Discrimination methodology between error processes and bilinear processes

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Abstract

A new methodology, based on the asymptotic separation of probability laws, was introduced by Gonçalves, Jacob and Mendes-Lopes (2000) in the development of the statistical inference of bilinear models, namely in the construction of a consistent decision procedure for the simple bilinear ones.

This paper presents a generalisation of that study by introducing in the procedure a smoother decision statistics.

The aim of this decision method is to discriminate between an error process and a simple bilinear model. So, we use it as a consistent test and its consistence is obtained by establishing the asymptotic separation of the sequences of probability laws defined by each hypothesis.

The convergence rate of the procedure is studied under the truthfulness of the error process hypothesis. An exponential decay is obtained.

AMS Classification:

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1 Introduction

In Gonçalves, Jacob and Mendes-Lopes (2000) a new methodology of statistical decision to discriminate between an error process and a diagonal simple bilinear model was presented. This methodology was inspired in an asymptotic separation result obtained, in 1976, by Geffroy, which appeared particularly useful to construct consistent tests and estimators for detecting a signal in a white noise (Pieczinsky, 1986, Moché, 1989).

Let $X = (X_t, t \in \mathbb{Z})$ be a real stochastic process with a law which belongs to a set of parametric laws $(P_\theta, \theta \in \Theta)$, with $\Theta = \{\theta_1, \theta_2\}$. Following Geffroy (1976), we say that the two laws P_{θ_1} and P_{θ_2} are asymptotically separated if there exists a sequence of Borel sets of $\mathbb{R}^T$, $(A_T, T \in \mathbb{N})$, such that

$$\begin{cases} P_{\theta_0}^T(A_T) \underset{T\to+\infty}{\longrightarrow} 1 \\ P_{\theta_1}^T(A_T) \underset{T\to+\infty}{\longrightarrow} 0, \end{cases}$$

where P_θ^T denotes the probability law of $(X_1, X_2, ..., X_T)$.

In this way, a consistent decision rule was defined and studied in Gonçalves, Jacob and Mendes-Lopes (2000) to separate the hypothesis "$H_0 : X$ follows an error process" against "$H_1 : X$ follows a diagonal bilinear model".

With the aim of improving the rate of convergence of the decision procedure we present, in this paper, a generalisation of that study in which a more smooth statistics is considered in the definition of the sequence of acceptance regions $(A_T)_{T\in\mathbb{N}.}$ In fact, unlike what we have considered in that pioneer study, the statistics here considered is, in general, a continuous function of the sample.

2 General properties and hypotheses

Let us consider the diagonal bilinear model $X = (X_t, t \in \mathbb{Z})$ defined by

$$X_t = \varphi X_{t-1}\,\varepsilon_{t-1} + \varepsilon_t, \tag{1}$$

where φ is a real number and $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ a real stochastic process.

We are going to construct a decision procedure to discriminate between the hypotheses $H_0 : \varphi = 0$ against $H_1 : \varphi = \beta$ $(\beta > 0, \text{fixed})$.

Let us denote the process $X = (X_t, t \in \mathbb{Z})$ distribution and the corresponding expectation by P_φ and E_φ respectively, when the parameter of the model is equal to φ.

We suppose that

$\mathcal{H}1$: $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a strictly stationary and ergodic process.
$\mathcal{H}2$: $E|\log|\varepsilon_t|| < \infty$ and $E(\log|\varepsilon_t|) + \log|\varphi| < 0$.

Under these conditions, model (1) has a strictly stationary and ergodic solution, $P_\varphi - a.s.$ unique, given by

$$X_t = \varepsilon_t + \sum_{n=1}^{+\infty} \varphi^n \varepsilon_{t-n} \prod_{j=0}^{n-1} \varepsilon_{t-1-j} \;\; (a.s.), \;\; t \in \mathbb{Z}.$$

If, in adition, we have

$\mathcal{H}3$: $E|\log|X_t|| < \infty.$ and $E(\log|X_t|) + \log|\varphi| < 0$,

then model (1) is invertible and

$$\varepsilon_t = X_t + \sum_{n=1}^{+\infty} (-\varphi)^n X_{t-n} \prod_{j=0}^{n-1} X_{t-1-j} \;\; (a.s.), \;\; t \in \mathbb{Z}.$$

Under conditions $\mathcal{H}1$, $\mathcal{H}2$ and $\mathcal{H}3$ we deduce, in view of the two equalities above, that $\underline{X}_t = \underline{\varepsilon}_t$, where $\underline{X}_t$ and $\underline{\varepsilon}_t$ denote the σ-fields generated by $(X_t, X_{t-1}, \ldots)$ and $(\varepsilon_t, \varepsilon_{t-1}, \ldots)$ respectively.

