Sequential topological conditions in $\mathbb{R}$ in the absence of the axiom of choice

Gonçalo Gutierres*

Departamento de Matemática da Universidade de Coimbra

Apartado 3008, 3001-454 Coimbra, Portugal

ggutc@mat.uc.pt

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Abstract

It is known that – assuming the axiom of choice – for subsets A of $\mathbb{R}$ the following hold:

(a) A is compact iff it is sequentially compact,

(b) A is complete iff it is closed in $\mathbb{R}$,

(c) $\mathbb{R}$ is a sequential space.

We will show that these assertions are not provable in the absence of the axiom of choice, and that they are equivalent to each other.

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0 Introduction

There are many topological results in *Zermelo-Fraenkel set theory including the axiom of choice* (**ZFC**) that are not true in the absence of choice, i.e. in **ZF**. Even if we restrict our attention to $\mathbb{R}$, many "familiar" topological results are not provable in **ZF**, although in most cases their validity follows from a weaker version of the axiom of choice, **CC**($\mathbb{R}$).

Definition 0.1 The *axiom of countable choice* (**CC**) states that every countable family of non-empty sets has a choice function.

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Definition 0.2 $\mathbf{CC}(\mathbb{R})$ is the axiom of countable choice restricted to families of sets of real numbers.

Under $\mathbf{CC}(\mathbb{R})$, it is known to be true that *"a subspace of $\mathbb{R}$ is compact if and only if it is sequentially compact"*. In this paper, we will investigate under which conditions this equivalence remains valid and we will exhibit a list of equivalent conditions to this one.

In [4 - p.128], Felscher has a partial answer to this question by saying that $\mathbf{CC}(\mathbb{R})$ is equivalent to this condition together with the idempotence of the sequential closure in $\mathbb{R}$.

We will see that $\mathbf{CC}(\mathbb{R})$ is not necessary to prove the equivalence between compact and sequentially compact subspaces of $\mathbb{R}$, after proving it from a weaker form of choice.

The equivalence is not provable in **ZF**, once that implies that *"every Dedekind-finite subset of $\mathbb{R}$ is finite"*, known to be not provable in **ZF** (basic Cohen model).

We call a set *finite* if it is either empty or equipollent to a natural number. Otherwise the set is *infinite*.

Definitions 0.3 A set X is *Dedekind-finite* if no proper subset of X is equipollent to X. Otherwise X is *Dedekind-infinite*.

Proposition 0.4 *A set X is Dedekind-infinite if and only if has a countable subset (i.e. there is an injection from $\mathbb{N}$ to X).*

Throughout this paper we work in **ZF**, *Zermelo-Fraenkel set theory without the axiom of choice*.

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1 Compactness

The definition of compactness we will use here is usually called Heine-Borel compactness.

Definition 1.1 A topological space X is *compact* if every open cover of X has a finite subcover.

Theorem 1.2 (Heine-Borel Theorem) *A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Definition 1.3 A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence.

Proposition 1.4 *Every first countable, compact space is sequentially compact.*

In Theorem 7.1.3 of [11] it is proved that every first countable, (countably) compact space is sequentially compact. With a slight modification, that proof is valid in **ZF**.

Under the assumption of $\mathbf{CC}(\mathbb{R})$, these two notions of compactness are equivalent for subsets of $\mathbb{R}$. The equivalence does not remain true in **ZF** (see form 74 in [9]). This fact was pointed out by T. J. Jech in [10 - p.24]. For pseudometric spaces – and for second countable spaces – sequential compactness and compactness coincide if and only if **CC** holds (see [1]).

Theorem 1.5 *If every sequentially compact subspace of* $\mathbb{R}$ *is compact then every infinite subset of* $\mathbb{R}$ *is Dedekind-infinite.*

Proof: Let A be an infinite subset of $\mathbb{R}$.

Suppose first that A is not compact, hence A is not sequentially compact, which means that there is a sequence in A with no convergent subsequence and consequently A is Dedekind-infinite.

Suppose next that A is an infinite, closed and bounded subspace of $\mathbb{R}$. Since A is infinite and bounded, it has an accumulation point x. Without loss of generality, we consider x an accumulation point from the left.

