EFFECTIVE DESCENT MORPHISMS IN CATEGORIES OF LAX ALGEBRAS

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Abstract. In this paper we investigate effective descent morphisms in categories of reflexive and transitive lax algebras. We show in particular that open and proper maps are effective descent, result that extends the corresponding results for the category of topological spaces and continuous maps.

Introduction

A morphism $p : E \to B$ in a category $\mathbf{C}$ with pullbacks is called *effective descent* if it allows a description of structures over the base B as algebras on structures over the extension E of B. Here the meaning of "structure over B" might depend on the category $\mathbf{C}$; however, in this paper we define it simply to be a morphism with codomain B. In that particular case $p : E \to B$ is effective descent if and only if the pullback functor $p^* : (\mathbf{C} \downarrow B) \to (\mathbf{C} \downarrow E)$ is monadic. In locally cartesian closed categories effective descent morphisms are easy to describe: they are exactly the regular epimorphisms. Such a characterization is far from being true in an arbitrary category; in general it can be quite a hard problem to find necessary and sufficient conditions for a morphism to be effective descent (see, for instance, [12] for the topological case). In order to obtain such conditions, it is often useful to embed our category into a category which has an easy description of effective descent morphisms, and then apply the pullback criterion of Theorem 1.1 below; this will be the basic technique of this paper.

Following a suggestion of George Janelidze, we investigate effective descent morphisms in categories of reflexive and transitive lax algebras $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ when $\mathbf{V}$ is a lattice, providing this way a unified treatment of descent theory for various categories. In particular, we characterize effective descent morphisms between quasi-metric spaces and, moreover, show that (suitably defined) open and proper maps are effective descent in $\mathrm{Alg}(\mathsf{T};\mathbf{V})$, encompassing the results for topological spaces obtained by Moerdijk [10, 11] and Sobral [13].

2000 *Mathematics Subject Classification.* 18D05, 18C20, 18D10.

Key words and phrases. monad, effective descent morphism, lax algebra.

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT and Centro de Matemática da Universidade de Aveiro/FCT.

1. Preliminaries

Throughout we will be working in the setting described in [5], restricted to the case of a non-degenerated lattice $\mathbf{V}$. More precisely,

- $\mathbf{V}$ is a complete symmetric monoidal closed (non-degenerated) lattice, with tensor product $\otimes$ and unit I, and
- $\mathsf{T} = (T, e, m)$ is a monad on **Set** lax-extended to $\mathrm{Mat}(\mathbf{V})$.

We recall that $\mathrm{Mat}(\mathbf{V})$ is the bicategory with sets as objects, with 1-cells $r : X \nrightarrow Y$ given by $X \times Y$ $\mathbf{V}$-matrices (that is, r is a map $X \times Y \to \mathbf{V}$), and with 2-cells determined by the componentwise lattice-order:

$$(r : X \nrightarrow Y) \leq (s : X \nrightarrow Y) \text{ if } \forall (x, y) \in X \times Y \quad r(x, y) \leq s(x, y) \text{ in } \mathbf{V}.$$

Composition of 1-cells is given by matrix multiplication, so that, for $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$,

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

$\mathrm{Mat}(\mathbf{V})$ has a natural pseudo-involution, given by matrix transposition:

$$\text{for } r : X \nrightarrow Y,\ r^\circ : Y \nrightarrow X \text{ is defined by } r^\circ(y, x) := r(x, y).$$

The category **Set** can be naturally embedded into $\mathrm{Mat}(\mathbf{V})$, assigning to each map $f : X \to Y$ the matrix with (x, y)-entry I in case $y = f(x)$ and 0 otherwise. By a lax-extension of the monad T into $\mathrm{Mat}(\mathbf{V})$ we mean a lax functor $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ that extends the endofunctor T of **Set** and such that the natural transformations e and m become op-lax; this means that:

- $Ts \cdot Tr \leq T(s \cdot r)$,
- $e_Y \cdot r \leq Tr \cdot e_X$ and
- $m_Y \cdot T^2 r \leq Tr \cdot m_X$

for all $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ in $\mathrm{Mat}(\mathbf{V})$. In addition we require that T preserves the pseudo-involution. As it is observed in [3], from this property it follows that T is functorial with respect to composition with maps on the right. We refer to this situation as our *basic setting.*