Hereafter we assume these general hypotheses concerning the stationarity, ergodicity, and invertibility of model (1). We also define the process $Y = (Y_t, t \in \mathbb{Z})$ by

$$Y_t = X_t \left(X_t + \sum_{n=1}^{\infty} (-\varphi)^n X_{t-n} \prod_{j=0}^{n-1} X_{t-1-j} \right) \;\; (a.s.).$$

This process is also strictly stationary and ergodic. We will denote it by $Y_t(\varphi)$, if its dependence on the parameter φ is to be emphasized.

We note that $X_t = \varphi Y_{t-1} + \varepsilon_t$, according to (1). Otherwise, taking into account that $E|\log|\varepsilon_t|| < +\infty$ and $E|\log|X_t|| < +\infty$, we have $Y_t(\varphi) \neq 0, a.e., \forall \varphi$.

3 A consistent test

We are going to construct a decision procedure to distinguish, in model (1), the hypotheses

$$H_0 : \varphi = 0 \text{ against } H_1 : \varphi = \beta \;\; (\beta > 0)$$

from T observations of the process X, denoted by $x_1, x_2, \ldots, x_T$.

The procedure we are proposing is based on the notion of asymptotic separation of two families of probability laws (Geffroy (1980), Moché (1989)) and it generalises recent works (Gonçalves, Jacob, Mendes-Lopes (2000), Gonçalves, Martins, Mendes-Lopes (2001)).

First of all, we establish the asymptotic separation of the families of probability laws associated to the hypotheses in study by presenting a sequence of Borel sets of $\mathbb{R}^T$, $(A_T, T \in \mathbb{N})$, called separation regions, such that

$$\begin{cases} P_0^T(A_T) \underset{T \to +\infty}{\longrightarrow} 1 \\ P_\beta^T(A_T) \underset{T \longrightarrow +\infty}{\longrightarrow} 0 \end{cases}$$

where P_φ^T denotes the probability law of $(X_1, ..., X_T)$ when the parameter is equal to φ.

We will accept $H_0 : \varphi = 0$ against $H_1 : \varphi = \beta > 0$ if $(x_1, ..., x_T) \in A_T$.

The separation regions that we are going to propose are inspired in previous works. In those papers the set

$$\begin{aligned} D &= \left\{(u,v) \in \mathbb{R}^2 : u > 0,\, v < \frac{\beta}{2}u\right\} \cup \left\{(u,v) \in \mathbb{R}^2 : u < 0,\, v > \frac{\beta}{2}u\right\} \\ &= \left\{(u,v) \in \mathbb{R}^2 : u\left(\frac{\beta}{2}u - v\right) > 0\right\} \end{aligned}$$

is very important in the construction of a convergent test for the same hypotheses. The test takes into account the number of times that $u\left(\frac{\beta}{2}u - v\right) > 0$ when $(u,v) = (y_{t-1}, x_t)$, y_t denoting the particular value of Y_t, $t = 1, ..., T$.

The generalization here studied presents a test statistic with the same basical idea but defined through a more smooth function, eventually a continuous one.

From the definition of D we have

$$(y_{t-1}, x_t) \in D \Longleftrightarrow \left(y_{t-1} > 0, \tfrac{\beta}{2}y_{t-1} - x_t > 0\right) \text{ or } \left(y_{t-1} < 0, \tfrac{\beta}{2}y_{t-1} - x_t < 0\right).$$

So, if we consider a distribution function F of a symmetrical law we have

$$x > 0 \Longrightarrow F(x) \geq \tfrac{1}{2}.$$

Then,

$$\begin{aligned} (y_{t-1}, x_t) \in D \Rightarrow & \left(2F(y_{t-1}) - 1 \geq 0, 2F\left(\tfrac{\beta}{2}y_{t-1} - x_t\right) - 1 \geq 0\right) \text{ or} \\ & \left(2F(y_{t-1}) - 1 \leq 0, 2F\left(\tfrac{\beta}{2}y_{t-1} - x_t\right) - 1 \leq 0\right) \\ \Rightarrow & \left[2F(y_{t-1}) - 1\right]\left[2F\left(\tfrac{\beta}{2}y_{t-1} - x_t\right) - 1\right] \geq 0. \end{aligned}$$

The study here presented takes into account this product. Moreover, a great degree of generality is achieved as the distribution function considered in the first factor may be different from that appearing in the second one.

