# How accessible are categories of algebras?

**Jiří Adámek**[1] **and Lurdes Sousa**[2]

**Abstract**

For locally finitely presentable categories $\mathcal{K}$ it is well-known that categories of $F$-algebras, where $F$ is a finitary endofunctor, are also locally finitely presentable. We prove that this generalizes to locally finitely multipresentable categories. But it fails, in general, for finitely accessible categories: we even present an example of a strongly finitary functor $F$ (one that preserves finitely presentable objects) whose category of $F$-algebras is not finitely accessible. On the other hand, categories of $F$-algebras are proved to be $\omega_1$-accessible for all strongly finitary functors – and it is an open problem whether this holds for all finitary functors.

## 1 Introduction

The aim of the present paper is to study properties of the categories

$$\mathrm{Alg}F$$

of $F$-algebras for a given endofunctor $F : \mathcal{A} \to \mathcal{A}$. Recall that an $F$-algebra is a pair $(A, \alpha)$ consisting of an object $A$ of $\mathcal{A}$ and a morphism $\alpha : FA \to A$; morphisms from $(A, \alpha)$ to $(B, \beta)$, called homomorphisms, are those morphisms $f : A \to B$ of $\mathcal{A}$ for which the following square

$$\begin{array}{ccc} FA & \xrightarrow{\alpha} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow{\beta} & B \end{array}$$

commutes. These categories $\mathrm{Alg}F$ naturally generalize the classical categories of $\Sigma$-algebras for a signature $\Sigma$ (i.e., a collection of operation symbols $\sigma$ endowed with an arity function assigning to $\sigma$ a natural number $\mathrm{ar}(\sigma)$). In fact, the "polynomial" endofunctor $F_\Sigma$ of $\mathcal{S}et$ defined on objects by

$$F_\Sigma A = \coprod_{\sigma \in \Sigma,\, \mathrm{ar}(\sigma) = n} A^n \,,$$

and analogously on morphisms, has the property that $\mathrm{Alg}F_\Sigma$ is the category of $\Sigma$-algebras and $\Sigma$-homomorphisms. More generally, given an $S$-sorted signature $\Sigma$, where the arity function maps every operation symbol $\sigma$ to a pair $\mathrm{ar}(\sigma) =$

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$(s_1...s_n, s)$ consisting of a word $s_1...s_n$ over $S$ and an element $s$ of $S$ (so that $\sigma$ has $n$ variables, the $i$-th one of sort $s_i$, and it has results of sort $s$), then the corresponding "polynomial" endofunctor of $\mathcal{S}et^S$ is defined by

$$(F_\Sigma A)_s = \coprod_{\sigma\in\Sigma,\, \mathrm{ar}(\sigma)=(s_1...s_n,s)} A_{s_1} \times ... \times A_{s_n} .$$

Again, Alg$F_\Sigma$ is the category of $S$-sorted $\Sigma$-algebras in the usual sense, see e.g. [10].

If $\mathcal{A}$ is a locally finitely presentable category in the sense of Gabriel and Ulmer, see [8] and [4], then Alg$F$ is also a locally finitely presentable category, whenever $F$ is *finitary*, i.e., preserves filtered colimits. In fact, Alg$F$ is complete and has filtered colimits since the forgetful functor

$$U : \mathrm{Alg}F \rightarrow \mathcal{A}$$

creates limits (always) and filtered colimits (whenever $F$ preserves them). Moreover, $U$ is a right adjoint, i.e., every object $A$ of $\mathcal{A}$ generates a free $F$-algebra. In fact, as proved in [1], a free $F$-algebra is a colimit of the $\omega$-chain

$$A \xrightarrow{i} FA + A \xrightarrow{Fi+A} F(FA + A) + A \xrightarrow{F(Fi+A)+A} ...$$

Now the collection of free $F$-algebras on all finitely presentable objects of $\mathcal{A}$ is easily seen to form a strong generator of Alg$F$, which proves that Alg$F$ is locally finitely presentable. More generally: if $\mathcal{A}$ is locally $\lambda$-presentable then so is Alg$F$ whenever $F$ preserves $\lambda$-filtered colimits.

We concentrate in our paper on accessible categories in the sense of Lair [9] and Makkai-Paré [11]. Recall that a category $\mathcal{A}$ is called $\lambda$-*accessible* if $\lambda$ is a regular cardinal (i.e., an infinite cardinal which is not a sum of less than $\lambda$ cardinals smaller than $\lambda$) such that

(i) $\mathcal{A}$ has $\lambda$-filtered colimits

and

(ii) there exists a (small) set $\mathcal{A}_\lambda$ of $\lambda$-presentable objects of $\mathcal{A}$ whose closure under $\lambda$-filtered colimits is all of $\mathcal{A}$.

If $\lambda = \omega_0$ we speak about *finitely accessible* categories. Recall further that a functor $F$ is called $\lambda$-*accessible* provided that its domain and codomain are $\lambda$-accessible categories, and $F$ preserves $\lambda$-filtered colimits. And "accessible" means "$\lambda$-accessible for some regular cardinal $\lambda$". It follows from results of Makkai and Paré in [11] that for every accessible endofunctor $F$ the category Alg$F$ is accessible. However, we cannot sharpen this to a given $\lambda$: for every regular cardinal $\lambda$ an example of a $\lambda$-accessible endofunctor is presented below whose category of algebras is not $\lambda$-accessible.

