Convergence rates for the estimation of two-dimensional distribution functions under association and estimation of the covariance of the limit empirical process*

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Abstract

Let X_n, $n \geq 1$, be an associated and strictly stationary sequence of random variables, having a marginal distribution function F. Under some conditions on the covariance structure of those variables, the empirical process converges in distribution to a centered Gaussian process Z, with covariance function defined by an infinite sum of terms of the form $\varphi_k(s,t) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - F(s)F(t)$. Under suitable conditions on the decrease rate of the covariances $\mathrm{Cov}(X_1, X_n)$, $n \geq 2$, we prove an exponential type inequality, from which a convergence rate for the almost sure convergence of the estimator of $\varphi_k(s,t)$ is derived. Finally, we find an estimator for the infinite sum that defines the covariance function of the limit process Z, and prove that the decrease rates on $\mathrm{Cov}(X_1, X_n)$ are also sufficient for the strong consistency of that estimator.

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1 Introduction, assumptions and definitions

Let X_n, $n \geq 1$, be a strictly stationary sequence of real-valued random variables with common continuous distribution function F. The empirical process induced by the sequence X_n, $n \geq 1$, is defined by

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(I_{[-\infty,t]}(X_i) - F(t) \right),$$

where I_A represents the indicator function of the set A. The limit behaviour of the empirical process has been intensively studied in recently years due to the importance of this function to many statistical applications. It is well known that the study of convergence of Z_n can be carried out supposing the variables X_n, $n \geq 1$, to be uniformly distributed on $[0,1]$. This case we will be referred to as the uniform empirical process. For independent random variables, the uniform empirical process converges in the Skorohod space $D[0,1]$ to the Brownian bridge, a centered Gaussian process with covariance function $\Gamma(s,t) = s \wedge t - st$. For dependent sequences, under

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certain conditions, the limit of the uniform empirical process still is a centered Gaussian process, but the covariance function changes to

$$\Gamma(s,t) = s \wedge t - st + \sum_{k=1}^{\infty} \left(\mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - st\right) + \sum_{k=1}^{\infty} \left(\mathrm{P}\left(X_1 \leq t, X_{k+1} \leq s\right) - st\right) , \tag{1}$$

due to the presence of covariances between the original random variables.

The problem of characterizing the limit distribution of the uniform empirical process has been studied under different conditions on the structure of dependence of the variables of the sequence X_n, $n \geq 1$. In this paper we consider associated random variables, a dependence concept introduced by Esary *et al.* [3] which we recall here. The random variables X_n, $n \geq 1$, are associated if

$$\mathrm{Cov}\left(f\left(X_1, ..., X_n\right),\, g\left(X_1, ..., X_n\right)\right) \geq 0$$

for any $n \in \mathbb{N}$ and any real-valued coordinatewise increasing functions f and g for which the covariance above exists.

For this kind of dependence structure, and supposing strict stationarity of the sequence, convergence results in $D(0,1)$ for the uniform empirical process were first obtained by Yu [15] under the assumption $\mathrm{Cov}(X_1, X_n) = O(n^{-r})$ with $r > 7.5$, later improved by Shao and Yu [14] requiring only that $r > (3+\sqrt{33})/2 \approx 4.373$. The best known rate has been proved by Louhichi [7] requiring that $r > 4$.

Oliveira and Suquet studied this problem in the space $L^2[0,1]$ (see [10]) and later in $L^p[0,1]$, with $p \geq 2$ (see [11]). Again under the strict stationarity of the sequence, the convergence of the uniform empirical process in these spaces follows from:

- $\sum_n \mathrm{Cov}^{1/3}(X_1, X_n) < \infty$, in the $L^2[0,1]$ space;
- $\mathrm{Cov}(X_1, X_n) = O(n^{-r})$ with $r > 3p/2$, in the $L^p[0,1]$ space.

A necessary and sufficient condition for the $L^2[0,1]$ convergence of the uniform empirical process has been proved by Morel and Suquet [8] requiring that

$$\sum_{n=2}^{\infty} \left(\frac{2}{3} - \mathrm{E}\max(X_1,\, X_n)\right) < \infty\,.$$

Here we will not use any further this necessary and sufficient condition as we will concentrate on assumptions on the covariance structure.