3.1 Acceptation regions definition: notation, hypotheses

We define

$$g(u,v) = \left[2G(u) - 1\right]\left[2F\left(\tfrac{\beta}{2}u - v\right) - 1\right], (u,v) \in \mathbb{R}^2,$$

where F and G are distribution functions of symmetrical laws with decreasing densities on $\mathbb{R}^+$.

Let us consider the following regions

$$A_T = \left\{ (x_1, x_2, \ldots, x_T) \in \mathbb{R}^T : \sum_{t=2}^{T} g(y_{t-1}(\beta), x_t) \geq 0 \right\}.$$

In what follows, we take $g_t = g(y_{t-1}(\beta), x_t)$ and $\overline{g}_T = \frac{1}{T} \sum_{t=2}^{T} g_t$ and we assume the hypothesis:

$\mathcal{H}_4$: the law of ε_t given $\underline{\varepsilon}_{t-1}$ is symmetrical.

We have the following result:

Lemma 1. (i) Under the hypothesis $\varphi = 0$, $\underset{T}{lim}\ \overline{g}_T = E_0(g_2) > 0$.
(ii) Under the hypothesis $\varphi = \beta > 0$, $\underset{T}{lim}\ \overline{g}_T = E_\beta(g_2) < 0$.

Proof. By the ergodic theorem we have

$$\underset{T}{lim}\ \overline{g}_T = E_\varphi(g_2), a.e.$$

with

$$E_\varphi(g_2) = E_\varphi(g(Y_1(\beta), X_2)) = E_\varphi\left([2G(Y_1(\beta)) - 1]\left[2F\left(\tfrac{\beta}{2}Y_1(\beta) - X_2\right) - 1\right]\right).$$

Let us now study the sign of the limit under each one of the hypotheses H_0 and H_1. In what follows, we take $Y_1(\beta) = Y_1$.

Under $\varphi = 0$ we have $X_2 = \varepsilon_2$ and so

$$\begin{aligned}
E_0(g_2) &= E_0\left([2G(Y_1) - 1]\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right]\right) \\
&= E_0\left([2G(Y_1) - 1]\, E_0\left\{\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right] / \underline{\varepsilon}_1\right\}\right) \\
&= E_0\left([2G(Y_1) - 1]\, \mathbb{I}_{\{Y_1 > 0\}} E_0\left\{\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right] / \underline{\varepsilon}_1\right\}\right) + \\
&\quad + E_0\left([2G(Y_1) - 1]\, \mathbb{I}_{\{Y_1 < 0\}} E_0\left\{\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right] / \underline{\varepsilon}_1\right\}\right)
\end{aligned}$$

When $Y_1 > 0$, we have $2G(Y_1) - 1 > 0$ and $E_0\left\{\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right] / \underline{\varepsilon}_1\right\} > 0$ taking into account the symmetry of the law of -ε_t given $\underline{\varepsilon}_{t-1}$; if $Y_1 < 0$ then $2G(Y_1) - 1 < 0$ and $E_0\left\{\left[2F\left(\frac{\beta}{2}Y_1 - \varepsilon_2\right) - 1\right] / \underline{\varepsilon}_1\right\} < 0$.

So, $E_0(g_2) > 0$.

Under $\varphi = \beta > 0$ we have $Y_1 = X_1\varepsilon_1$, $X_2 = \beta X_1\varepsilon_1 + \varepsilon_2$ and then

$$\begin{aligned} E_\beta(g_2) &= E_\beta\left([2G(X_1\varepsilon_1)-1]\left[2F\left(\frac{\beta}{2}X_1\varepsilon_1 - \beta X_1\varepsilon_1 - \varepsilon_2\right)-1\right]\right) \\ &= E_\beta\left([2G(X_1\varepsilon_1)-1]\,E_\beta\left\{\left[2F\left(-\frac{\beta}{2}X_1\varepsilon_1 - \varepsilon_2\right)-1\right]/\underline{\varepsilon}_1\right\}\right) \\ &= E_\beta\left([2G(X_1\varepsilon_1)-1]\,\mathbb{I}_{\{X_1\varepsilon_1>0\}}E_\beta\left\{\left[2F\left(-\frac{\beta}{2}X_1\varepsilon_1 - \varepsilon_2\right)-1\right]/\underline{\varepsilon}_1\right\}\right) \\ &\quad + E_\beta\left([2G(X_1\varepsilon_1)-1]\,\mathbb{I}_{\{X_1\varepsilon_1<0\}}E_\beta\left\{\left[2F\left(-\frac{\beta}{2}X_1\varepsilon_1 - \varepsilon_2\right)-1\right]/\underline{\varepsilon}_1\right\}\right). \end{aligned}$$