We say that a *diagram*

$$\begin{array}{ccc} W & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{1}$$

in **Set** *has the Beck-Chevalley Property* (BCP) if

$$g^\circ \cdot f = h \cdot k^\circ.$$

Beck-Chevalley Property of T : **Set** $\to$ **Set** means that, whenever diagram (1) is a pullback, its image by T has (BCP), i.e.

$$Tg^\circ \cdot Tf = Th \cdot Tk^\circ.$$

In order to describe (classes of) effective descent morphisms in $\mathrm{Alg}(\mathsf{T};\mathbf{V})$, we intend to apply the following

Theorem 1.1 (Janelidze and Tholen, [8])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be categories satisfying*

(a) $\mathbf{B}$ *has pullbacks and coequalizers and* $\mathbf{A}$ *is a full subcategory of* $\mathbf{B}$ *closed under pullbacks, and*

(b) *every regular epimorphism in* $\mathbf{B}$ *is an effective descent morphism.*

Then a morphism $p : E \to B$ *in* $\mathbf{A}$*, which is effective descent in* $\mathbf{B}$*, is an effective descent morphism in* $\mathbf{A}$ *if and only if*

$$E \times_B A \in \mathbf{A} \Rightarrow A \in \mathbf{A}$$

holds for every pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_1} & A \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

in $\mathbf{B}$.

In our situation $\mathbf{A}$ will be the category $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ of reflexive and transitive lax algebras and lax homomorphisms, and $\mathbf{B} = \mathrm{Alg}(T, e;\mathbf{V})$ the category of reflexive lax algebras and lax homomorphisms. That is, objects of $\mathbf{B}$ are pairs (X, a) where X is a set and $a : TX \nrightarrow X$ is a 1-cell in $\mathrm{Mat}(\mathbf{V})$ such that

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & \underset{\mathrm{id}_X}{\searrow} \quad \leq & \big\downarrow{\scriptstyle a} \\ & & X, \end{array}$$

and morphisms $(X, a) \to (Y, b)$ are maps $f : X \to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\big\downarrow & \leq & \big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

The category $\mathbf{A}$ is the full subcategory of $\mathbf{B}$ whose objects (X, a) satisfy in addition

$$\begin{array}{ccc} TX & \xleftarrow{Ta} & T^2X \\ {\scriptstyle a}\big\downarrow & \leq & \big\downarrow{\scriptstyle m_X} \\ X & \xleftarrow[a]{} & TX. \end{array}$$

In order to apply Theorem 1.1 we have to analyse its hypotheses:

(a) is obviously fulfilled: both $\mathrm{Alg}(T, e;\mathbf{V})$ and $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ are complete categories and, moreover, $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ is a reflective subcategory of $\mathrm{Alg}(T, e;\mathbf{V})$ (see [3] for details).

(b) holds if, for instance, $\mathrm{Alg}(T,e;\mathbf{V})$ is locally cartesian closed. It is shown in [4] that local cartesian closedness of $\mathrm{Alg}(T,e;\mathbf{V})$ is guaranteed by
   - $\mathbf{V}$ *is an Heyting algebra*[1] and
   - *the functor* $T:\mathbf{Set}\to\mathbf{Set}$ *has the Beck-Chevalley Property.*

We remark that Theorem 1.1 implies

Corollary 1.2. *Assume that in our basic situation* $\mathbf{V}$ *is an Heyting algebra and* $T:\mathbf{Set}\to\mathbf{Set}$ *has (BCP), and let* $\mathcal{E}$ *be a class of morphisms in* $\mathrm{Alg}(T,e;\mathbf{V})$*. Then* $\mathcal{E}\cap\mathrm{Alg}(\mathsf{T};\mathbf{V})$ *is a class of effective descent morphisms in* $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ *provided that*

(1) *each* f *in* $\mathcal{E}\cap\mathrm{Alg}(\mathsf{T};\mathbf{V})$ *is a regular epimorphism in* $\mathrm{Alg}(T,e;\mathbf{V})$*;*
(2) $\mathcal{E}$ *is stable under pullbacks, and*
(3) $\mathcal{E}$*-morphisms preserve transitivity; that is, with* $f:(X,a)\to(Y,b)$ *in* $\mathcal{E}$ *and* (X,a) *transitive, also* (Y,b) *is transitive.*