As it was mentioned above, these results can easily be extended to the case where the random variables X_n, $n \geq 1$, are not uniformly distributed on $[0,1]$, the limit process of Z_n being now a centered Gaussian process with covariance function given by

$$\begin{aligned}\Gamma(s,t) &= F(s \wedge t) - F(s)F(t) + \\ &+ \sum_{k=1}^{\infty} \left(\mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - F(s)F(t)\right) + \sum_{k=1}^{\infty} \left(\mathrm{P}\left(X_1 \leq t, X_{k+1} \leq s\right) - F(s)F(t)\right) .\end{aligned} \tag{2}$$

For practical proposes we need to be able to approximate the sum of the series in the expression of $\Gamma(s,t)$. Such an example is the Cramér-von Mises test statistic, which is the $L^2[0,1]$ norm of the uniform empirical process. The convergence of the uniform empirical process to the centered

Gaussian process Z, implies that of the Cramér-von Mises test statistic to the $L^2[0,1]$ norm of Z. So we have an asymptotic distribution for this test statistic, but we can not characterize it completely because we do not know the covariance function of Z.

Under the assumptions of association and strict stationarity of the sequence X_n, $n \geq 1$, Henriques and Oliveira [5] studied the histogram estimator for the distribution function of (X_1, X_{k+1}), namely,

$$\widehat{F}_{k,n}(s,t) = \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[-\infty,s]}(X_i) I_{[-\infty,t]}(X_{i+k}) \right) . \tag{3}$$

The strong consistency of the estimator follows if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}^{1/3}(X_1, X_j) = 0 \,,$$

but no rates are provided. In the present paper we establish an exponential inequality from which a convergence rate for the almost sure convergence of that estimator is derived. This is done in Theorems 1 and 2 of Section 3.

For the estimation of the terms $\varphi_k(s,t) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - F(s)F(t)$, with $k \in \mathbb{N}$ fixed, we consider the following estimator

$$\widehat{\varphi}_{k,n}(s,t) = \widehat{F}_{k,n}(s,t) - F_n(s)F_n(t) \,, \tag{4}$$

where F_n is the empirical distribution function defined by $F_n(s) = \frac{1}{n} \sum_{j=1}^{n} I_{[-\infty,s]}(X_i)$. Finally, the infinite sum in the expression of $\Gamma(s,t)$ is estimated by

$$\sum_{k=1}^{q_n} \widehat{\varphi}_{k,n}(s,t) \,, \tag{5}$$

where $q_n \to +\infty$, in a manner to be precised later.

A convergence rate for the almost sure convergence of $\widehat{\varphi}_{k,n}(s,t)$ is obtained in Theorem 3 of Section 3. This theorem also establishes the strong consistency of the estimator (5).

For easy reference we now present the assumptions to be considered throughout this paper.

Assumptions

(S1) X_n, $n \geq 1$, is an associated and strictly stationary sequence of random variables having density function bounded by B_0;

(S2) Set $C(k) = \mathrm{Cov}(X_1, X_{k+1})$. We assume that $C(k)$ is nonincreasing as $k \to \infty$.

Given assumption (S1) we define the constant $B_1 = 2\max(2/\pi^2,\, 45B_0)$.

2 Notation and preliminary results

In this section we state and prove some preliminary results needed for the proof of the theorems of the next section. These auxiliary results take care of most of the technical aspects of the proof of the exponential inequality to be proved below in Theorem 1, which is one of the main results of this article. The proof technique is similar to the approach of Ioannides and Roussas [4] who where the first authors to prove an exponential inequality for associated variables.

Before proceeding to more specific notations and results we quote a general lemma of interest in the course of proof of our lemmas.

Lemma 1 (Devroye [1]) *Let X be a centered random variable. If there exist $a, b \in \mathbb{R}$ such that* $\mathrm{P}(a \le X \le b) = 1$*, then, for every $\lambda > 0$,*

$$\mathrm{E}(e^{\lambda X}) \le \exp\left(\frac{\lambda^2 (b-a)^2}{8}\right).$$

For the formulation of the next results we need to introduce some additional notation. Let q_n be a sequence of positive integers such that $q_n \to \infty$. For each $n \in \mathbb{N}$ and each $k \in \{0, 1, ..., q_n\}$ divide the set $\{1, ..., n-k\}$ into subsets, each one containing p_n elements, where p_n is a sequence of positive integers satisfying $1 \le p_n \le n - q_n$, $p_n > q_n$ and $p_n \to \infty$. The number of subsets with p_n elements is given by $2r_{k,n}$, where $r_{k,n}$ is the largest integer such that

$$0 < r_{k,n} < n - k, \qquad 2r_{k,n} \le \frac{n-k}{p_n}.$$

We will suppose that, for each $k \in \{0, 1, ..., q_n\}$, $r_{k,n} \to \infty$. The last subset in the partition of $\{1, ..., n-k\}$ will have $n - k - 2r_{k,n}p_n < 2p_n$ elements. The choice of these sequences is crucial for the proof of the exponential inequality. These sequences must be well tuned with the behaviour of the covariance structure of the variables X_n, $n \ge 1$. Assuming a convenient decrease rate on the covariances, an example showing the constructibility of such sequences, satisfying the assumptions that will be introduced below, is given in Section 4.