Define the sets $A_n := [x-\frac{1}{n}, x) \cap A$. Each A_n is not empty and closed at the left, so there is $a_n := \min A_n$. This defines an infinite sequence (a_n) in A and completes the proof. ■

See also T. J. Jech [10 - Corollary 10.4].

Proposition 1.6 *For subspaces of* $\mathbb{R}$, *the following conditions are equivalent:*

(i) *sequentially compact* $\Longleftrightarrow$ *compact,*

(ii) *sequentially compact* $\Longrightarrow$ *closed,*

(iii) *sequentially compact* $\Longrightarrow$ *bounded.*

Proof: Since compact, i.e. closed and bounded (Theorem 1.2), implies sequentially compact (Proposition 1.4), it is clear that (i) is equivalent to the conditions (ii) and (iii) together. So, it is enough to prove the equivalence between (ii) and (iii).

(ii) ⇒ (iii) Let A be a sequentially compact, unbounded subspace of $\mathbb{R}$. We need to show that there is a sequentially compact, not closed subspace of $\mathbb{R}$.

There is an homeomorphism $f : \mathbb{R} \longrightarrow (0,1)$ between the reals and the open interval $(0,1)$. The set $f(A)$ is also sequentially compact, but since A is unbounded either 0 or 1 belongs to the closure of $f(A)$.

(iii) ⇒ (ii) Let now A be a sequentially compact, not closed subspace of $\mathbb{R}$ and x a point in the closure of A but not in A. Without loss of generality, we assume that x is an accumulation point from the left. The set $B := (-\infty, x) \cap A$ is sequentially compact. There is an increasing homeomorphism $g : (-\infty, x) \longrightarrow \mathbb{R}$. We have that $g(B)$ is sequentially compact but it is unbounded because x is in the closure of B. ■

2 Completeness

Another interesting question, in this context, is the following: under which conditions do the complete and the closed subspaces of $\mathbb{R}$ coincide? We will compare this property and the equivalence between the sequentially compact and the compact subspaces of $\mathbb{R}$.

Definition 2.1 A metric space X is *complete* if every Cauchy sequence in X converges.

Definition 2.2 Let A be a subspace of a topological space X. The *sequential closure* of A in X is defined by:

$$\sigma_X(A) := \{x \in X \mid \exists (x_n) \in A^{\mathbb{N}} : (x_n) \to x\}.$$

Using a similar notation, the (usual) *Kuratowski closure* of A in X is denoted by $k_X(A)$.

Proposition 2.3 *Let X be a complete metric space. A subspace A of X is complete if and only if $\sigma_X(A) = A$.*

Corollary 2.4 *If A is a closed subspace of a complete metric space then it is complete.*

Lemma 2.5 *A bounded subspace of $\mathbb{R}$ is sequentially compact if and only if it is complete.*

Proof: The fact that a sequentially compact space is complete is always true.

For the reverse implication, let A be a bounded subset of $\mathbb{R}$ and $\sigma_{\mathbb{R}}(A) = A$ (A is complete). Since A is bounded, $k(A)$ is bounded and hence compact. By Proposition 1.4, $k(A)$ is sequentially compact. So, every sequence (a_n) in A has a subsequence (a_{n_i}) converging to some x in $k(A)$. But since $\sigma_{\mathbb{R}}(A) = A$, we have that $x \in A$. ■

Theorem 2.6 *For subspaces of* $\mathbb{R}$*, the following conditions are equivalent:*

(i) *complete* $\Longleftrightarrow$ *closed,*

(ii) *sequentially compact* $\Longrightarrow$ *closed.*

Proof: **(i)** $\Rightarrow$ **(ii)** By the fact that sequential compactness implies completeness.

(ii) $\Rightarrow$ **(i)** One implication is in Corollary 2.4 and the other implication follows from Lemma 2.5 and the equalities: $k(A) = \bigcup_{n\in\mathbb{N}} k([-n,n] \cap A)$ and $\sigma(A) = \bigcup_{n\in\mathbb{N}} \sigma([-n,n] \cap A)$. ■

Theorem 2.6 together with Theorem 1.5 and Proposition 1.6 show us that there are models of **ZF** with complete not closed subspaces of $\mathbb{R}$ (basic Cohen model – $\mathcal{M}1$ in [9]).