There is a natural generalization of the above result on locally $\lambda$-presentable categories: it concerns the *locally $\lambda$-multipresentable categories* of Diers [7]. Recall that a $\lambda$-accessible category is called locally $\lambda$-multipresentable iff it has

connected limits

or, equivalently, it has

*multicolimits*, i.e. for every small diagram $D$ there is a small set of cocones of $D$ such that every cocone of $D$ factors through a unique member of that set, and the factorization is also unique.

For every $\lambda$-accessible endofunctor $F$ of a locally $\lambda$-multipresentable category we prove that Alg$F$ is also locally $\lambda$-multipresentable. Analogously for algebras over a monad: if the monad is $\lambda$-accessible, then its Eilenberg-Mac Lane category of algebras is proved to be locally $\lambda$-multipresentable.

Surprisingly, we have not been able to find any example of a $\lambda$-accessible endofunctor $F$ which would need, for Alg$F$, a raise of accessibility of more than one step. That is, let $\lambda^+$ denote the cardinal successor of $\lambda$, then we do not know whether the following implication

$$F \text{ is } \lambda\text{-accessible} \Rightarrow \mathrm{Alg}F \text{ is } \lambda^+\text{-accessible}$$

holds in general. We do present a proof of the special case when $F$ preserves the $\lambda$-presentability of objects – such functors are called *strongly $\lambda$-accessible.*

## 2 Locally multipresentable categories

**2.1** Locally multipresentable categories have been introduced by Y. Diers [7] as a class of categories incorporating, besides the "well-behaved" locally presentable categories of P. Gabriel and F. Ulmer (see [8] or [4]), important examples such as

$\mathcal{L}in$ : linearly ordered sets and order-preserving functions
$\mathcal{F}ld$ : fields and (ring-)homomorphisms

or

$\mathcal{M}od\, \mathcal{S}$ : models of a limit-coproduct sketch $\mathcal{S}$ (i.e., a small category with specified cones and discrete cocones).

Locally $\lambda$-multipresentable categories do not have colimits, in general, but they have $\lambda$-filtered colimits and multicolimits of all small diagrams $D$.

Before turning to properties of algebras in locally multipresentable categories, let us mention a useful criterion:

**2.2 Lemma** *A category is locally $\lambda$-multipresentable iff it has multicolimits, $\lambda$-filtered colimits, and a strong generator formed by $\lambda$-presentable objects.*

**Proof** Every locally $\lambda$-multipresentable category has all above properties. Conversely, let $\mathcal{A}$ have those properties, and let $\mathcal{B}$ be a strong generator formed by $\lambda$-presentable objects. Then the closure of $\mathcal{B}$ under $\lambda$-small multicolimits, i.e., the smallest full subcategory $\bar{\mathcal{B}}$ of $\mathcal{A}$ containing $\mathcal{B}$ and such that a multicolimit of a diagram on less than $\lambda$ morphisms in $\bar{\mathcal{B}}$ has all components in $\bar{\mathcal{B}}$, is essentially small. (In fact, we have $\bar{\mathcal{B}} = \cup_{i<\lambda}\mathcal{B}_i$ where $\mathcal{B}_0 = \mathcal{B}$, $\mathcal{B}_{i+1}$ is a one-step closure of $\mathcal{B}_i$ under $\lambda$-small multicolimits, and $\mathcal{B}_i = \cup_{j<i}\mathcal{B}_j$ for limit ordinals $i < \lambda$. Since $\mathcal{B}_0$ is small, each $\mathcal{B}_i$ is essentially small.) We claim that

a. every object of $\bar{\mathcal{B}}$ is $\lambda$-presentable

and

b. $\mathcal{A}$ is a closure of $\bar{\mathcal{B}}$ under $\lambda$-filtered colimits.

The first statement follows from the fact that a $\lambda$-small multicolimit of $\lambda$-presentable objects has all components $\lambda$-presentable (which is proved analogously to 1.3 in

[4]). To prove the latter, we proceed similarly to 1.11 in [4]: for every object $K$ we form the canonical diagram

$$D : \bar{\mathcal{B}}/K \to \mathcal{A}$$

assigning to every arrow $f : A \to K$ ($A \in \bar{\mathcal{B}}$) the domain. Since $\bar{\mathcal{B}}$ is closed under $\lambda$-small multicolimits, $D$ is clearly $\lambda$-filtered. Therefore, it has a colimit $K^*$ with colimit cocone

$$f^* : A \to K^* \qquad \text{(for all } f : A \to K \text{ in } \bar{\mathcal{B}}/K).$$

The canonical cocone of all $f$'s factors through this cocone, i.e., there exists a morphism $m : K^* \to K$ with

$$mf^* = f \qquad \text{for all } f \in \bar{\mathcal{B}}/K.$$

Since $\mathcal{B}$ is a strong generator, $m$ is an extremal epimorphism. Analogously to the last part of the proof of 1.1 in [4] it is proved that $m$ is a monomorphism. Consequently, $K \cong K^*$ is in the closure of $\bar{\mathcal{B}}$ under $\lambda$-filtered colimits.