Finally, we recall that regular epimorphisms in $\mathrm{Alg}(T,e;\mathbf{V})$ were described in [3] as those morphisms $f:(X,a)\to(Y,b)$ such that

$$\forall\mathfrak{y}\in TY\ \forall y\in Y\quad b(\mathfrak{y},y)=f\cdot a\cdot(Tf)^\circ(\mathfrak{y},y)=\bigvee_{\substack{\mathfrak{x}\in Tf^{-1}(\mathfrak{y})\\ x\in f^{-1}(y)}}a(\mathfrak{x},x).$$

2. The Identity monad

Our first aim is to study effective descent morphisms in categories of the form $\mathrm{Alg}(\mathsf{Id};\mathbf{V})$, for Id the identity monad; that is, in categories of $\mathbf{V}$-enriched categories (see [9]). Note that the identity functor has obviously (BCP); hence we only need to assume that, in our basic setting, $\mathbf{V}$ is a Heyting algebra. We are going to show that for every effective descent morphism $f:(X,a)\to(Y,b)$ in $\mathrm{Alg}(\mathsf{Id};\mathbf{V})$,

$$\forall y_2,y_1,y_0\in Y\qquad b(y_2,y_1)\otimes b(y_1,y_0)=\bigvee_{\substack{x_i\in f^{-1}(y_i)\\ i=0,1,2}}a(x_2,x_1)\otimes a(x_1,x_0).\qquad(*)$$

Moreover, we will establish conditions under which this equality is also sufficient for a morphism in $\mathrm{Alg}(\mathsf{Id};\mathbf{V})$ to be effective descent. Throughout the text, for simplicity, we will omit "$i=\ldots$" whenever it is clear from the context which indexing set is meant.

Following [12], a surjective morphism in $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$ is called a **-quotient map* if it satisfies condition (*).

Lemma 2.1. *Every effective descent morphism* $f:(X,a)\to(Y,b)$ *in* $\mathrm{Alg}(\mathsf{Id};\mathbf{V})$ *is a regular epimorphism in* $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$*.*

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[1] We point out that a complete lattice is locally cartesian closed if and only if it is cartesian closed.

Proof. Let $f : (X, a) \to (Y, b)$ be effective descent in $\mathrm{Alg}(\mathsf{Id}; \mathbf{V})$. Recall that, being effective descent, f is necessarily a pullback stable regular epimorphism in $\mathrm{Alg}(\mathsf{Id}; \mathbf{V})$. Let $y_1, y_0 \in Y$ be given. Since f is a lax homomorphism, we have

$$\alpha := \bigvee_{x_i \in f^{-1}(y_i)} a(x_1, x_0) \leq b(y_1, y_0) =: \beta.$$

We define reflexive and transitive structures b_α and b_β on $2 = \{0, 1\}$ as follows:

$$b_\alpha(1, 0) = \alpha,\ b_\beta(1, 0) = \beta,\ b_\alpha(1, 1) = b_\beta(1, 1) = I = b_\alpha(0, 0) = b_\beta(0, 0).$$

For $g : 2 \to Y$ with $g(i) = y_i$ $(i = 0, 1)$, consider the pullback

$$\begin{array}{ccc} (X', a') & \xrightarrow{f'} & (2, b_\beta) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Y, b) \end{array}$$

in $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$. Since $g' : (X', a') \to (X, a)$ is a lax homomorphism it holds

$$\bigvee_{x'_i : f'(x'_i) = i} a'(x'_1, x'_0) \leq \bigvee_{x_i : f(x_i) = y_i} a(x_1, x_0) = \alpha.$$

Therefore the underlying map of f' defines a lax homomorphism $f'' : (X', a') \to (2, b_\alpha)$. But, as a pullback of f, $f' : (X', a') \to (2, b_\beta)$ is a regular epimorphism, and so $\mathrm{id}_2 : (2, b_\beta) \to (2, b_\alpha)$ is a lax homomorphism which implies that $\beta \leq \alpha$. Hence f is a regular epimorphism in $\mathrm{Alg}(\mathrm{Id}, \mathrm{id}; \mathbf{V})$ as claimed. ☐

Proposition 2.2. *Every effective descent morphism in* $\mathrm{Alg}(\mathsf{Id}; \mathbf{V})$ *is a *-quotient.*