As previously, $2G(X_1\varepsilon_1) - 1 > 0$ and $E_\beta\left\{\left[2F\left(-\frac{\beta}{2}X_1\varepsilon_1 - \varepsilon_2\right)-1\right]/\underline{\varepsilon}_1\right\} < 0$ when $X_1\varepsilon_1 > 0$; on the other hand, $2G(X_1\varepsilon_1) - 1 < 0$ and $E_\beta\left\{\left[2F\left(-\frac{\beta}{2}X_1\varepsilon_1 - \varepsilon_2\right)-1\right]/\underline{\varepsilon}_1\right\} > 0$, when $X_1\varepsilon_1 < 0$.
Then $E_\beta(g_2) < 0$.

We immediately deduce, by the bounded convergence theorem, the following result:

Corollary. (i) If $\varphi = 0$, $P_0(\overline{g}_T \geq 0) \longrightarrow 1$, as $T \longrightarrow +\infty$.
(ii) If $\varphi = \beta > 0$, $P_\beta(\overline{g}_T \geq 0) \longrightarrow 0$, as $T \longrightarrow +\infty$.

Taking into account the definition of regions A_T, we conclude that the probability laws of process $(X_t, t \in \mathbb{Z})$ defined by the hypotheses $H_0 : \varphi = 0$ and $H_1 : \varphi = \beta > 0$ are asymptotically separated.

So, A_T is the acceptance region of a consistent test for these hypotheses.

4 Convergence rate of the decision procedure

The convergence rate of the decision procedure, presented in the previous paragraph as a test, may be evaluated when we consider, in the acceptance regions A_T, the true value of Y_t, i.e., $Y_t(\varphi)$, and we assume that the null hypothesis is true. Let us denote these borelians by $A_T(\varphi)$.

This convergence rate study has been developped under the hypothesis of absolute continuity and symmetry of the distribution laws involved. So, in a previous paragraph we establish several lemmas concerning distribution functions of symmetrical densities.

4.1 Preliminary results

Lemma 1. Let f be a symmetrical density decreasing on $\mathbb{R}^+$ with distribution function F. Let a and b be fixed real numbers, with $a > 0$. Then the function $\widetilde{G}$ defined by

$$\widetilde{G}(v) = \int_{-\infty}^{+\infty} F(av - bu) f(u)\, du$$

is the distribution function of a law with symmetrical density g decreasing on $\mathbb{R}^+$.

Proof. As we can derivate under the $\int$,we obtain

$$\tfrac{d}{dv}\widetilde{G}(v) = \int_{-\infty}^{+\infty} af(av - bu) f(u)\, du.$$

Then, as f is symmetrical,

$$\begin{aligned}
\frac{d}{dv}\widetilde{G}(-v) &= \int_{-\infty}^{+\infty} af(-av - bu) f(u)\, du \\
&= \int_{-\infty}^{+\infty} af(av + bu) f(u)\, du \\
&= \int_{-\infty}^{+\infty} af(av - by) f(y)\, dy \\
&= \int_{-\infty}^{+\infty} af(av - bu) f(u)\, du \\
&= \frac{d}{dv}\widetilde{G}(v).
\end{aligned}$$

Denoting $\frac{d}{dv}\widetilde{G} = g$, g is a symmetrical function. Let us prove that g is a density function and $\widetilde{G}$ the distribution function of density g.

¿From Fubini, we obtain

$$\int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} af(av - bu) f(u)\, du = \int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} af(av - bu)\, dv\right) du.$$

But

$$\int_{-\infty}^{+\infty} af(av - bu)\, dv = \int_{-\infty}^{+\infty} af(z) \tfrac{1}{a} dz = 1.$$

Then

$$\int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} af(av - bu) f(u)\, du = 1.$$

On the other hand, again from Fubini,

$$\begin{aligned}
\int_{-\infty}^{x} g(v)dv &= \int_{-\infty}^{x} \left(\int_{-\infty}^{+\infty} af\left(av-bu\right)du \right) dv \\
&= \int_{-\infty}^{+\infty} f(u) \left[\int_{-\infty}^{x} af\left(av-bu\right)dv \right] du \\
&= \int_{-\infty}^{+\infty} f(u) \left[\int_{-\infty}^{ax-bu} af\left(z\right)\frac{1}{a}dz \right] du \\
&= \int_{-\infty}^{+\infty} F\left(ax-bu\right)f(u)du \\
&= \widetilde{G}\left(v\right).
\end{aligned}$$

¿From the definition of g and as f is decreasing on $\mathbb{R}^+$, it is obvious that g is decreasing on $\mathbb{R}^+$.