For the sequences just defined we have, for each $k \in \{0, 1, ..., q_n\}$,

$$1 \le \frac{n-k}{2r_{k,n}p_n} \le \frac{2p_n + 2r_{k,n}p_n}{2r_{k,n}p_n} = \frac{1 + r_{k,n}}{r_{k,n}},$$

so that,

$$\frac{n-k}{2r_{k,n}p_n} \longrightarrow 1. \tag{6}$$

Note also that, for each $n \in \mathbb{N}$, the set $\{1, ..., n-1\}$ has more $q_n - 1$ elements than $\{1, ..., n - q_n\}$ and, by definition, $q_n < p_n$. So, the partition of the first set will have at most two more subsets than the partition of the last one. This means that, for each $k \in \{0, 1, .., q_n\}$, we have

$$r_{k,n} = r_{1,n} \quad \vee \quad r_{k,n} = r_{1,n} - 1. \tag{7}$$

Let us define the sets $E_i = \{2(i-1)p_n + 1, ..., (2i-1)p_n\}$, $O_i = \{(2i-1)p_n + 1, ..., 2ip_n\}$, for each $i = 1, ...r_{k,n}$, and $R = \{2r_{k,n}p_n + 1, ..., n-k\}$.

Then we may partition the set $\{1, \ldots, n-k\}$ as follows:

$$\{\underbrace{1, ..., p_n}_{E_1}, \underbrace{p_n + 1, ..., 2p_n}_{O_1}, ..., \underbrace{(2r_{k,n} - 2)p_n + 1, ..., (2r_{k,n} - 1)p_n}_{E_{r_{k,n}}},$$
$$\underbrace{(2r_{k,n} - 1)p_n + 1, ..., 2r_{k,n}p_n}_{O_{r_{k,n}}}, \underbrace{2r_{k,n}p_n + 1, ..., n-k}_{R}\}.$$

For each $n \in \mathbb{N}$, $k \in \{0, 1, .., q_n\}$ and fixed $s, t \in \mathbb{R}$, define the random variables

$$W_{k,n} = I_{(-\infty,s]}(X_n) I_{(-\infty,t]}(X_{k+n}) - \mathrm{P}(X_1 \le s, X_{k+1} \le t).$$

Note that the random variables $W_{k,n}$, $n \ge 1$, are centered and bounded by 1. Additionally, since the sequence X_n, $n \ge 1$, is associated and strictly stationary and the $W_{k,n}$ are decreasing functions of the X_n, the sequence $W_{k,n}$, $n \ge 1$, is also associated and strictly stationary.

To obtain the exponential inequality the sum in

$$\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t) = \frac{1}{n-k} \sum_{i=1}^{n-k} W_{k,i}$$

is decomposed into three parts, one consisting of the sum for indexes in the blocks E_i, another consisting of the sum for indexes in the blocks O_i and the last one consisting of the sum for indexes in R. To do this, define the random variables

$$U_{k,i} = \sum_{j \in E_i} W_{k,j}\,, \qquad V_{k,i} = \sum_{j \in O_i} W_{k,j}\,, \qquad i = 1, ...r_{k,n} \qquad \text{and} \qquad Z_{k,n} = \sum_{j \in R} W_{k,j}\,.$$

Now, set

$$\overline{U}_{k,n} = \frac{1}{n-k} \sum_{i=1}^{r_{k,n}} U_{k,i}\,, \qquad \overline{V}_{k,n} = \frac{1}{n-k} \sum_{i=1}^{r_{k,n}} V_{k,i} \qquad \text{and} \qquad \overline{Z}_{k,n} = \frac{1}{n-k} Z_{k,n}\,,$$

so that

$$\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t) = \overline{U}_{k,n} + \overline{V}_{k,n} + \overline{Z}_{k,n}\,. \tag{8}$$

We now present three lemmas that will be employed in the proof of the theorems of the next section. The first lemma follows from Newman's inequality [9] (for a detailed proof see Dewan and Prakasa Rao [2]).