The next proposition will give us a better insight into the way these spaces may look like.

Theorem 2.7 *If there is a complete, non-closed subspace of* $\mathbb{R}$ *then there is a dense, complete, non-closed subspace of* $\mathbb{R}$*.*

Proof: Let A be a complete, non-closed subspace of $\mathbb{R}$ and suppose, without loss of generality, that 0 is in $k(A) \setminus A$ and it is an accumulation point from the right.

Since $\mathbb{Q}$ is countable, there is $r \in \mathbb{Q}^+$ such that $[0,r] \cap A \cap \mathbb{Q} = \emptyset$. Otherwise, we could construct a sequence of rationals converging to 0.

Define $I_n := (r_n, s_n)$, with $r_n := \frac{r}{n+1}$ and $s_n := \frac{r}{n}$, and $A_0 := [0,r] \cap A = \bigcup_{n\in\mathbb{N}}(I_n \cap A)$. The space A_0 is complete but not closed.

Some of the sets $I_n \cap A$ might be empty but infinitely many of them that are not. This just means that the set $\mathbb{M}$ defined by $\mathbb{M} := \{m \in \mathbb{N} \,|\, I_m \cap A \neq \emptyset\}$ has the same cardinal as $\mathbb{N}$. Let us consider the base $(B_n)_{n\in\mathbb{M}}$ of $\mathbb{R}$ consisting of open intervals $((a_n, b_n))_{n\in\mathbb{M}}$ with rational endpoints.

For each $n \in \mathbb{M}$, there is a bijection $f_n : I_n \longrightarrow (a_n, b_n)$ defined by $f_n(x) := a_n + \frac{b_n - a_n}{s_n - r_n}(x - r_n)$. Now, we can define $f : A_0 \longrightarrow \mathbb{R}$ by $f(x) := f_n(x)$ if $x \in I_n$. The function is well-defined because the I_n's are disjoint.

The space $B := f(A)$ is the intended space. It is dense in $\mathbb{R}$ since, for every $n \in \mathbb{M}$, $B_n \cap B \supseteq f_n(I_n \cap A) \neq \emptyset$. It is not closed because $B \cap \mathbb{Q} = \emptyset$, since f sends irrationals to irrationals. It remains to be shown that B is complete.

Let (b_n) be a Cauchy sequence in B and define $\varphi : \mathbb{N} \longrightarrow \mathbb{M}$ by $\varphi(n) := \min\{k \in \mathbb{M} \mid f^{-1}(\{b_n\}) \cap I_k \neq \emptyset\}$. Since $f_{\varphi(n)}$ is bijective, the set $f^{-1}(\{b_n\}) \cap I_{\varphi(n)}$ is singular. The single elements of these sets define a sequence (a_n) in A_0.

By Lemma 2.5, A_0 is sequentially compact. Consequently (a_n) has a subsequence (a_{n_i}) converging to $a \in A_0$. This subsequence only meets finitely many I_k's, otherwise it would converge to $0 \notin A_0$, which implies that the subsequence is eventually in $I_j \cap A_0$ for a fixed $j \in \mathbb{M}$ and with $a \in I_j \cap A_0$.

The continuity of f_j implies the convergence $f(a_{n_i}) \rightarrow f(a)$ in B. The convergence of a subsequence of the Cauchy sequence (b_n) implies that it has to converge to the same point as well. ■

Remarks 2.8

(a) The sets B and $\mathbb{R} \setminus B$ of the above proof are both uncountable.

(b) In [2], N. Brunner had shown that, if there is an infinite, Dedekind-finite subset of $\mathbb{R}$, then there is a dense, Dedekind-finite subset of $\mathbb{R}$.

Definitions 2.9 A topological space is:

(a) a *sequential space* if, for $A \subseteq X$, $\sigma_X(A) = A \Leftrightarrow k_X(A) = A$;

(b) a *Fréchet-Urysohn space* if, for $A \subseteq X$, $k_X(A) = \sigma_X(A)$.