**2.3 Theorem** *If $\mathcal{A}$ is a locally $\lambda$-multipresentable category, then AlgF is also locally $\lambda$-multipresentable for every $\lambda$-accessible endofunctor $F$.*

**Proof** The forgetful functor

$$U : \mathrm{Alg}F \to \mathcal{A}$$

creates connected limits (since they exist in $\mathcal{A}$) and $\lambda$-filtered colimits (since they are preserved by $F$). By Theorem 2.1 of [2], $U$ is a right multiadjoint. That is, for every object $A$ of $\mathcal{A}$ there exists a collection of $F$-algebras $C_k$ ($k \in K$) and arrows $\eta_k : A \to UC_k$ with the following multiuniversal property:

> for every $F$-algebra $C$ and every arrow $f : A \to UC$ there exists a unique $k \in K$ such that $f = h \cdot \eta_k$ for some homomorphism $h : C_k \to C$, and moreover $h$ is also unique.

If $A$ is $\lambda$-presentable in $\mathcal{A}$, then each $C_k$ is $\lambda$-presentable in Alg$F$. This follows easily from $U$ preserving $\lambda$-filtered colimits. Consequently, if $A$ is $\lambda$-presentable, then the indexing family $K$ is small. Let $\mathcal{A}_\lambda$ be a set of representatives for $\lambda$-presentable objects of $\mathcal{A}$, and for every $X \in \mathcal{A}_\lambda$ let $(C_k)_{k \in K(X)}$ be a multifree family on $X$; put $\mathcal{B}_\lambda = \bigcup\limits_{X \in \mathcal{A}_\lambda} \{C_k\}_{k \in K(X)}$. Then $\mathcal{B}_\lambda$ is a set of $\lambda$-presentable $F$-algebras. And $\mathcal{B}_\lambda$ is a strong generator of Alg$F$: given a proper subobject $m : B \to A$ in Alg$F$ there exists (since $\mathcal{A}_\lambda$ is a strong generator of $\mathcal{A}$) a morphism $x : X \to A$, $X \in \mathcal{A}_\lambda$, which does not factor through $m$ in $\mathcal{A}$. Let $x = \bar{x} \cdot \eta_k$ be a factorization of $x$ through a homomorphism $\bar{x} : C_k \to A$ (where $C_k \in \mathcal{B}_\lambda$), then $\bar{x}$ does not factor through $m$ in $\mathcal{A}$, thus, nor in Alg$F$.

Now Alg$F$ has $\lambda$-filtered colimits, connected limits and a strong generator formed by $\lambda$-presentable objects.

Consequently, Alg$F$ is locally $\lambda$-multipresentable by 2.2.

**2.4 Remark** Analogously, for every $\lambda$-accessible monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{A}$ (locally $\lambda$-multipresentable) the Eilenberg-Mac Lane category $\mathcal{A}^{\mathbf{T}}$ is locally $\lambda$-multipresentable. In fact, the free $\mathbb{T}$-algebras on $\lambda$-presentable objects of $\mathcal{A}$ form a strong generator of $\lambda$-presentable objects of $\mathcal{A}^{\mathbf{T}}$.

**2.5 Remark** If a finitely accessible category has finite coproducts, then, as remarked in the introduction, Alg$F$ has free algebras for every finitary endofunctor. Now although free algebras play a central role in the proof of finite local presentability of Alg$F$, and analogously multifree algebras have been used in the above proof, it is not clear whether they are sufficient in the present case:

**2.6 Open problem** Let $\mathcal{A}$ be a finitely accessible category with finite coproducts. Is Alg$F$ finitely accessible for every finitary endofunctor $F$?

**2.7 Example**: Multivarieties. Recall from [5] that given a (possibly infinitary) signature $\Sigma$ and a set $X$ of variables, a *multiequation* is an expression

$$\nabla_{i\in I}(u_i = v_i)$$

where $u_i$ and $v_i$ are $\Sigma$-terms on $X$, and $\nabla$ denotes exclusive or (i.e., disjoint disjunction). A $\Sigma$-algebra $A$ *satisfies* the above multiequation provided that for every homomorphism $h : \phi_\Sigma X \to A$ from the free $\Sigma$-algebra on $X$ into $A$ there exists precisely one $i \in I$ with $h(u_i) = h(v_i)$. A *multivariety* of $\Sigma$-algebras is a full subcategory of Alg$\Sigma$ specified by a collection of multiequations. We call it $\lambda$-*ary* provided that all arities are less than $\lambda$ and the sets $X$ of variables used in all multiequations have cardinalities less than $\lambda$.