Proof. Let $f : (X, a) \to (Y, b)$ be effective descent in $\mathrm{Alg}(\mathsf{Id}; \mathbf{V})$. From the lemma above we know that f is a regular epimorphism and hence effective descent in $\mathrm{Alg}(\mathrm{Id}, \mathrm{id}; \mathbf{V})$. Given elements $y_2, y_1 \in Y$, it holds (with $y_0 := y_1$)

$$\begin{aligned} b(y_2, y_1) \otimes b(y_1, y_1) &= b(y_2, y_1) \\ &= \bigvee_{x_i \in f^{-1}(y_i)} a(x_2, x_1) \\ &= \bigvee_{x_i \in f^{-1}(y_i)} a(x_2, x_1) \otimes a(x_1, x_1) \\ &\leq \bigvee_{x_i \in f^{-1}(y_i)} a(x_2, x_1) \otimes a(x_1, x_0), \end{aligned}$$

where in the first equality "$\leq$" follows from transitivity and "$\geq$" from reflexivity of b. In a similar way we obtain

$$b(y_1, y_1) \otimes b(y_1, y_0) \leq \bigvee_{x_i \in f^{-1}(y_i)} a(x_2, x_1) \otimes a(x_1, x_0)$$

for any $y_1, y_0 \in Y$ and $y_2 := y_1$.
Assume now that there exist three elements $y_2, y_1, y_0 \in Y$ such that

$$\alpha := \bigvee_{x_i \in f^{-1}(y_i)} a(x_2, x_1) \otimes a(x_1, x_0) < b(y_2, y_1) \otimes b(y_1, y_0) \le b(y_2, y_0).$$

We define the following reflexive and non-transitive structure b_0 on Y:

- $\forall\, y \in Y \quad b_0(y, y) := I$,
- $b_0(y_2, y_1) := b(y_2, y_1)$, $b_0(y_1, y_0) := b(y_1, y_0)$, $b_0(y_2, y_0) := \alpha$ and
- $b_0(y, y') := 0$ otherwise.

Then the identity map $i_0 : (Y, b_0) \to (Y, b)$ is a lax homomorphism. Regarding Theorem 1.1, in the pullback in $\mathrm{Alg}(\mathrm{Id}, \mathrm{id}; \mathbf{V})$

$$\begin{array}{ccc} (X, a_0) & \xrightarrow{f} & (Y, b_0) \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle i_0} \\ (X, a) & \xrightarrow[f]{} & (Y, b) \end{array}$$

(X, a_0) is not transitive. Hence there exist elements $x_2, x_1, x_0 \in X$ such that

$$a_0(x_2, x_1) \otimes a_0(x_1, x_0) \not\le a_0(x_2, x_0),$$

which is only possible if $x_i \in f^{-1}(y_i)$ $(i = 0, 1, 2)$. Since

$$a_0(x_2, x_0) = a(x_2, x_0) \wedge b_0(y_2, y_0) = a(x_2, x_0) \wedge \alpha$$

and

$$a_0(x_2, x_1) \otimes a_0(x_1, x_0) \le a(x_2, x_1) \otimes a(x_1, x_0),$$

we have

$$a(x_2, x_1) \otimes a(x_1, x_0) \not\le a(x_2, x_0) \wedge \alpha.$$

From the transitivity of a we obtain $a(x_2, x_1) \otimes a(x_1, x_0) \not\le \alpha$, a contradiction. □

Lemma 2.3. *Let $f : (X, a) \to (Y, b)$ be a *-quotient map in* $\mathrm{Alg}(\mathrm{Id}, \mathrm{id}; \mathbf{V})$ *and assume that I is terminal in* $\mathbf{V}$ *or b is transitive. Then f is a regular epimorphism.*

Proof. Obviously, any *-quotient map in $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$ must be surjective. Given y_1, y_0 in Y:

$$\begin{aligned} b(y_1,y_0) &= b(y_1,y_1)\otimes b(y_1,y_0) \\ &= \bigvee_{x_i\in f^{-1}(y_i)} a(x_2,x_1)\otimes a(x_1,x_0) \\ &\le \bigvee_{x_2\in f^{-1}(y_2)} \bigvee_{x_i\in f^{-1}(y_i)} I\otimes a(x_1,x_0) \\ &\le \bigvee_{x_i\in f^{-1}(y_i)} a(x_1,x_0). \end{aligned}$$

□

Lemma 2.4. *Every *-quotient map preserves transitivity.*

Proof. Given a *-quotient map $f:(X,a)\to(Y,b)$ in $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$ with a transitive and $y_2,y_1,y_0\in Y$:

$$\begin{aligned} b(y_2,y_1)\otimes b(y_1,y_0) &= \bigvee_{x_i\in f^{-1}(y_i)} a(x_2,x_1)\otimes a(x_1,x_0) \\ &\le \bigvee_{x_i\in f^{-1}(y_i)} a(x_2,x_0) \\ &\le b(y_2,y_0). \end{aligned}$$

□

Lemma 2.5. *If* $\otimes=\wedge$ *then the class of *-quotient maps is stable under pullbacks in* $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$.