Theorem 2.10 *$\mathbb{R}$ is a sequential space if and only if every complete subspace of $\mathbb{R}$ is closed.*

Proof: Follows directly from Proposition 2.3. ■

Theorem 2.11 [4 - p.124] *$\mathbb{R}$ is a Fréchet-Urysohn space if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Now, after Theorems 2.6, 2.10 and 2.11, it is easier to understand Felscher's result (see Introduction). It is a consequence from the fact that $\mathbb{R}$ is a Fréchet-Urysohn space if and only if $\mathbb{R}$ is a sequential space and $\sigma_{\mathbb{R}}$ is idempotent.

3 A weak form of choice

In this section we will see that a "very" weak form of choice, ω–$\mathbf{CC}(\mathbb{R})$, implies that *"$\mathbb{R}$ is a sequential space"*. Moreover, this form of choice is independent from $\mathbf{CC}(\mathbb{R})$ (Feferman/Levi model – $\mathcal{M}9$ in [9]).

Definition 3.1 ω–$\mathbf{CC}(\mathbb{R})$ states that for every countable family $(X_n)_{n\in\mathbb{N}}$ of non-empty sets of real numbers, there is a family $(A_n)_{n\in\mathbb{N}}$ of non-empty at most countable sets with $A_n \subseteq X_n$ for each n.

Theorem 3.2 *If ω–$\mathbf{CC}(\mathbb{R})$ holds then $\mathbb{R}$ is a sequential space.*

Proof: Let $A = \sigma_{\mathbb{R}(A)}$ be a sequentially closed subset of $\mathbb{R}$ and $x \in k_{\mathbb{R}}(A)$. Define the sets $A_n := [x - \frac{1}{n}, x + \frac{1}{n}] \cap A$. For each $n \in \mathbb{N}$, the set A_n is not empty and sequentially closed. By ω–$\mathbf{CC}(\mathbb{R})$ there is a family $(B_n)_{n\in\mathbb{N}}$ of non-empty at most countable sets such that, for each $n \in \mathbb{N}$, $B_n \subseteq A_n$. Since each B_n is countable, $\sigma_{\mathbb{R}}(B_n) = k_{\mathbb{R}}(B_n) \subseteq \sigma_{\mathbb{R}}(A_n) = A_n$. The elements $x_n := \min k_{\mathbb{R}}(B_n)$ belong to $A_n \subseteq A$, for each $n \in \mathbb{N}$, and the sequence (x_n) converges to x. This means that x is the limit of a sequence in A, and so $x \in \sigma_{\mathbb{R}}(A) = A$. ■

Following the proof of Theorem 3.2, it is not difficult to prove that *"$\mathbb{R}$ is a sequential space"* is also equivalent to a choice principle.

Proposition 3.3 *Equivalent are:*

(i) *$\mathbb{R}$ is a sequential space,*

(ii) *the axiom of (countable) choice restricted to families of sequentially closed (=complete) subspaces of $\mathbb{R}$.*

Theorem 3.4 *If $\mathbb{R}$ is the countable union of countable sets then ω–$\mathbf{CC}(\mathbb{R})$ does hold.*

Proof: Let $\mathbb{R}$ be the countable union of countable sets, $\mathbb{R} = \bigcup_{n\in\mathbb{N}} A_n$, with each A_n countable, and consider $X \subseteq \mathbb{R}$.

The set X is equal to $\bigcup_{n\in\mathbb{N}} A_n \cap X$. Define $s := \min\{n \in \mathbb{N} \mid A_n \cap X \neq \emptyset\}$; the set $A_s \cap X$ is at most countable. This process can be done for infinitely many sets at the same time, which concludes the proof. ■

Alonzo Church [3] proved that $\mathbf{CC}(\mathbb{R})$ implies that the first uncountable ordinal is not the limit of a sequence of countable ordinals, which implies that $\mathbb{R}$ is not the countable union of countable sets (see [10 - p.148]).

In the Feferman/Levi model, $\mathbb{R}$ is the countable union of countable sets and then ω–$\mathbf{CC}(\mathbb{R})$ holds in this model but $\mathbf{CC}(\mathbb{R})$ not.

After these considerations, the next Corollary is clear.

Corollary 3.5 *If $\mathbb{R}$ is the countable union of countable sets then the axiom of countable choice restricted to families of countable sets of reals does not hold.*

There is a different proof to this fact in [8].