We are going to show that $\lambda$-ary multivarieties are locally $\lambda$-multipresentable. This holds over much more general categories than $\mathcal{S}et$. Before we introduce these general multivarieties, let us translate multiequations categorically following the translation of equations by B. Banaschewski and H. Herrlich [6]. Denote by

$$e_i : \phi_\Sigma X \to E_i$$

the quotient of the free algebra modulo the congruence generated by $u_i = v_i$ (for $i \in I$). Then we obtain a cone of regular epimorphisms of $\phi_\Sigma X$:

$$\begin{tikzcd} & \phi_\Sigma X \arrow[dl, "e_i"'] \arrow[d] \arrow[dr] & & \\ E_i & {} & \cdots & (i \in I) \end{tikzcd}$$

An algebra $A$ satisfies the above multiequation iff it is *orthogonal* to that cone, i.e., for every morphism $h : \phi_\Sigma X \to A$ there exists a unique $i \in I$ such that $h$ factors through $e_i$:

$$\begin{tikzcd} & \phi_\Sigma X \arrow[dl, "e_i"'] \arrow[d] \arrow[dr] \arrow[rr, "h"] & & A \\ E_i \arrow[urrr, dashed] & {} & {} & \end{tikzcd}$$

**2.8 Remark** If we restrict multivarieties to the finitary case, where all arities of operations in $\Sigma$ are finite, and the sets $X$ of variables are finite, then we see that

(i) the corresponding endofunctor $FX = \sum_{\sigma\in\Sigma,\, \mathrm{ar}\sigma = n} X^n$ of $\mathcal{S}et$ is finitary and

(ii) in the above cone the domain $\phi_\Sigma X$ and all the codomains $E_i$ $(i \in I)$ are finitely presentable $F$-algebras.

Thus, every finitary multivariety of $F$-algebras is an $\omega$-cone-orthogonality class in Alg$F$ in the following sense (introduced in [4]):

**2.9 Definition** A full subcategory $\mathcal{K}$ of a subcategory $\mathcal{L}$ is called a $\lambda$-*cone-orthogonality class* if it can be presented by orthogonality w.r.t. cones with $\lambda$-presentable domains and codomains. That is, there exists a collection $\mathcal{C}$ of such cones in $\mathcal{L}$ such that an object lies in $\mathcal{K}$ iff it is orthogonal w.r.t. every cone in $\mathcal{C}$.

**2.10 Corollary** *Let $F$ be a $\lambda$-accessible endofunctor of a locally $\lambda$-multipresentable category. Then every $\lambda$-cone-orthogonality class of $F$-algebras is also locally $\lambda$-multipresentable.*

This follows from Theorem 4.30 of [4] which states that for locally $\lambda$-presentable categories $\mathcal{L}$ all $\lambda$-orthogonality classes are locally $\lambda$-multipresentable. This generalizes immediately to $\mathcal{L}$ locally $\lambda$-multipresentable.

# 3 Strongly accessible functors

**3.1 Definition** A functor $F : \mathcal{K} \to \mathcal{L}$ is called *strongly* $\lambda$-*accessible* provided that $\lambda$ is a regular cardinal such that $F$ is $\lambda$-accessible and preserves $\lambda$-presentable objects.

**Remark** For every infinite cardinal $\lambda$, the cardinal successor $\lambda^+$ is always regular.

The method of the proof of the following theorem is similar to the proof in [3] that categories of coalgebras over $\lambda$-accessible functors are $\lambda$-accessible (for all uncountable regular cardinals $\lambda$).

**3.2 Theorem** *For every strongly $\lambda$-accessible endofunctor $F$ the category of $F$-algebras is $\lambda^+$-accessible.*

**Proof** Let $F : \mathcal{A} \to \mathcal{A}$ be strongly $\lambda$-accessible.

(A) We prove that every algebra $FA \xrightarrow{\alpha} A$ with a $\lambda^+$-presentable underlying object $A$ is $\lambda^+$-presentable in Alg$F$.

Let a $\lambda^+$-filtered colimit in Alg$F$ be given:

$$\begin{tikzcd} FB_i \arrow[r, "\beta_i"] \arrow[d, "Fb_i"'] & B_i \arrow[d, "b_i"] \\ FB \arrow[r, "\beta"] & B \end{tikzcd}$$

where $i$ ranges through a small $\lambda^+$-filtered category $I$. For every homomorphism $f : A \to B$ in Alg$F$ there exists a factorization $f = b_i g$ for some $i \in I$ and $g : A \to B_i$ in $\mathcal{A}$, and it is our task to prove that we can choose this factorization so that $g$ is a homomorphism in Alg$F$. The essential uniqueness of the factorization in Alg$F$ is obvious; thus, $A$ will be proved to be $\lambda^+$-presentable in Alg$F$.