Lemma 2 *Let $X_1, X_2, ..., X_n$ be associated random variables that are bounded by a constant M. Then, for any $\theta > 0$,*

$$\left| \mathrm{E}\left(e^{\theta \sum_{i=1}^{n} X_i}\right) - \prod_{i=1}^{n} \mathrm{E}\left(e^{\theta X_i}\right) \right| \le \theta^2 e^{n\theta M} \sum_{1 \le i < j \le n} \mathrm{Cov}(X_i, X_j)\,.$$

The following lemma establishes an exponential probability inequality for the variables $\overline{U}_{k,n}$ defined above. A similar lemma holds for $\overline{V}_{k,n}$.

Lemma 3 *Let $\varepsilon > 0$. Suppose that* (S1) *and* (S2) *are satisfied and there exists a constant $C_1 > 0$ such that*

$$\exp\left(8 r_{k,n} \frac{\varepsilon}{q_n}\right) C^{1/3} (p_n - k) \le C_1\,. \tag{9}$$

Then

$$\mathrm{P}\left(|\overline{U}_{k,n}| \ge \frac{\varepsilon}{q_n}\right) \le 2 C_2 \exp\left(-2 r_{k,n} \frac{\varepsilon^2}{q_n^2}\right),$$

where $C_2 = 1 + 2B_1 C_1$.

Proof: For each $k \in \{0, 1, ..., q_n\}$, the variables $U_{k,1}, U_{k,2}, ..., U_{k,r_{k,n}}$, being the sum of associated random variables, are associated. Additionally, $|U_{k,i}| \le p_n$ for every $i = 1, \ldots, r_{k,n}$. Then we may apply Lemma 2 to obtain that, given $\lambda > 0$,

$$\begin{aligned} \mathrm{E}\left(e^{\lambda \overline{U}_{k,n}}\right) &= \mathrm{E}\left(e^{\frac{\lambda}{n-k} \sum_{i=1}^{r_{k,n}} U_{k,i}}\right) \le \\ &\le \prod_{i=1}^{r_{k,n}} \mathrm{E}\left(e^{\frac{\lambda}{n-k} U_{k,i}}\right) + \frac{\lambda^2}{(n-k)^2} \exp\left(r_{k,n} \frac{\lambda}{n-k} p_n\right) \sum_{1 \le i < j \le r_{k,n}} \mathrm{Cov}(U_{k,i}, U_{k,j})\,. \end{aligned} \tag{10}$$

As the density of the variables is supposed to be bounded by B_0, it follows from Corollary to Theorem 1 in Sadikova [13] and relation (21) in Newman [9] (see Lemma 2.6 in Roussas [12] for details) that

$$\mathrm{Cov}\left(I_{[-\infty,s]}(X_i), I_{[-\infty,t]}(X_j)\right) \leq B_1 \mathrm{Cov}^{1/3}\left(X_i, X_j\right), \qquad s,t \in \mathbb{R}, \tag{11}$$

where B_1 has been defined earlier. Now applying a classical inequality by Lebowitz [6] and taking account of (11), we find

$$\mathrm{Cov}(W_{k,l}, W_{k,m}) \leq B_1 \Big[\mathrm{Cov}^{1/3}(X_l, X_m) + \mathrm{Cov}^{1/3}(X_l, X_{k+m}) + \\ + \mathrm{Cov}^{1/3}(X_{k+l}, X_m) + \mathrm{Cov}^{1/3}(X_{k+l}, X_{k+m})\Big]\,,$$

so, using (S2), it follows that

$$\mathrm{Cov}(W_{k,l}, W_{k,m}) \leq 4B_1 C^{1/3}(p_n - k)\,.$$

Therefore

$$\begin{aligned}\sum_{1\leq i<j\leq r_{k,n}} \mathrm{Cov}(U_{k,i}, U_{k,j}) &= \sum_{i=1}^{r_{k,n}-1} \sum_{j=i+1}^{r_{k,n}} \sum_{l\in E_i} \sum_{m\in E_j} \mathrm{Cov}(W_{k,l}, W_{k,m}) \\ &\leq \sum_{i=1}^{r_{k,n}-1} \sum_{j=i+1}^{r_{k,n}} \sum_{l\in E_i} \sum_{m\in E_j} 4B_1 C^{1/3}(p_n - k) \\ &= \sum_{i=1}^{r_{k,n}-1} \sum_{j=i+1}^{r_{k,n}} 4p_n^2 B_1 C^{1/3}(p_n - k) \\ &= 2r_{k,n}(r_{k,n}-1)p_n^2 B_1 C^{1/3}(p_n - k).\end{aligned}$$

Inequality (10) then becomes

$$\mathrm{E}\left(e^{\lambda \overline{U}_{k,n}}\right) \leq \prod_{i=1}^{r_{k,n}} \mathrm{E}\left(e^{\frac{\lambda}{n-k} U_{k,i}}\right) + \exp\left(\frac{r_{k,n} p_n \lambda}{n-k}\right) \frac{2r_{k,n} p_n^2 \lambda^2}{(n-k)^2}\left(r_{k,n}-1\right) B_1 C^{1/3}(p_n - k)\,. \tag{12}$$