Proof. Let

$$\begin{array}{ccc} (X\times_Z Y,d) & \xrightarrow{\pi_f} & (Y,b) \\ {\scriptstyle\pi_g}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Z,c) \end{array}$$

be a pullback in $\mathrm{Alg}(\mathrm{Id},\mathrm{id};\mathbf{V})$ with f a *-quotient map. For $y_2,y_1,y_0\in Y$, we have

$$\begin{aligned} b(y_2,y_1)\wedge b(y_1,y_0) &\le c(g(y_2),g(y_1))\wedge c(g(y_1),g(y_0)) \\ &= \bigvee_{x_i:f(x_i)=g(y_i)} a(x_2,x_1)\wedge a(x_1,x_0) \end{aligned}$$

and therefore

$$\begin{aligned} b(y_2,y_1)\wedge b(y_1,y_0) &= \left(\bigvee_{x_i:f(x_i)=g(y_i)} a(x_2,x_1)\wedge a(x_1,x_0)\right)\wedge(b(y_2,y_1)\wedge b(y_1,y_0))\\ &= \bigvee_{x_i:f(x_i)=g(y_i)} a(x_2,x_1)\wedge b(y_2,y_1)\wedge a(x_1,x_0)\wedge b(y_1,y_0) \qquad (\diamond)\\ &= \bigvee_{(x_i,y_i)\in\pi_f^{-1}(y_i)} d((x_2,y_2),(x_1,y_1))\wedge d((x_1,y_1),(x_0,y_0)). \end{aligned}$$

□

Remark 2.6. The only place where we make use of the hypothesis $\otimes=\wedge$ is $(\diamond)$. Hence it would have been enough to assume

$$(\alpha_1\wedge\beta_1)\otimes(\alpha_2\wedge\beta_2)=(\alpha_1\otimes\alpha_2)\wedge(\beta_1\otimes\beta_2)$$

for all $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbf{V}$; however, a tensor $\otimes$ with this property must be equal to $\wedge$.

So far we have seen that, in case $\otimes=\wedge$, the class of *-quotient maps satisfies the hypotheses of Corollary 1.2 and we obtain

Theorem 2.7. *If $\mathbf{V}$ is a complete Heyting algebra and $\otimes=\wedge$, then a morphim $f:(X,a)\to(Y,b)$ in $\mathrm{Alg}(\mathsf{Id};\mathbf{V})$ is effective descent if and only if*

$$\forall y_2,y_1,y_0\in Y \qquad b(y_2,y_1)\otimes b(y_1,y_0)=\bigvee_{x_i\in f^{-1}(y_i)} a(x_2,x_1)\otimes a(x_1,x_0).$$

One important example beyond the scope of the theorem above is $\mathbf{V}=[0,\infty]$ with the order given by "greater or equal" and the monoidal structure given by addition, where we obtain the category **QMet** of quasi-metric spaces and non-expansive maps as $\mathrm{Alg}(\mathsf{Id};[0,\infty])$ (see [9]). Nevertheless, effective descent morphisms in this category can still be characterized as exactly the *-quotient maps, as we will show below. Observe that a non-expansive map $f:(X,a)\to(Y,b)$ in **QMet** is a *-quotient map if and only if

$$\forall y_2,y_1,y_0\in Y \qquad b(y_2,y_1)+b(y_1,y_0)=\inf_{x_i\in f^{-1}(y_i)} a(x_2,x_1)+a(x_1,x_0).$$

In order to apply Corollary 1.2, the only missing property is the pullback stability of *-quotient maps.

Lemma 2.8. *Let $f:(X,a)\to(Y,b)$ be a non-expansive map in $\mathrm{Alg}(\mathrm{Id},\mathrm{id};[0,\infty])$. For all $y_2,y_1,y_0\in Y$ with $b(y_2,y_1)\neq\infty\neq b(y_1,y_0)$, the following conditions are equivalent:*

(1) $b(y_2,y_1)+b(y_1,y_0)=\inf_{x_i\in f^{-1}(y_i)} a(x_2,x_1)+a(x_1,x_0)$.