Since $\lambda \triangleleft \lambda^+$ (see [4], 2.13(1)), every $\lambda^+$-presentable object is a $\lambda$-filtered colimit of a diagram of $\lambda$-presentable objects in $\mathcal{A}$ having at most $\lambda$ morphisms – this is proved in [11], 2.3.11. It follows that since $F$ preserves $\lambda$-presentability and $\lambda$-filtered colimits, it also preserves $\lambda^+$-presentability. Thus, $FA$ is $\lambda^+$-presentable. Now the desired equation for $g$ being a homomorphism is $g \cdot \alpha = \beta_i \cdot Fg : FA \to B_i$:

$$\begin{tikzcd}
 & FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] \arrow[dddl, "Fg"'] & A \arrow[d, "f"] \arrow[dddr, "g"] & \\
 & FB \arrow[r, "\beta"] \arrow[d, "Fb_j"'] & B \arrow[d, "b_j"] & \\
 & FB_j \arrow[r, "\beta_j"] & B_j & \\
FB_i \arrow[uur, "Fb_i"] \arrow[ur, "Fu"] \arrow[rrr, "\beta_i"] & & & B_i \arrow[uul, "b_i"'] \arrow[ul, "u"']
\end{tikzcd}$$

Since $I$ is $\lambda^+$-filtered, and since the colimit morphism $b_i$ merges the two sides of our equation,

$$b_i \cdot g \cdot \alpha = f \cdot \alpha = \beta \cdot Ff = \beta \cdot Fb_i \cdot Fg = b_i \cdot \beta_i \cdot Fg\,,$$

there exists, due to the $\lambda^+$-presentability of $FA$, a connecting morphism

$$u : B_i \to B_j \quad (j \in I)$$

of the given diagram which also merges the two sides:

$$u \cdot g \cdot \alpha = u \cdot \beta_i \cdot Fg\,.$$

It is now sufficient to substitute $B_i$ with $B_j$, and $g$ with $g' = ug$: we have a factorization

$$f = b_i g = b_j u g = b_j g'$$

and $g'$ is a homomorphism:

$$g' \cdot \alpha = u \cdot \beta_i \cdot Fg = \beta_j \cdot Fu \cdot Fg = \beta_j \cdot Fg'\,.$$

Thus, A is a $\lambda^+$-presentable $F$-algebra.

(B) We prove that every algebra $FC \xrightarrow{\gamma} C$ is a $\lambda^+$-filtered colimit of algebras of type (A). To this end, let $\mathcal{A}_0$ denote a set of representatives of $\lambda^+$-presentable objects of $\mathcal{A}$. Since $\mathcal{A}$ is $\lambda^+$-accessible (recall that $\lambda \triangleleft \lambda^+$), every object $C$ is a canonical colimit of the diagram of all arrows with domains in $\mathcal{A}_0$. More precisely, let

$$D_C : \mathcal{A}_0/C \to \mathcal{A}$$

denote the diagram on the comma-category $\mathcal{A}_0/C$ assigning to every arrow $a : A \to C$ ($A \in \mathcal{A}_0$) the domain $A$. It has the canonical colimit $(A \xrightarrow{a} C)_{(A,a) \in \mathcal{A}_0/C}$.

Thus, in order to prove (B), it is sufficient to verify that the subdiagram of $D_C$ formed by all $F$-algebra homomorphisms is cofinal. More precisely, the forgetful functor $U : \mathrm{Alg}F \to \mathcal{A}$ induces a forgetful functor

$$U_C : (\mathrm{Alg}F)_0 \,/\, (C, \gamma) \to \mathcal{A}_0/C$$

where $(\mathrm{Alg}F)_0$ denotes the category of all $F$-algebras carried by objects of $\mathcal{A}_0$. We are going to prove that $U_C$ is a cofinal functor – since $U$ creates $\lambda^+$-filtered colimits (because $F$ preserves $\lambda^+$-filtered colimits), we conclude from

$$C = \mathrm{colim} D_C U_C$$

that (B) holds. In order to prove the cofinality of $U_C$, we verify the following facts:

(a) For every object $A \xrightarrow{a} C$ of $\mathcal{A}_0/C$ there is an object

$$\begin{tikzcd} FB \arrow[r, "\beta"] \arrow[d, "Fb"'] & B \arrow[d, "b"] \\ FC \arrow[r, "\gamma"] & C \end{tikzcd}$$

of $(\mathrm{Alg}F)_0/(C,\gamma)$ such that $a$ factors through $b$:

$$\begin{tikzcd} A \arrow[rr, dashed, "h"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & C & \end{tikzcd}$$

(b) Given two such factorizations

$$\begin{tikzcd} & A \arrow[dl, "h"'] \arrow[dd, "a"] \arrow[dr, "h'"] & \\ B \arrow[dr, "b"'] & & B' \arrow[dl, "b'"] \\ & C & \end{tikzcd}$$

where $b' : (B', \beta') \to (C, \gamma)$ is another object of $(\mathrm{Alg}F)_0/(C,\gamma)$, there exists a pair of homomorphisms

$$(B, \beta) \xrightarrow{u} (\tilde{B}, \tilde{\beta}) \xleftarrow{u'} (B', \beta')$$

in $(\mathrm{Alg}F)_0$ with $uh = u'h'$ together a homomorphism $\tilde{b} : (\tilde{B}, \tilde{\beta}) \to (C, \gamma)$ in $\mathrm{Alg}F$ such that the following diagram

$$\begin{tikzcd} (B,\beta) \arrow[r, "u"] \arrow[dr, "b"'] & (\tilde{B},\tilde{\beta}) \arrow[d, "\tilde{b}"] & (B',\beta') \arrow[l, "u'"'] \arrow[dl, "b'"] \\ & (C,\gamma) & \end{tikzcd}$$

commutes.