By construction of the sequences $r_{k,n}$ and p_n, we have $2r_{k,n}p_n \leq n-k$. It follows then that $\frac{2p_n^2 r_{k,n}}{(n-k)^2} \leq \frac{1}{2r_{k,n}}$ and $\frac{p_n r_{k,n}}{n-k} \leq \frac{1}{2} < 1$. From (12) we then get

$$\mathrm{E}\left(e^{\lambda \overline{U}_{k,n}}\right) \leq \prod_{i=1}^{r_{k,n}} \mathrm{E}\left(e^{\frac{\lambda}{n-k} U_{k,i}}\right) + e^{\lambda}\lambda^2 \frac{r_{k,n}-1}{2r_{k,n}} B_1 C^{1/3}(p_n - k)\,. \tag{13}$$

Applying the inequality $xe \leq e^x, x \in \mathbb{R}$, with $x = \lambda/2$, we find $\lambda^2 \leq 4e^{\lambda}e^{-2} \leq 4e^{\lambda}$, and consequently from (13) it follows

$$\mathrm{E}\left(e^{\lambda \overline{U}_{k,n}}\right) \leq \prod_{i=1}^{r_{k,n}} \mathrm{E}\left(e^{\frac{\lambda}{n-k} U_{k,i}}\right) + 2B_1 e^{2\lambda} C^{1/3}(p_n - k)\,. \tag{14}$$

Suppose now that there exists a constant $C_0 > 0$ such that

$$2B_1 e^{2\lambda} C^{1/3}(p_n - k) \leq C_0\,. \tag{15}$$

Under this assumption, applying Lemma 1 it follows from (14) that

$$\mathrm{E}\left(e^{\lambda \overline{U}_{k,n}}\right) \leq \exp\left[r_{k,n}\frac{\lambda^2}{8}\left(\frac{2p_n}{n-k}\right)^2\right] + C_0 \leq e^{\frac{\lambda^2}{8r_{k,n}}} + C_0\,, \tag{16}$$

as $\frac{4p_n^2 r_{k,n}}{(n-k)^2} \leq \frac{1}{r_{k,n}}$. Then, by the Markov inequality, we obtain, for each $\varepsilon > 0$,

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{q_n}\right) \leq e^{-\lambda\frac{\varepsilon}{q_n}+\frac{\lambda^2}{8r_{k,n}}} + C_0 e^{-\lambda\frac{\varepsilon}{q_n}}\,.$$

In order to minimize the first term in the right of the previous inequality we choose $\lambda = 4r_{k,n}\frac{\varepsilon}{q_n}$. We find then that

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{q_n}\right) \leq e^{-2r_{k,n}\frac{\varepsilon^2}{q_n^2}} + C_0 e^{-4r_{k,n}\frac{\varepsilon^2}{q_n^2}} \leq (1+C_0)e^{-2r_{k,n}\frac{\varepsilon^2}{q_n^2}}\,. \tag{17}$$

Note that for this choice of λ the requirement (15) becomes

$$2B_1 \exp\left(\frac{8r_{k,n}\varepsilon}{q_n}\right) C^{1/3}(p_n - k) \leq C_0\,, \tag{18}$$

which is equivalent to (9), with $C_1 = \frac{C_0}{2B_1}$.

Since the variables $-U_{k,i}$, $i = 1,..,r_{k,n}$, have the same properties as the variables $U_{k,i}$, $i = 1,..,r_{k,n}$, inequality (17) also holds for $-\overline{U}_{k,n}$. Then we have, under (9), that

$$\mathrm{P}\left(|\overline{U}_{k,n}| \geq \frac{\varepsilon}{q_n}\right) \leq \mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{q_n}\right) + \mathrm{P}\left(-\overline{U}_{k,n} \geq \frac{\varepsilon}{q_n}\right) \leq 2C_2 e^{-2r_k\frac{\varepsilon^2}{q_n^2}}. \tag{19}$$

■

Let $\alpha > 1$ and let ε_n be defined by:

$$\varepsilon_n = \left(\frac{9}{2}\alpha\right)^{1/2} q_n \left(\frac{\log\, n}{r_{1,n}-1}\right)^{1/2}. \tag{20}$$

Lemma 4 *Let ε_n be defined as in (20), then, under assumptions* (S1) *and* (S2)*,*

$$\mathrm{P}\left(|\overline{Z}_{k,n}| \geq \frac{\varepsilon_n}{3q_n}\right) = 0\,,$$

for every sufficiently large n and every $k \in \{0,1,...,q_n\}$.

