded closed intervals are connected with respect to $\mathcal{T}_Y$:

Assume that there are sets $O, \tilde{O} \in \mathcal{T}_Y$ and an interval $I = [c,d]$ such that $\emptyset \neq O \cap I$, $\emptyset \neq \tilde{O} \cap I$, $I \subset O \cup \tilde{O}$, $I \cap O \cap \tilde{O} = \emptyset$. Assume further w.l.o.g. $c \in O$. We consider two cases:

α) There is a maximal half-open interval $[c,\xi) \subset O \cap I$. Clearly $\xi \neq d$ because otherwise $\tilde{O} \cap I \neq \emptyset$ implies $d \in \tilde{O}$, and then $\tilde{O} \in \mathcal{T}_Y$ and the NLE-property of Y imply $\tilde{O} \cap O \cap I \neq \emptyset$, a contradiction. In the same way we conclude $\xi \notin \tilde{O}$. Hence $\xi \in O$. But then ξ has to be in Y, because otherwise a neighbourhood of the type $(\xi - \eta, \xi + \eta)$ has to be in O for some $\eta > 0$, which contradicts to the maximality of ξ.

Since $\xi \in Y \cap O$ and $\xi < d$ we find a $\delta > 0$ such that $[\xi, \xi+\delta) \cap Y \subset O \cap I$. Then the NLE-property of Y implies that there exists a $\xi_1 \in Y$, $\xi < \xi_1 < \xi + \delta$. Hence $\xi_1 \in O \cap I$. For the interval $[c, \xi_1) \subset I$ we consider the disjoint partition $([c,\xi_1) \cap O) \cup ([c,\xi_1) \cap \tilde{O})$. By construction $[c,\xi_1) \cap Y \subset [c,\xi_1) \cap O$ and $[c,\xi_1) \cap \tilde{O} = [\xi,\xi_1) \cap \tilde{O} \subset R \setminus Y$. For $x \in [\xi,\xi_1) \cap \tilde{O} = (\xi,\xi_1) \cap \tilde{O} \neq \emptyset$ we consider χ maximal such that $[x,\chi) \subset (\xi,\xi_1) \cap \tilde{O}$. This exists because $x \notin Y$. Applying the same argument like in the first paragraph, only interchanging the roles of O and $\tilde{O}$ we see that $\chi \notin O$. Hence $\chi < \xi_1$ and as above it has to be in $Y \cap \tilde{O}$, giving a contradiction to $Y \cap \tilde{O} \cap [\xi, \xi_1) = \emptyset$.

β) If there is no interval of the type $[c,\xi)$ in $O \cap I$, then $c \in Y$. Applying the same argument as in the second half of case α) we shall get a contradiction again. Hence the partition of I assumed above is not possible showing that $[c,d]$ is connected with respect to $\mathcal{T}_Y$.

Since every other type of interval can be obtained as a monotonically increasing union of closed intervals, general results for connected sets imply that all intervals are connected with respect to $\mathcal{T}_Y$. $\bowtie$

Proposition 2: Let, for $Y \subset R$, $x \in Y$ be a point such that $[x, x+\varepsilon) \setminus Y \neq \emptyset$ and $(x-\varepsilon, x] \setminus Y \neq \emptyset$ for all $\varepsilon > 0$. Then $\{x\}$ is the path component of x with respect to $\mathcal{T}_Y$.

Proof: Let C_x be the path component of x, $y \in C_x$ and $w : [0,1] \to R$ a path from x to y. Let $A := w^{-1}(x)$. Then $A \neq \emptyset$ because $0 \in A$. For $t \in A$ and $\varepsilon > 0$ $w^{-1}((x+\varepsilon, x-\varepsilon) \cap Y)$ contains an interval $K := (t-\eta, t+\eta) \cap [0,1]$ for a suitable $\eta > 0$, because w is continous and $(x+\varepsilon, x-\varepsilon) \cap Y$ is a neighborhood of x with respect to $\mathcal{T}_Y$. By continuity $w(K)$ has to be an interval. Obviously $x \in w(K) \subset (x+\varepsilon, x-\varepsilon) \cap Y$. From the assumption on the location of x in Y we get that $w(K) = \{x\}$ and hence A is open in [0,1]. Clearly A is closed in $[0,1]$, which implies $A = [0,1]$ and thus $x = y$. $\bowtie$

Corollary 1: Let Q be the set of rationals. The $\mathcal{T}_Q$ and $\mathcal{T}_{R \setminus Q}$ are totally path disconnected topologies on R which have all intervals as connected sets.