Proof of (a): We define a $\lambda$-chain

$$\begin{tikzcd}
A = B_0 \arrow[r, "u_{01}"] \arrow[drrr, "a = b_0"'] & B_1 \arrow[r] \arrow[drr, "b_1"] & \cdots & B_i \arrow[r, "u_{ij}"] \arrow[d, "b_i"'] & B_j \arrow[r] \arrow[dl, "b_j"] & \cdots \\
 & & & C & &
\end{tikzcd}$$

in $\mathcal{A}_0/C$ together with morphisms

$$v_i : FB_i \to B_{i+1} \qquad (i < \lambda)$$

satisfying

(i) $$b_{i+1} \cdot v_i = \gamma \cdot Fb_i \qquad (i < \lambda)$$

and the following naturality conditions

(ii) $$v_i \cdot Fu_{ji} = u_{j+1,i+1} \cdot v_j \qquad (j \leq i < \lambda)$$

by transfinite induction as follows:

Isolated step: Given $B_i \xrightarrow{b_i} C$ in $\mathcal{A}_0/C$, the object $FB_i$ is $\lambda^+$-presentable, see (A) above; therefore, since $\mathcal{A}_0/C$ is filtered, the morphisms

$$\gamma \cdot Fb_i : FB_i \to C \quad \text{and} \quad b_i : B_i \to C$$

factorize through some morphism forming an object of $\mathcal{A}_0/C$, say, through $B_{i+1} \xrightarrow{b_{i+1}} C$. That is, there is a commutative diagram as follows:

$$\begin{tikzcd}
FB_i \arrow[r, "v_i"] \arrow[d, "Fb_i"'] & B_{i+1} \arrow[d, "b_{i+1}"'] & B_i \arrow[l, "u_{i,i+1}"'] \arrow[dl, "b_i"] \\
FC \arrow[r, "\gamma"'] & C &
\end{tikzcd}$$

It remains to attend to the equation (ii):

$$v_i \cdot Fu_{ji} = u_{j+1,i+1} \cdot v_j : FB_j \to B_{i+1} \quad (\text{for } j < i)$$

If this does not hold for some $j$, observe that $FB_j$ is $\lambda^+$-presentable, see (A), and that the colimit morphism $b_{i+1}$ of the $\lambda^+$-filtered colimit $C = \operatorname{colim} D_C$ merges both sides:

$$\begin{aligned}
b_{i+1} \cdot (v_i \cdot Fu_{ji}) &= \gamma \cdot F(b_i \cdot u_{ji}) && \text{by (i)} \\
&= \gamma \cdot Fb_j \\
&= b_{j+1} \cdot v_j && \text{by (i)} \\
&= b_{i+1} \cdot (u_{j+1,i+1} \cdot v_j)
\end{aligned}$$

Consequently, there is a connecting morphism $v : (B_{i+1}, b_{i+1}) \longrightarrow (B'_{i+1}, b'_{i+1})$ of $\mathcal{A}_0/C$ which also merges the two sides:

$$v \cdot (v_i \cdot Fu_{ji}) = v \cdot (u_{j+1,i+1} \cdot v_j)\,.$$

It is sufficient to "improve" our choice of $(B_{i+1}, b_{i+1})$ by changing it to $(B'_{i+1}, b'_{i+1})$ (and changing $v_i$ to $v \cdot v_i$ and $u_{i,i+1}$ to $v \cdot u_{i,i+1}$). This improvement can be performed for all $j < i$ at once, since the number of such $j$'s is at most $\operatorname{card} i < \lambda$ and $\mathcal{A}_0/C$ is $\lambda^+$-filtered. Thus, we can suppose, without loss of generality, that the desired condition (ii) holds.

Limit step: Let $j$ be a limit ordinal ($j < \lambda$). Since, for $i < j$, all $(B_i, b_i)$ belong to the $\lambda^+$-filtered category $\mathcal{A}_0/C$, there is some $(B_j, b_j)$ in $\mathcal{A}_0/C$ and morphisms $u_{ij} : B_i \to B_j$ such that $b_j \cdot u_{ij} = b_i$.

Let $(B_i \xrightarrow{\bar{b}_i} \bar{B})_{i<\lambda}$ be the colimit of the above chain in $\mathcal{A}$, then the above cocone $b_i : B_i \to C$ gives us a unique $\bar{b} : \bar{B} \to C$ with $\bar{b}\bar{b}_i = b_i$ ($i < \lambda$). The natural transformation $v_i$ yields a morphism $\bar{\beta} : F\bar{B} \to \bar{B}$ with $\bar{\beta} \cdot F\bar{b}_i = \bar{b}_{i+1} \cdot v_i$ ($i < \lambda$) since $F\bar{B} = \operatorname{colim} FB_i$. Then $\bar{b}$ is a homomorphism: the equality

$$\bar{b}\bar{\beta} = \gamma \cdot F\bar{b} : F\bar{B} \to C$$

follows from

$$\begin{aligned}
(\bar{b} \cdot \bar{\beta}) \cdot F\bar{b}_i &= \bar{b} \cdot \bar{b}_{i+1} \cdot v_i && \text{by definition of } \bar{\beta} \\
&= b_{i+1} \cdot v_i && \text{by definition of } \bar{b} \\
&= \gamma \cdot Fb_i && \text{by definition of } v_i \\
&= (\gamma \cdot F\bar{b}) \cdot F\bar{b}_i && \text{since } \bar{b}\bar{b}_i = b_i
\end{aligned}$$