Proof: Note that $\overline{Z}_{k,n}$, being the sum of $(n-k) - 2r_{k,n}p_n < 2p_n$ variables, which are bounded by 1, satisfies

$$|\overline{Z}_{k,n}| \leq \frac{2p_n}{n-k}\,,$$

and consequently

$$\mathrm{P}\left(|\overline{Z}_{k,n}| \geq \frac{\varepsilon_n}{3q_n}\right) \leq \mathrm{P}\left(\frac{2p_n}{n-k} \geq \frac{\varepsilon_n}{3q_n}\right) = \mathrm{P}\left(\frac{\varepsilon_n(n-k)}{6p_nq_n} \leq 1\right).$$

The last probability is equal to zero for every sufficiently large n and $k \in \{0, 1, ..., q_n\}$. In fact, on account of (7), we have, for each $k \in \{0, 1, ..., q_n\}$,

$$\begin{aligned}
\frac{\varepsilon_n(n-k)}{6p_nq_n} &= \left(\frac{9}{2}\,\alpha\right)^{1/2} q_n \left(\frac{\log\, n}{r_{1,n}-1}\right)^{1/2} \frac{n-k}{6p_nq_n} \\
&> \left(\frac{9}{2}\,\alpha\right)^{1/2} \left(\frac{\log\, n}{r_{1,n}-1}\right)^{1/2} \frac{n-k}{6p_n} \times \frac{r_{1,n}-1}{r_{k,n}} \\
&= \left(\frac{9}{2}\,\alpha\right)^{1/2} ((r_{1,n}-1)\,\log\, n)^{1/2}\, \frac{n-k}{6p_nr_{k,n}}\,,
\end{aligned}$$

which converges to $+\infty$ due to (6). ■

3 Main results

The next theorem establishes an exponential probability inequality which will be used, in Theorem 2, to obtain a convergence rate for the almost sure convergence of the estimator $\widehat{F}_{k,n}(s,t)$.

Theorem 1 *Let $\alpha > 1$, let ε_n be defined as in (20), and suppose that assumptions* (S1) *and* (S2) *hold. If there exists a constant $C_1 > 0$ such that, for every $n \geq n_0$,*

$$\exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n-q_n) \leq C_1\,, \tag{21}$$

then, for every $n \geq n_0$ and $k \in \{0, 1, ..., q_n\}$,

$$\mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \leq s, X_{k+1} \leq t)\right| \geq \frac{\varepsilon_n}{q_n}\right) \leq C_3 \exp\left(-\frac{2}{9}\, r_{k,n}\,\frac{\varepsilon_n^2}{q_n^2}\right)\,,$$

where $C_3 = 4 + 8B_1C_1$.

Proof: Note that, on account of (7) and (S2), we have, for each $k \in \{0, 1, ..., q_n\}$ and $n \geq n_0$,

$$\exp\left(8r_{k,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n-k) \leq \exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n-q_n) \leq C_1\,,$$

using (21). That is, condition (9) of Lemma 3 is satisfied, so we may apply this lemma, which obviously also holds for $\overline{V}_{k,n}$. Therefore, taking account of decomposition (8), we obtain, from Lemma 3 and Lemma 4,

$$\begin{aligned}
&\mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \leq s, X_{k+1} \leq t)\right| \geq \frac{\varepsilon_n}{q_n}\right) \leq \\
&\quad\leq \mathrm{P}\left(|\overline{U}_{k,n}| \geq \frac{\varepsilon_n}{3q_n}\right) + \mathrm{P}\left(|\overline{V}_{k,n}| \geq \frac{\varepsilon_n}{3q_n}\right) + \mathrm{P}\left(|\overline{Z}_{k,n}| \geq \frac{\varepsilon_n}{3q_n}\right) \leq \\
&\quad\leq 4C_2 \exp\left(-\frac{2}{9}\, r_{k,n}\,\frac{\varepsilon_n^2}{q_n^2}\right)\,,
\end{aligned}$$

for every $n \geq n_0$ and $k \in \{0, 1, ..., q_n\}$. ■

Theorem 2 *Let $\alpha = 1+\delta$, for some $\delta > 0$, and $0 < \beta < \delta$. Choose $q_n = O(n^\beta)$, $r_{1,n} = o\left(\frac{1}{q_n^2 \log\, n}\right)$ and ε_n defined as in (20). Then, under* (S1), (S2) *and (21), it holds*

$$q_n \left(\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t)\right) \longrightarrow 0 \qquad a.s., \tag{22}$$

for each $k \in \{0, 1, ..., q_n\}$, and also,

$$\sum_{k=1}^{q_n} \left(\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t)\right) \longrightarrow 0 \qquad a.s.. \tag{23}$$