4 Lindelöf spaces

It is known that in **ZFC** every subspace of $\mathbb{R}$ is Lindelöf. H. Herrlich and G. E. Strecker ([7]) proved that this fact is equivalent to $\mathbf{CC}(\mathbb{R})$.

The question that triggered the investigation of this section was to find out under which conditions there are non-Lindelöf, sequentially compact subspaces of $\mathbb{R}$.

Definition 4.1 A topological space X is *Lindelöf* if every open cover of X has an at most countable subcover.

Theorem 4.2 *Every unbounded, Lindelöf subspace of $\mathbb{R}$ contains an unbounded sequence.*

Proof: Let A be an unbounded, Lindelöf subspace of $\mathbb{R}$. Without loss of generality, consider A with no upper bound. Define the cover $\mathcal{U} := \{(-\infty, a) \cap A \mid a \in A\}$ of A. It is a cover because A has no upper bound. But A is Lindelöf, so there is a countable subcover $\{(-\infty, a_n) \cap A \mid a_n \in A \text{ and } n \in \mathbb{N}\}$ of $\mathcal{U}$. The a_n's are uniquely determined by the sets, since if $a < b$ then $a \in (-\infty, b) \cap A$ and $a \notin (-\infty, a) \cap A$.

The sequence (a_n) is the desired sequence. ■

Remark 4.3 The condition *"every unbounded subset of $\mathbb{R}$ contains an unbounded sequence"* is equivalent to $\mathbf{CC}(\mathbb{R})$ (see [7]).

Corollary 4.4 *The negation of $\mathbf{CC}(\mathbb{R})$ is equivalent to the statement that for subspaces of $\mathbb{R}$ the properties compact and Lindelöf coincide.*

Proof: If **CC**($\mathbb{R}$) holds then every subspace of $\mathbb{R}$ is Lindelöf.

Suppose that there is a Lindelöf, non-compact subspace of $\mathbb{R}$. Hence there is $B \subseteq \mathbb{R}$ Lindelöf and unbounded. We can prove this fact in an analogous way to (iii)$\Rightarrow$(ii) in Proposition 1.6. By Theorem 4.2, B has an unbounded sequence (b_n). Without loss of generality, consider (b_n) with no upper bound. Define $s(n) := \min\{k \in \mathbb{N} \,|\, b_k \geq n\}$ and $C := \{b_{s(n)} \,|\, n \in \mathbb{N}\}$. The space C is closed, because it is discrete, and a closed subspace of a Lindelöf space is Lindelöf. The space C is countable and discrete, which implies that C is homeomorphic to $\mathbb{N}$ and consequently $\mathbb{N}$ is Lindelöf. Finally, $\mathbb{N}$ Lindelöf is equivalent to **CC**($\mathbb{R}$). (For this last equivalence, see [7].) ∎

Remark 4.5 In [6], H. Herrlich has a generalization of this fact saying that compact and Lindelöf T_1-spaces coincide if and only if **CC**($\mathbb{R}$) fails.

Corollary 4.6 *Every sequentially compact, Lindelöf subspace of* $\mathbb{R}$ *is compact.*

Proof: If **CC**($\mathbb{R}$) holds then compact coincides with sequentially compact and if **CC**($\mathbb{R}$) fails compact coincides with Lindelöf. ∎

Corollary 4.7 *For subspaces of* $\mathbb{R}$, *the following conditions are equivalent:*

(i) *sequentially compact* $\Longrightarrow$ *compact,*

(ii) *sequentially compact* $\Longrightarrow$ *Lindelöf.*

Proof: **(i)** $\Rightarrow$ **(ii)** Obvious.

(ii) $\Rightarrow$ **(i)** Follows directly from Corollary 4.6. ∎

Remark 4.8 The condition *"complete implies Lindelöf"* is equivalent to **CC**($\mathbb{R}$). This is easily deduced from the fact that $\mathbb{R}$ is complete and $\mathbb{R}$ is Lindelöf if and only if **CC**($\mathbb{R}$) holds. This last fact is in [7].

5 Separability

Definition 5.1 A topological space is *separable* if it contains an at most countable dense subset.