Lemma 2. Let $h\left(x\right) = c\left[2R\left(a-dx\right)-1\right], x \in \mathbb{R}$, where c, a, d are positive numbers and R is the distribution function of a symmetrical and decreasing on $\mathbb{R}^+$ density, r.

Let $H(x) = e^{-h(x)}$. Then $H(x)+H(-x)$ is increasing on $\mathbb{R}^+$.

Proof. We have

$$\begin{aligned}
\frac{d}{dx}\left[H(x)+H(-x)\right] &= \frac{d}{dx}\left[e^{-h(x)}+e^{-h(-x)}\right] \\
&= \left[-h'(x)e^{-h(x)}+h'(-x)e^{-h(-x)}\right] \\
&= 2cdr\left(a-dx\right)e^{-h(x)} - 2cdr\left(a+dx\right)e^{-h(-x)}.
\end{aligned}$$

Let us show that this derivative is non negative.

As c and d are positive, it is enough to show that

$$\left\{ \begin{array}{ll} r\left(a-dx\right) \geq r\left(a+dx\right), & \forall x \geq 0 \\ e^{-h(x)} \geq e^{-h(-x)}, & \forall x \geq 0 \end{array} \right. .$$

As $a>0$ and $d>0$ and r is decreasing on $\mathbb{R}^+$, we have $r\left(a-dx\right) \geq r\left(a+dx\right)$, for every $x \geq 0$ such that $a-dx>0$.

But, as r is symmetrical, r is increasing on $\mathbb{R}^-$ and if $a-dx<0$ we have

$$r\left(a-dx\right) = r\left(dx-a\right) \geq r\left(a+dx\right),$$

as $0 \leq dx - a < dx + a$.

Moreover, as r is a symmetrical density, the function

$$2R(x) - 1 = R(x) - R(-x)$$

is odd and obviously increasing on $\mathbb{R}^+$.

As c and d are positive we can conclude, by an analogous way, that for every $x \geq 0$

$$c[2R(a - dx) - 1] \leq c[2R(a + dx) - 1]$$

that is

$$h(x) \leq h(-x)$$

and, in consequence,

$$e^{-h(x)} \geq e^{-h(-x)}.$$

Lemma 3. Let φ and f be two symmetrical densities and $a > 0$ such that $\varphi > f$ on $[0, a[$ and $\varphi < f$ on $]a, +\infty[$.

Let T be a positive and increasing function, defined on $\mathbb{R}^+$. Then

$$\int_0^{+\infty} \varphi(x) T(x) \, dx < \int_0^{+\infty} f(x) T(x) \, dx.$$

Proof. We have

$$\begin{aligned} \int_0^{+\infty} [\varphi(x) - f(x)] T(x) \, dx &= \int_{[0,a[} [\varphi(x) - f(x)] T(x) \, dx + \\ &\quad + \int_{]a,+\infty[} [\varphi(x) - f(x)] T(x) \, dx \\ &< T\left(a^-\right) \int_{[0,a[} [\varphi(x) - f(x)] \, dx + \\ &\quad + T\left(a^+\right) \int_{]a,+\infty[} [\varphi(x) - f(x)] \, dx \end{aligned}$$

as T is an increasing function and where $T(a^-)$ denotes the left limit and $T(a^+)$ the right limit on a.

As the first quantity is positive, we have

$$\int_0^{+\infty} [\varphi(x) - f(x)] T(x) \, dx < T(a^+) \int_{]0,+\infty[} [\varphi(x) - f(x)] \, dx = 0,$$

taking into account that φ and f are symmetrical densities.

Lemma 4. Let h be the function of lemma 2, φ and f the probability densities of lemma 3 and Y and Z real random variables with densities f and φ, respectively.

Then

$$E\left[e^{-h(Z)}\right] \leq E\left[e^{-h(Y)}\right].$$

Proof. We have

$$\begin{aligned}
E\left[e^{-h(Z)}\right] &= \int_{-\infty}^{+\infty} e^{-h(z)}\varphi(z)\,dz \\
&= \int_{-\infty}^{+\infty} e^{-h(z)}\varphi(-z)\,dz \\
&= \int_{-\infty}^{+\infty} e^{-h(-u)}\varphi(u)\,du \\
&= E\left[e^{-h(-Z)}\right].
\end{aligned}$$

Then, with $H(x) = e^{-h(x)}$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} H(x)\,\varphi(x)\,dx &= \int_{-\infty}^{+\infty} H(-x)\,\varphi(x)\,dx \\
&= \int_{-\infty}^{+\infty} \frac{H(x)+H(-x)}{2}\varphi(x)\,dx \\
&= \int_{-\infty}^{0} \frac{H(x)+H(-x)}{2}\varphi(x)\,dx + \int_{0}^{+\infty} \frac{H(x)+H(-x)}{2}\varphi(x)\,dx \\
&= \int_{0}^{+\infty} \left(H(x)+H(-x)\right)\varphi(x)\,dx,
\end{aligned}$$

as φ is symmetrical.