Proof: The statement on the connected sets is a consequence from Proposition 1. Proposition 2 implies that for $\mathcal{T}_Q$ the rationals consist of 1-point path components. But then also the irrationals give 1-point path components only. The same applies to $\mathcal{T}_{R \setminus Q}$. $\bowtie$

3. Self-homeomorphisms and Orbit Spaces

Proposition 3: Let Y be an NLE-set in R and $x \in Y$ such that $(x, x+\varepsilon) \cap (R \setminus Y) \neq \emptyset$ and $(x-\varepsilon, x) \cap (R \setminus Y) \neq \emptyset$ for all $\varepsilon > 0$. Let $f : R \to R$ be a continuous self-map with respect to $\mathcal{T}_Y$. Then for any $y \in f^{-1}(x) \cap (R \setminus Y)$ there is an $\eta > 0$ such that $f|(y-\eta, y+\eta)$ is constant.

Proof: Let $\varepsilon > 0$. Then $V_\varepsilon := (x-\varepsilon, x+\varepsilon) \cap Y \in \mathcal{T}_Y$ and $\{x\}$ is the component of x in V_ε. Let $y \in f^{-1}(x) \cap (R \setminus Y)$. By continuity of f there is an $O \in \mathcal{T}_Y$ such that $f(O) \subset V_\varepsilon$ and $y \in O$. Hence there are $O_1, O_2 \in \mathcal{T}$ such that $y \in O_1 \cup (O_2 \cap Y) \subset O$. Since y is not in Y we conclude $O_1 \neq \emptyset$. Hence $y \in O_1 \subset O$. Hence there is an $\eta > 0$ such that $(y-\eta, y+\eta) \subset O$ which implies with $f(y) = x$ and the connectedness argument above $f((y-\eta, y+\eta)) = \{x\}$. $\bowtie$

Corollary 2: Let Y be an NLE-set in R with empty interior. Then every self-homeomorphism of R with respect to $\mathcal{T}_Y$ is monotonically increasing or decreasing and maps Y to Y and $R \setminus Y$ to $R \setminus Y$.

Proof: Since the interior of Y is empty the assumptions of Proposition 3 are satisfied, which implies the invariance of Y and its complement. The monotonicity is a consequence of Proposition 1. $\bowtie$

Let us denote by $Hom(R, \mathcal{T}_Y)$ the group of self-homeomorphisms of $(R, \mathcal{T}_Y)$.

Proposition 4: The quotient $(R, \mathcal{T}_{(R\setminus Q)})/Hom(R, \mathcal{T}_{(R\setminus Q)})$ is the Sierpinski space.

Proof: We prove that there are just two orbits, one consisting of the rational numbers, which are not open in $\mathcal{T}_{(R\setminus Q)}$, and the other one formed by the irrational numbers, which are open in this topology.

If $q_1, q_2 \in Q$ it is clear that there is a homeomorphism $f : R \to R$ with respect to $\mathcal{T}_{(R\setminus Q)}$ such that $f(q_1) = q_2$. In fact the translation of R which maps q_1 to q_2 will do.

If $q \in Q$ and $i \in R \setminus Q$ then there is not a homeomorphism $f : R_I \to R_I$ such that $f(q) = i$. This follows from Corollary 2.

It remains to show that given $i_1, i_2 \in R \setminus Q$ there is a homeomorphism $f : R \to R$ with respect to $\mathcal{T}_{(R\setminus Q)}$ such that $f(i_1) = i_2$. We will construct a homeomorphism $f : R \to R$ with respect to $\mathcal{T}$ with this property such that $f(Q) = Q$.

Let $(x_i)_{i\in N}$ (resp. $(y_i)_{i\in N}$) be an increasing (resp. decreasing) sequence of rational numbers converging from below (resp. above) to i_1. Similarly let now $(z_i)_{i\in N}, (w_i)_{i\in N}$ be analogous sequences but converging to i_2 this time. We define $f : R \to R$ by

$$f(x) = \begin{cases} i_2, & \text{if } x = i_1 \\ z_1 - x_1 + x, & \text{if } x \in (-\infty, x_1] \\ \frac{z_{i+1} - z_i}{x_{i+1} - x_i}\,(x - x_i) + z_i, & \text{if } x \in [x_i, x_{i+1}], i \in N \\ \frac{-w_{i+1} + w_i}{-y_{i+1} + y_i}\,(x - y_{i+1}) + w_{i+1}, & \text{if } x \in [y_{i+1}, y_i], i \in N \\ w_1 - y_1 + x, & \text{if } x \in [y_1, +\infty) \end{cases}$$

Then f is an increasing homeomorphism with the required properties. Obviously we could have defined f in such a way that it would be decreasing.

$\bowtie$

Similarly $(R, \mathcal{T}_Q)/Hom(R, \mathcal{T}_Q)$ is the Sierpinski space as well.