In **ZF**, we cannot prove that every Lindelöf metric space is separable ([5]).
For pseudometric spaces the statement is equivalent to **CC** ([1]).
In the particular case of subspaces of $\mathbb{R}$ it is a theorem of **ZF**.

Theorem 5.2

(a) *Every closed subset of* $\mathbb{R}$ *is separable.*

(b) *Every compact subspace of* $\mathbb{R}$ *is separable.*

(c) *Every Lindelöf subspace of* $\mathbb{R}$ *is separable.*

Proof: **(a)** Let $A \neq \emptyset$ be a closed subset of $\mathbb{R}$ and $b := \sup A$ (b can be $+\infty$). Define now $\mathbb{P} := \{r \in \mathbb{Q} \,|\, r < b\}$ and, for each $r \in \mathbb{P}$, $\phi(r) := \min([r, +\infty) \cap A)$, which exists because A is closed. The set $\{\phi(r) \,|\, r \in \mathbb{P}\}$ is countable and dense in A.

(b) Follows from (a).

(c) If $\mathbf{CC}(\mathbb{R})$ holds then every subspace of $\mathbb{R}$ is separable. In the case of the failure of $\mathbf{CC}(\mathbb{R})$, by Corollary 4.4 every Lindelöf subspace of $\mathbb{R}$ is compact, hence separable by (b). ∎

Remark 5.3 [7] Every subspace of $\mathbb{R}$ is separable if and only if $\mathbf{CC}(\mathbb{R})$ holds.

Theorem 5.4 *For subspaces of* $\mathbb{R}$, *the following conditions are equivalent:*

(i) *sequentially compact* $\Longrightarrow$ *closed,*

(ii) *complete* $\Longrightarrow$ *closed,*

(iii) *complete* $\Longrightarrow$ *separable,*

(iv) *sequentially compact* $\Longrightarrow$ *separable.*

Proof: **(i)** $\Leftrightarrow$ **(ii)** Theorem 2.6.

(ii) $\Rightarrow$ **(iii)** By Theorem 5.2(a).

(iii) $\Rightarrow$ **(iv)** Every sequentially compact subspace of $\mathbb{R}$ is complete.

(iv) $\Rightarrow$ **(i)** Let $A \subseteq \mathbb{R}$ be a sequentially compact space. By (iv) there is $B = \{x_n \,|\, n \in \mathbb{N}\}$ such that $k_A(B) = A$, and then $k_{\mathbb{R}}(A) = k_{\mathbb{R}}(B)$.

Let x be an element of $k_{\mathbb{R}}(A) = k_{\mathbb{R}}(B)$. For every $n \in \mathbb{N}$ we have: $(x - \frac{1}{n}, x + \frac{1}{n}) \cap B \neq \emptyset$. Define $\varphi(n) := \min\{k \in \mathbb{N} \,|\, x_k \in (x - \frac{1}{n}, x + \frac{1}{n})\}$. The sequence $(x_{\varphi(n)})$ converges to x.

Since A is sequentially compact (and thus complete), by Proposition 2.3, $\sigma_{\mathbb{R}}(A) = A$, that is $x \in A$. ∎

6 Summary

Here is a list of conditions that are equivalent to each other. They follow from ω–$\mathbf{CC}(\mathbb{R})$ and they imply the equivalence between finitness and Dedekind-finitness for subsets of $\mathbb{R}$. I do not know if one of the reverse implications is true.

- $A \subseteq \mathbb{R}$ is sequentially compact iff it is compact.
- If $A \subseteq \mathbb{R}$ is sequentially compact then it is closed.
- If $A \subseteq \mathbb{R}$ is sequentially compact then it is bounded.
- $A \subseteq \mathbb{R}$ is complete iff it is closed.
- If $A \subseteq \mathbb{R}$ is complete and $k(A) = \mathbb{R}$ then $A = \mathbb{R}$.
- $\mathbb{R}$ is a sequential space.
- The axiom of (countable) choice restricted to families of complete subspaces of $\mathbb{R}$.
- If $A \subseteq \mathbb{R}$ is sequentially compact then it is Lindelöf.
- If $A \subseteq \mathbb{R}$ is sequentially compact then it is separable.
- If $A \subseteq \mathbb{R}$ is complete then it is separable.

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