In the same way, we have

$$E\left[e^{-h(Y)}\right] = \int_0^{+\infty}\left[H(x)+H(-x)\right]f(x)\,dx.$$

As, by lemma 2, the function $H(x)+H(-x)$ is increasing on $\mathbb{R}^+$, we can apply lemma 3 to obtain

$$\int_0^{+\infty}\varphi(x)\left[H(x)+H(-x)\right]dx < \int_0^{+\infty} f(x)\left[H(x)+H(-x)\right]dx,$$

that's to say,

$$E\left[e^{-h(Z)}\right] < E\left[e^{-h(Y)}\right].$$

Lemma 5. Let g be a symmetrical function, increasing on $\mathbb{R}^+$, equal to zero in the origin and bounded. Let Y be a real random variable with a symmetrical and decreasing on $\mathbb{R}^+$ density f. Let h be the function of lemma 2. Then

$$E\left[e^{-g(Y)-h(Y)}\right] < E\left[e^{-g(Y)}\right] E\left[e^{-h(Y)}\right].$$

Proof. Let us take

$$\frac{1}{b} = \int_{-\infty}^{+\infty} e^{-g(x)} f(x)\, dx.$$

We note that $b > 1$, as $e^{-g} < 1$ almost everywhere.

We consider

$$\varphi(x) = b e^{-g(x)} f(x).$$

Then φ is a symmetrical density.

On the other hand, as $b > 1$ and $g(0) = 0$, we obtain

$$\varphi(0) = b e^{-g(0)} f(0) > f(0).$$

Moreover

$$\varphi(x) = f(x) \Longleftrightarrow b e^{-g(x)} = 1.$$

As g is monotone increasing, there is a unique root $a > 0$ such that $\varphi > f$ in $[0, a[$ and $\varphi < f$ in $]a, +\infty[$.

Let Z be a real random variable with density φ. From lemma 4 we have

$$E\left[e^{-h(Z)}\right] \leq E\left[e^{-h(Y)}\right] \Leftrightarrow \int_{-\infty}^{+\infty} e^{-h(x)} \varphi(x)\, dx \leq \int_{-\infty}^{+\infty} e^{-h(x)} f(x)\, dx.$$

As $\varphi(x) = b e^{-g(x)} f(x)$, we obtain

$$b \int_{-\infty}^{+\infty} e^{-h(x)} e^{-g(x)} f(x)\, dx \leq \int_{-\infty}^{+\infty} e^{-h(x)} f(x)\, dx$$

or, using the b definition,

$$\int_{-\infty}^{+\infty} e^{-h(x)} e^{-g(x)} f(x)\, dx \leq \int_{-\infty}^{+\infty} e^{-h(x)} f(x)\, dx \int_{-\infty}^{+\infty} e^{-g(x)} f(x)\, dx$$

which is equivalent to

$$E\left[e^{-g(Y)-h(Y)}\right] \leq E\left[e^{-g(Y)}\right] E\left[e^{-h(Y)}\right].$$

4.2 Evaluation of the (R) convergence rate

We are going to evaluate the convergence rate of $P_0\left(\overline{A}_T(\varphi)\right)$.

We have

$$\begin{aligned}
P_0\left(\overline{A}_T(\varphi)\right) &= P_0\left(\sum_{t=2}^{T} g_t < 0\right) \\
&\leq E_0\left[\exp\left(-\sum_{t=2}^{T} g_t\right)\right] \\
&= E_0\left\{E_0\left[\exp\left(-\sum_{t=2}^{T} g_t\right)/\underline{\varepsilon}_{T-1}\right]\right\} \\
&= E_0\left\{\exp\left(-\sum_{t=2}^{T-1} g_t\right)E_0\left[\exp\left(-g_T\right)/\underline{\varepsilon}_{T-1}\right]\right\}.
\end{aligned}$$

Let us study $E_0\left[g_t/\underline{\varepsilon}_{t-1}\right], t \in \mathbb{Z}$.