Examples: With similar methods some other orbit spaces could be computed:

a) Let A_+ (resp. A_-) be the positive (resp. negative) rationals, B_+ (resp. B_-) be the positive (resp. negative) irrationals, and $C := \{0\}$. Then, taking $Y := A_- \cup B_+$ we get as a basis for the quotient topology of $(R, \mathcal{T}_Y)/Hom(R, \mathcal{T}_Y)$

$$\{\emptyset, \{A_-\}, \{A_-, B_-\}, \{B_+\}, \{A_+, B_+\}, \{A_-, A_+, B_-, B_+, C\}\},$$

i.e. we have two Sierpinski 3-spaces matched at C, where by a Sierpinski 3-space we mean a space based on a set $\{a, b, c\}$ for which the non-trivial open sets are $\{a\}, \{a, b\}$.
b) With the notations above take $Y := A_- \cup B_+ \cup C$. Then we get as a basis for the quotient topology of $(R, \mathcal{T}_Y)/Hom(R, \mathcal{T}_Y)$

$$\{\emptyset, \{A_-\}, \{A_-, B_-\}, \{B_+\}, \{A_+, B_+\}, \{A_-, B_+, C\}\}$$

which contains two three point locally Sierpinski spaces (see below, §5) matched at C again.
c) With the notations above take $Y := A_-$ and set $D := (0, \infty)$. Then we get as a basis for the quotient topology of $(R, \mathcal{T}_Y)/Hom(R, \mathcal{T}_Y)$

$$\{\emptyset, \{A_-\}, \{A_-, B_-\}, \{D\}, \{A_-, B_-, C, D\}\}$$

matching a Sierpinski space and a Sierpinski 3-space at C.
d) Extend the Cantor set by reflections and similarities to all of R, leading to a set of the same type with the analogous self-similarities like they are exhibited by the original candidate in $[0, 1]$. This set C contains two types of points. The first type consists of local extrema in C which are endpoints of open intervals in the complement of C. Call the set of these points C_0. Furthermore there is a second type of points which are not local extrema. Call this non-empty set $C_1 = C \setminus C_0$. There are self-similarities of C mapping a given point of C_0 to another given point of C_0. The same applies to C_1. Calling $D := R \setminus C$ and setting $Y := C_1$ we get as a basis for the quotient topology of $(R, \mathcal{T}_Y)/Hom(R, \mathcal{T}_Y)$

$$\{\emptyset, \{C_1\}, \{D\}, \{C_1, C_0, D\}\}.$$

4. The Michael Line

This section is introduced just for the sake of completeness.

The Michael line $(R, \mathcal{T}_M)$ is the topological space with R as base set and the unions $A \cup A'$, with A open in R and A' a subset of the irrational numbers, as open sets.

It follows that if $f : R \to R$ is an injective, continuous map with respect to $\mathcal{T}_M$ then $f(Q) \subset Q$. Also if $f : R \to R$ is continuous with respect to $\mathcal{T}$ and $f(Q) \subset Q$ then $f : R \to R$ is also continuous with respect to $\mathcal{T}_M$.

Proposition 5: $(R, \mathcal{T}_M)/Hom(R, \mathcal{T}_M)$ is the Sierpinski space.

Proof: By one of the remarks above there is no homeomorphism mapping a rational to an irrational or vice versa.

If $i_1, i_2, \in R \setminus Q$ then we can take as homeomorphism with respect to $\mathcal{T}_M$ the map $f : R \to R$ given by $f(i_1) = i_2, f(i_2) = i_1, f(x) = x$, for $x \neq i_1, i_2$.

If $q_1, q_2, \in Q$ take the translation of R that maps q_1 to q_2. $\bowtie$

If we replace $R \setminus Q$ by Q in the definition of the Michael line then a statement similar to the one we have just proved still holds. A map like the one in the proof of Proposition 4 could be used to establish it.

5. Locally Sierpinski Spaces

A topological space X is *locally Sierpinski* (*l. S.* in short) if, for $x \in X$, there is an open neighbourhood homeomorphic to the Sierpinski space [1].

Examples of connected l. S. spaces are obtained as follows. Let X be a set with, at least, two points. Fix $p \in X$ and define a set to be open if it is either the empty set or contains p.

L. S. spaces can be characterized as locally (path-)connected spaces whose (path-)components are subspaces of the same type as X [2].

A more general question than the one we answered above is to know which l. S. spaces can be obtained as the quotient of any of the mentioned spaces by a group of homeomorphisms. We do not have the answer but we point out that l. S. spaces other than the Sierpinski space can be obtained. For example, if G is the group of homeomorphisms of $f : R \to R$ with respect to $\mathcal{T}_Q$ such that $f(Z) = Z$ (resp. $f(Z) = Z$ and $f(Z + \frac{1}{2}) = Z + \frac{1}{2}$) then the orbit space is an l. S. space with 3 (resp. 4) points.

This question for the Sorgenfrey line was dealt with in [2].

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