Observe that $\bar{B}$ is $\lambda^+$-presentable since it is a colimit of a $\lambda^+$-small diagram of $\lambda^+$-presentable objects (see 1.16 of [4]). Thus, the desired factorization is

$$\begin{tikzcd}
A \arrow[rr, "\bar{b}_0"] \arrow[dr, "a"'] & & \bar{B} \arrow[dl, "\bar{b}"] \\
& C &
\end{tikzcd}$$

Proof of (b): Before proving (b) we observe that (a) has the following strengthening: Suppose that besides $A \xrightarrow{a} C$ also two $F$-algebras $(D_1, \delta_1)$ and $(D_2, \delta_2)$ in $(\mathrm{Alg}F)_0$ and two morphisms $d_i : D_i \to A$ in $\mathcal{A}$ , $i = 1, 2$, are given. Then assuming that $ad_i : (D_i, \delta_i) \to (C, \gamma)$ are homomorphisms for $i = 1, 2$, we conclude that the above data $(B, \beta)$, $b$ and $h$ can be chosen in such a way such that $hd_i : (D_i, \delta_i) \to (B, \beta)$ are homomorphisms for $i = 1, 2$. In fact,
all we have to do is to choose the first step $B_1$ and $v_0 : FA \to B_1$ above so that the following squares

(iii) $$\begin{tikzcd}
FD_i \arrow[r, "\delta_i"] \arrow[d, "Fd_i"'] & D_i \arrow[d, "u_{01}d_i"] \\
FA \arrow[r, "v_0"] & B_1
\end{tikzcd}$$

commute for $i = 1, 2$. It then follows that, as requested, $\bar{b}_0 d_i$ are homomorphisms (recall that the role of $h : A \to B$ is played by $\bar{b}_0 : A \to \bar{B}$), since the following is a commutative diagram:

$$\begin{tikzcd}
FD_i \arrow[rr, "\delta_i"] \arrow[d, "Fd_i"'] & & D_i \arrow[dl, "u_{01}d_i"'] \arrow[d, "d_i"] \\
FA \arrow[r, "v_0"] \arrow[d, "F\bar{b}_0"'] & B_1 \arrow[dr, "\bar{b}_1"'] & A \arrow[d, "\bar{b}_0"] \\
F\bar{B} \arrow[rr, "\bar{\beta}"] & & \bar{B}
\end{tikzcd}$$

And the reason why the first step can be chosen so that (iii) commutes is the following: We have $C$ as a filtered colimit of $D_C : \mathcal{A}_0/C \to \mathcal{A}$, and the colimit morphism $b_1 : B_1 \to C$ merges the two sides of those squares:

$$\begin{aligned}
b_1(v_0 \cdot Fd_i) &= \gamma \cdot Fa \cdot Fd_i && \text{by (i)} \\
&= a \cdot d_i \cdot \delta_i && \text{since } a \cdot d_i \text{ is a homomorphism} \\
&= b_1(u_{01} \cdot d_i \cdot \delta_i) && \text{by } a = b_0 = b_1 \cdot u_{01}.
\end{aligned}$$

Now the domain, $FD_i$, of the two morphisms in (iii) is $\lambda^+$-presentable (since $F$ preserves $\lambda^+$-presentability). Therefore, there exists a connecting morphism of the diagram $D_C$ which also merges the two sides of (iii); that is, there exists

$$\begin{tikzcd}
B_1 \arrow[rr, "r"] \arrow[dr, "b_1"'] & & B_1^\star \arrow[dl, "b_1^\star"] \\
& B &
\end{tikzcd}$$

in $\mathcal{A}_0$ with

$$r \cdot (v_0 \cdot Fd_i) = r \cdot (u_{01} \cdot d_i \cdot \delta_i) \qquad (i = 1, 2)\,.$$

It is now sufficient to modify the first step of the induction above by taking $B_1^\star$ instead of $B_1$ (and $b_1^\star$ instead of $b_1$, as well as $rv_0 : FA \to B_1^\star$ instead of $v_0$).