$$E_0\left[g_t/\underline{\varepsilon}_{t-1}\right] = \left[2G\left(\varepsilon_{t-1}^2\right) - 1\right]\left\{2E_0\left[F\left(\tfrac{\beta}{2}\varepsilon_{t-1}^2 - \varepsilon_t\right)/\underline{\varepsilon}_{t-1}\right] - 1\right\}.$$

Let us suppose that ε verifies the following condition, denoted by $\mathcal{H}_5$:

$$\varepsilon_t = \eta_{t-1} Z_t, t \in \mathbb{Z}$$

where

. η_t is a measurable and strictly positive function of $\varepsilon_t, \varepsilon_{t-1}, \ldots$ with $0 < m \leq \eta_t \leq M$;

. $(Z_t, t \in \mathbb{Z})$ is a sequence of independent and identically distributed real random variables, with distribution function F and density f that we suppose symmetrical and decreasing on $\mathbb{R}^+$. We also assume that Z_t is independent of $\underline{\varepsilon}_{t-1}$.

Let us analyse $E_0\left[F\left(\frac{\beta}{2}Y_{t-1} - \varepsilon_t\right)/\underline{\varepsilon}_{t-1}\right]$.

$$\begin{aligned}
E_0\left[F\left(\frac{\beta}{2}\varepsilon_{t-1}^2 - \varepsilon_t\right)/\underline{\varepsilon}_{t-1}\right] &= E_0\left[F\left(\frac{\beta}{2}\varepsilon_{t-1}^2 - \eta_{t-1}Z_t\right)/\underline{\varepsilon}_{t-1}\right] \\
&= \int_{-\infty}^{+\infty} F\left(\frac{\beta}{2}\varepsilon_{t-1}^2 - \eta_{t-1}u\right) f(u)du \\
&\geq \int_{-\infty}^{+\infty} F\left(\frac{\beta}{2}\varepsilon_{t-1}^2 - Mu\right) f(u)du.
\end{aligned}$$

Let 's choose the following function

$$G\left(v\right)=\int_{-\infty}^{+\infty}F\left(\frac{\beta}{2}v-Mu\right)f(u)du.$$

By lemma 1, G is the distribution function of a law with a symmetrical density, decreasing on $\mathbb{R}^{+}$. Moreover, we obtained

$$E_0\left[F\left(\frac{\beta}{2}\varepsilon_{t-1}^2-\varepsilon_t\right)/\underline{\varepsilon}_{t-1}\right]\geq G\left(\varepsilon_{t-1}^2\right)$$

and so

$$E_0\left[g_t/\underline{\varepsilon}_{t-1}\right]\geq\left[2G\left(\varepsilon_{t-1}^2\right)-1\right]^2.$$

From Hoeffding inequality (Hoeffding (1953)),

$$\begin{aligned}E_0\left[e^{-g_t}/\underline{\varepsilon}_{t-1}\right] &\leq e^{-E_0\left[g_t/\underline{\varepsilon}_{t-1}\right]+\frac{1}{2}\left[2G\left(\varepsilon_{t-1}^2\right)-1\right]^2}\\ &\leq e^{-\frac{1}{2}\left[2G\left(\varepsilon_{t-1}^2\right)-1\right]^2}.\end{aligned}$$

Then

$$\begin{aligned}P_0\left(\overline{A}_T\right) &\leq E_0\left\{\exp\left(-\sum_{t=2}^{T-1}g_t\right)\exp\left[-\frac{1}{2}\left(2G\left(\varepsilon_{T-1}^2\right)-1\right)^2\right]\right\}\\ &= E_0\left\{\exp\left(-\sum_{t=2}^{T-2}g_t\right)E_0\left[\exp\left(-g_{T-1}\right)\exp\left[-\frac{1}{2}\left(2G\left(\varepsilon_{T-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{T-2}\right]\right\}.\end{aligned}$$

From lemma 5 we have the following inequality, for every $t\in\mathbb{Z}$,

$$E_0\left\{\exp\left(-g_{t-1}\right)\exp\left[-\tfrac{1}{2}\left(2G\left(\varepsilon_{t-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{t-2}\right\}\leq$$

$$\leq E_0\left[\exp\left(-g_{t-1}\right)/\underline{\varepsilon}_{t-2}\right]E_0\left[\exp\left[-\frac{1}{2}\left(2G\left(\varepsilon_{t-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{t-2}\right]. \qquad (2)$$