Proof: For each $n \ge n_0$ and each k, it follows from Theorem 1 that, using (20) and (7),

$$\begin{aligned} \mathrm{P}\left(q_n \left|\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t)\right| > \varepsilon_n\right) &\le C_3 \exp\left(-\frac{2}{9}\, r_{k,n}\, \frac{\varepsilon_n^2}{q_n^2}\right) \le \\ &= C_3 \exp\left(-\frac{r_{k,n}}{r_{1,n}-1}\, \alpha\, \log\, n\right) \le \\ &\le C_3 e^{-\alpha\, \log\, n} = C_3 n^{-\alpha}\,, \end{aligned} \tag{24}$$

where $\alpha > 1$.

Since $r_{1,n} = o\left(\frac{1}{q_n^2 \log\, n}\right)$, we have

$$\varepsilon_n = \left(\frac{9}{2}\,\alpha \frac{q_n^2\, \log\, n}{r_{1,n}-1}\right)^{1/2} \longrightarrow 0\,. \tag{25}$$

The convergence (22) follows now from (24) and (25), using the Borel-Cantelli Lemma.

On account of (24) and using $q_n = O(n^\beta)$, we obtain

$$\begin{aligned} &\mathrm{P}\left(\left|\sum_{k=1}^{q_n} \left(\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t)\right)\right| > \varepsilon_n\right) \le \\ &\le \sum_{k=1}^{q_n} \mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t) - \mathrm{P}(X_1 \le s, X_{k+1} \le t)\right| > \frac{\varepsilon_n}{q_n}\right) \le \\ &\le q_n\, C_3\, n^{-\alpha} \le C_3\, n^{-(\alpha-\beta)}\,, \end{aligned} \tag{26}$$

where $\alpha - \beta > 1$. Now (25) and (26) imply the convergence (23). ■

The next theorem states results correspondent to the previous theorem but with respect to the centered estimators $\widehat{\varphi}_{k,n}(s,t)$ and $\sum_{k=1}^{q_n} \widehat{\varphi}_{k,n}(s,t)$.

Theorem 3 *Under the conditions of Theorem 2, it holds*

$$q_n \left(\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)\right) \longrightarrow 0 \qquad a.s., \tag{27}$$

for each $k \in \{0, 1, ..., q_n\}$, and also,

$$\sum_{k=1}^{q_n} \widehat{\varphi}_{k,n}(s,t) \longrightarrow \sum_{k=1}^{\infty} \varphi_k(s,t) \qquad a.s. \tag{28}$$

Proof: Setting in Theorem 2 $k=0$ and $s=t$ we find,

$$q_n\left(\widehat{F}_{0,n}(s,s)-F(s)\right)\longrightarrow 0 \qquad a.s., \tag{29}$$

where $\widehat{F}_{0,n}(s,s)=F_n(s)$, the one-dimensional empirical function.

Since

$$q_n(F(s)F(t)-F_n(s)F_n(t))=F(s)\,q_n(F(t)-F_n(t))+F_n(t)\,q_n(F(s)-F_n(s))\,,$$

and due to (29) it follows

$$q_n(F(s)F(t)-F_n(s)F_n(t))\longrightarrow 0 \qquad a.s.. \tag{30}$$

Now write

$$q_n\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right)=q_n\left(\widehat{F}_{k,n}(s,t)-\mathrm{P}(X_1\le s,X_{k+1}\le t)\right)+q_n(F(s)F(t)-F_n(s)F_n(t))\,.$$

Thus,(27) follows from (22) together with (30).

Analogously, (23) and (30) lead to

$$\begin{aligned}&\sum_{k=1}^{q_n}\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right)=\\&\quad=\textstyle\sum_{k=1}^{q_n}\left(\widehat{F}_{k,n}(s,t)-\mathrm{P}(X_1\le s,X_{k+1}\le t)\right)+q_n(F(s)F(t)-F_n(s)F_n(t))\longrightarrow 0 \quad a.s.,\end{aligned}$$

which is equivalent to (28), as

$$\sum_{k=1}^{q_n}\widehat{\varphi}_{k,n}(s,t)-\sum_{k=1}^{\infty}\varphi_k(s,t)=\sum_{k=1}^{q_n}\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right)-\sum_{k=q_n+1}^{\infty}\varphi_k(s,t)$$

the last term converging to zero, because it is the rest of a convergent series. ■