We are ready to prove (b). Since $\mathcal{A}_0/C$ is filtered, and $A$, $B$, $B' \in \mathcal{A}_0$, there exists an object $(\hat{B}, \hat{b})$ of $\mathcal{A}_0/C$ together with morphisms $u$ and $u'$ forming a commutative diagram as follows

$$\begin{tikzcd}
& A \arrow[dl, "h"'] \arrow[dr, "h'"] & \\
B \arrow[r, "u"] \arrow[dr, "b"'] & \hat{B} \arrow[d, "\hat{b}"] & B' \arrow[l, "u'"'] \arrow[dl, "b'"] \\
& C &
\end{tikzcd}$$

We now apply the strengthened formulation of (a) to the object

$$\hat{B} \xrightarrow{\hat{b}} C$$

of $\mathcal{A}_0/C$, and the morphisms

$$d_1 = u \qquad \text{and} \qquad d_2 = u'$$

which fulfil the requirement that $\hat{b}d_1 = b : (B, \beta) \to (C, \gamma)$ and $\hat{b}d_2 = b' : (B', \beta') \to (C, \gamma)$ are homomorphisms. Consequently, there exists an object $\tilde{b} : (\tilde{B}, \tilde{\beta}) \to (C, \gamma)$ of $(\mathrm{Alg}F)_0/(C, \gamma)$ and a morphism

$$\begin{tikzcd} \hat{B} \arrow[rr, "k"] \arrow[dr, "\hat{b}"'] & & \tilde{B} \arrow[dl, "\tilde{b}"] \\ & C & \end{tikzcd}$$

of $\mathcal{A}_0/C$ such that

$$ku : (B, \beta) \to (\tilde{B}, \tilde{\beta}) \quad \text{and} \quad ku' : (B', \beta') \to (\tilde{B}, \tilde{\beta})$$

are homomorphisms. Since

$$\tilde{b}(ku) = \hat{b}u = b$$

and

$$\tilde{b}(ku') = \hat{b}u' = b' ,$$

this concludes the proof of (b).

(C) $\mathrm{Alg}F$ is $\lambda^+$-accessible: it has $\lambda^+$-filtered colimits because $F$ preserves these colimits. And it has a set of $\lambda^+$-presentable algebras (see (A)) whose closure under $\lambda^+$-filtered colimits is all of $\mathrm{Alg}F$ (see (B)).

**3.3 Open Problem** Is $\mathrm{Alg}F$ $\lambda^+$-accessible for every $\lambda$-accessible endofunctor $F$?

**3.4 Example** of a strongly finitary endofunctor $F$ such that $\mathrm{Alg}F$ is not finitely accessible.

Let $X$ be an infinite set, and denote by $\mathcal{A}$ the power-set of $X$ (ordered by inclusion) with every element $M \subseteq X$, $M$ finite, split into two incomparable elements $M_1$, $M_2$. That is:

$$\mathcal{A} = \{K;\ K \subseteq X \text{ infinite}\} \cup \{M_i;\ i = 1, 2 \text{ and } M \subseteq X \text{ finite}\}$$

ordered as follows:

$$\begin{aligned} &M_i \leq M'_{i'} \text{ iff } M \subsetneq M' \text{ or } M = M', i = i', \\ &M_i \leq K \text{ iff } M \subseteq K, \\ &K \not\leq M_i, \end{aligned}$$

and

$$K \leq K' \text{ iff } K \subseteq K' .$$

Observe that every directed subset $D \subseteq \mathcal{A}$ has a join: either $D$ contains a largest element, or the union, $K$, of all subsets of $\mathcal{A}$ which (alone or with index 1 or 2) lie in $D$ is infinite. In the latter case we have $\bigvee D = K$.

From this description of directed joins it follows that all $M_i$ with $M \subseteq X$ finite are finitely presentable objects of $\mathcal{A}$. And every object is a directed join of finitely presentable objects $M_i$ ($M$ finite). Consequently, $\mathcal{A}$ is, as a category, finitely accessible. The functor $F : \mathcal{A} \to \mathcal{A}$ defined by

$$FM_1 = M_2,\ FM_2 = M_1, \text{ and } FK = K$$

for all $M \subseteq X$ finite and $K \subseteq X$ infinite is easily seen to be strongly finitary. However, Alg$F$, which is the subposet of exp$X$ on all infinite sets, is not finitely accessible: no element $K$ of Alg$F$ is finitely presentable in Alg$F$. In fact, let $K = \bigcup_{t \in T} K(t)$ be a disjoint decomposition of $K$ into infinitely many infinite sets. Then $K = \bigvee \{\bigcup_{s \in S} K(s);\ S \subseteq T,\ S \text{ finite}\}$ is a directed join which proves that $K$ is not finitely presentable.

**3.5 Remark** The above example generalizes immediately to any regular cardinal $\lambda$: we have a $\lambda$-accessible poset $\mathcal{A}$ and a strongly $\lambda$-accessible endofunctor $F$ with Alg$F$ failing to be $\lambda$-accessible. Take a set $X$ of cardinality $\lambda$, and let $\mathcal{A}$ be the poset obtained from the power-set of $X$ by splitting every $M \subseteq X$ of cardinality less than $\lambda$ to two elements $M_1$, $M_2$.

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Department of Theoretical Computer Science
Technical University of Braunschweig
Postfach 3329, 38023 Braunschweig, Germany
adamek@iti.cs.tu-bs.de

Departamento de Matemática
Escola Superior de Tecnologia de Viseu
Campus Politécnico, 3504-510 Viseu, Portugal
sousa@mat.estv.ipv.pt