In fact,

i) given $\underline{\varepsilon}_{t-2}$, $g_{t-1}=\left[2G\left(\varepsilon_{t-2}^2\right)-1\right]\left[2F\left(\frac{\beta}{2}\varepsilon_{t-2}^2-x_{t-1}\right)-1\right]$ has the form of the function $h(x)=c\left[2R\left(a-dx\right)-1\right]$ with c,a,d positives, and R the distribution function of a symmetrical density decreasing on $\mathbb{R}^{+}$, presented in lemma 2, as $x_{t-1}=\eta_{t-2}Z_{t-1}$ under H_0 and $c=2G\left(\varepsilon_{t-1}^2\right)-1>0$, $R=F$, $a=\frac{\beta}{2}\varepsilon_{t-1}^2\left(>0\right)$, and $d=\eta_{t-2}\left(>0\right)$.

ii) On the other hand, $\frac{1}{2}\left[2G\left(d^2x^2\right)-1\right]^2=\frac{1}{2}\left[G\left(d^2x^2\right)-G\left(-d^2x^2\right)\right]^2$ is a symmetrical function, increasing on $\mathbb{R}^{+}$, null in the origin and bounded. So, is the function considered on lemma 5.

As Z_{t-1} is independent of $\underline{\varepsilon}_{t-2}$, the inequality (2) takes the form

$$E_0\left[\exp\left(-h\left(Z_{t-1}\right)-g\left(Z_{t-1}\right)\right)\right]\leq E_0\left[\exp\left(-h\left(Z_{t-1}\right)\right)\right]E_0\left[\exp\left(-g\left(Z_{t-1}\right)\right)\right].$$

We can then write

$$\begin{aligned}
&P_0\left(\overline{A}_T\left(\varphi\right)\right)\\
&\leq E_0\left\{\exp\left(-\sum_{t=2}^{T-2}g_t\right)E_0\left[\exp\left(-g_{T-1}\right)/\underline{\varepsilon}_{T-2}\right]E_0\left[\exp\left[-\tfrac{1}{2}\left(2G\left(\varepsilon_{T-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{T-2}\right]\right\}\\
&=E_0\left\{E_0\left[\exp\left(-\sum_{t=2}^{T-1}g_t/\underline{\varepsilon}_{T-2}\right)E_0\left[\exp\left[-\tfrac{1}{2}\left(2G\left(\varepsilon_{T-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{T-2}\right]\right]\right\}\\
&\leq E_0\left\{E_0\left[\exp\left(-\sum_{t=2}^{T-1}g_t/\underline{\varepsilon}_{T-2}\right)E_0\left[\exp\left[-\tfrac{1}{2}\left(2G\left(m^2Z_{T-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{T-2}\right]\right]\right\}.
\end{aligned}$$

But $E_0\left[\exp\left[-\frac{1}{2}\left(2G\left(m^2Z_{T-1}^2\right)-1\right)^2\right]/\underline{\varepsilon}_{T-2}\right]$ is constant as Z_{t-1} is independent of $\underline{\varepsilon}_{T-2}, \forall t\in\mathbb{Z}$.

So,

$$P_0\left(\overline{A}_T\left(\varphi\right)\right)\leq E_0\left[\exp\left[-\frac{1}{2}\left(2G\left(m^2Z_{T-1}^2\right)-1\right)^2\right]\right]E_0\left[\exp\left(-\sum_{t=2}^{T-1}g_t/\underline{\varepsilon}_{T-2}\right)\right].$$

Using recursively the procedure conducive to

$$E_0\left[\exp\left(-\sum_{t=2}^{T}g_t\right)\right]\leq cE_0\left[\exp\left(-\sum_{t=2}^{T-1}g_t\right)\right]$$

we obtain

$$P_0\left(\overline{A}_T\left(\varphi\right)\right)\leq\left\{E_0\left[\exp\left[-\frac{1}{2}\left(2G\left(m^2Z^2\right)-1\right)^2\right]\right]\right\}^{T-1}$$

where Z is a random variable with the same law of Z_t.

Finally, we may state the following result:

Theorem. Let $X=(X_t,t\in\mathbb{Z})$ be a real stochastic process satisfying the model (1) subject to the general conditions $\mathcal{H}1$, $\mathcal{H}2$ and $\mathcal{H}3$.

If the error process satisfies condition $\mathcal{H}_5$ and the function G is defined by $G\left(v\right)=\int_{-\infty}^{+\infty}F\left(\frac{\beta}{2}v-Mu\right)f(u)du$ then the proposed decision rule satisfies

$$P_0\left(A_T\left(\varphi\right)\right)\geq 1-\left\{E_0\left[\exp\left[-\frac{1}{2}\left(2G\left(m^2Z^2\right)-1\right)^2\right]\right]\right\}^{T-1},\forall t\in\mathbb{N}.$$

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