4 An example

We now show an example of a covariance structure where the construction of the sequences p_n, q_n and $r_{k,n}$ satisfying all the requirements, and in particular (21), is achieved. This condition requires a quite fast decrease rate on the covariance function $C(k)$. Indeed, we shall see that if $C(k)$ decreases at a geometrical rate, condition (21) is satisfied for a convenient choice of q_n and $r_{1,n}$, but, if the covariance function $C(k)$ decreases only at a polynomial rate this condition is not attainable. This situation is not new. In fact, the examples given in Ioannides and Roussas [4] already show this behaviour: a geometrical decrease rate on $C(k)$ enables the identification of a convergence rate while a polynomial decrease rate does not.

Suppose that there exist $a_0>0$ and $a>1$ such that $C(k)=a_0a^{-k}$. Let $0<\delta\le 1/3$, $0<\beta<\delta$ and put $\alpha=1+\delta$. Choose $q_n=n^{\beta}$ and $r_{1,n}=n^{2\beta}(\log\, n)^{\gamma}$, for some $\gamma>1$. It is easily checked that for these choices of q_n and $r_{1,n}$ the requirements on these sequences of Theorems 2 and 3 are satisfied. Now, we will verify that condition (21) is also satisfied.

In the following discussion, c is a positive constant which may take different values in each appearance.

Due to (6), we can write $p_n = \frac{n-1}{2x_n r_{1,n}}$, for some $0 < x_n \to 1$, and then we have, using (20),

$$\begin{aligned} 0 \le \exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n - q_n) &= \exp\left[8r_{1,n}\left(\frac{9}{2}\,\alpha\,\frac{\log\, n}{r_{1,n}-1}\right)^{1/2}\right] C^{1/3}\left(\frac{n-1}{2x_n r_{1,n}} - q_n\right) = \\ &= \exp\left[\left(c\,\frac{r_{1,n}^2}{r_{1,n}-1}\,\log\, n\right)^{1/2}\right] a_0^{1/3} a^{-\frac{1}{3}\left(\frac{n-1}{2x_n r_{1,n}} - q_n\right)}. \end{aligned}$$

But, for n large enough, $\frac{r_{1,n}^2}{r_{1,n}-1} \le 2r_{1,n}$. Thus, we get, for $n \ge n_0$,

$$0 \le \exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n - q_n) \le a_0^{1/3} \exp\left[(c\, r_{1,n}\,\log\, n)^{1/2}\right] a^{-\frac{1}{3}\left(\frac{n-1}{2x_n r_{1,n}} - q_n\right)}, \tag{31}$$

For the exponents appearing on the right-hand side above, we have,

$$\frac{(c\, r_{1,n}\,\log\, n)^{1/2}}{\frac{1}{3}\left(\frac{n-1}{2x_n r_{1,n}} - q_n\right)} = \frac{c\, n^{\beta}\,(\log\, n)^{(\gamma+1)/2}}{\frac{1}{3}\left(\frac{n-1}{2x_n\, n^{2\beta}\,(\log\, n)^{\gamma}} - n^{\beta}\right)} = c\,\frac{n^{3\beta}(\log\, n)^{(3\gamma+1)/2} x_n}{n - 1 - 2x_n n^{3\beta} (log\, n)^{\gamma}} \longrightarrow 0\,,$$

since $3\beta < 1$ and $x_n \to 1$. Therefore, the sequence on the right-hand side of (31) converges to zero and consequently condition (21) is satisfied.

Now suppose that $C(k) = a_0 k^{-a}$, with $a_0 > 0$ and $a > 0$. Then, again using (20),

$$\begin{aligned} \exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n - q_n) &= \exp\left[\left(c\,\frac{r_1^2}{r_{1,n}-1}\,\log\, n\right)^{1/2}\right] a_0^{1/3}(p_n - q_n)^{-a/3} \ge \\ &\ge a_0^{1/3} \exp\left[\left(c\,\frac{r_1^2}{r_{1,n}-1}\,\log\, n\right)^{1/2}\right] n^{-a/3} \longrightarrow +\infty\,, \end{aligned}$$

since $\frac{r_{1,n}^2}{r_{1,n}-1}\,\log\, n \to +\infty$ as $\frac{r_{1,n}^2}{r_{1,n}-1} \sim r_{1,n} \to +\infty$.

Thus,

$$\exp\left(8r_{1,n}\frac{\varepsilon_n}{q_n}\right) C^{1/3}(p_n - q_n) \longrightarrow +\infty\,,$$

and consequently (21) can not be satisfied.

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