(2) $\forall\varepsilon>0\ \forall i=0,1,2\ \ \exists x_i\in f^{-1}(y_i)\ :\ \begin{cases} a(x_2,x_1)\le b(y_2,y_1)+\varepsilon \text{ and}\\ a(x_1,x_0)\le b(y_1,y_0)+\varepsilon.\end{cases}$

Proof. Obviously, condition (2) implies (1). Assume now that (1) holds and let $\varepsilon > 0$. Hence there exist $x_i \in f^{-1}(y_i)$ $(i = 0, 1, 2)$ such that

$$b(y_2, y_1) + b(y_1, y_0) + \varepsilon \geq a(x_2, x_1) + a(x_1, x_0).$$

Therefore,

$$a(x_2, x_1) \leq b(y_2, y_1) + (b(y_1, y_0) - a(x_1, x_0)) + \varepsilon \leq b(y_2, y_1) + \varepsilon$$

and

$$a(x_1, x_0) \leq b(y_1, y_0) + (b(y_2, y_1) - a(x_2, x_1)) + \varepsilon \leq b(y_1, y_0) + \varepsilon.$$

□

Lemma 2.9. *The class of *-quotient maps in* Alg(Id, id; [0, ∞]) *is stable under pullbacks.*

Proof. Let

$$\begin{array}{ccc} (X \times_Z Y, d) & \xrightarrow{\pi_f} & (Y, b) \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Z, c) \end{array}$$

be a pullback in Alg(Id, id; [0, ∞]) where f is *-quotient and let $y_2, y_1, y_0 \in Y$. The equality of condition (*) is obviously satisfied if one of the distances $b(y_2, y_1)$ and $b(y_1, y_0)$ is infinite, so we assume $b(y_2, y_1) \neq \infty \neq b(y_1, y_0)$. Let $\varepsilon > 0$. Since f is *-quotient, by the lemma above there exist $x_i \in f^{-1}(y_i)$ $(i = 0, 1, 2)$ such that

$$a(x_2, x_1) \leq c(g(y_2), g(y_1)) + \varepsilon \leq b(y_2, y_1) + \varepsilon$$

and

$$a(x_1, x_0) \leq c(g(y_1), g(y_0)) + \varepsilon \leq b(y_1, y_0) + \varepsilon.$$

This implies

$$d((x_2, y_2), (x_1, y_1)) = \max\{a(x_2, x_1), b(y_2, y_1)\} \leq b(y_2, y_1) + \varepsilon$$

and

$$d((x_1, y_1), (x_0, y_0)) = \max\{a(x_1, x_0), b(y_1, y_0)\} \leq b(y_1, y_0) + \varepsilon.$$

□

Theorem 2.10. *A morphism* $f : (X, a) \to (Y, b)$ *in* **QMet** *is effective descent if and only if*

$$\forall y_2, y_1, y_0 \in Y \qquad b(y_2, y_1) + b(y_1, y_0) = \inf_{x_i \in f^{-1}(y_i)} a(x_2, x_1) + a(x_1, x_0).$$

Remark 2.11. Recall that a quasi-metric $a : X \times X \to [0, \infty]$ is a metric if a is symmetric, a does not admit the distance ∞ and different points have non-zero distance. It is not too hard to see that we can also apply Theorem 1.1 to $\mathbf{B} = \mathrm{Alg}(\mathrm{Id}, \mathrm{id}; [0, \infty])$ and $\mathbf{A} = \mathbf{Met}$, the category of metric spaces and non-expansive maps. Moreover, a *-quotient map $f : (X, a) \to (Y, b)$ carries the additional properties of a metric from (X, a) to (Y, b). Therefore in **Met** we obtain the same characterization of effective descent maps as in **QMet**:

Corollary 2.12. *A morphism $f : (X, a) \to (Y, b)$ in* **Met** *is effective descent if and only if*

$$\forall y_2, y_1, y_0 \in Y \qquad b(y_2, y_1) + b(y_1, y_0) = \inf_{x_i \in f^{-1}(y_i)} a(x_2, x_1) + a(x_1, x_0).$$

3. Open and proper surjections are effective descent

In this section we turn to the case of an arbitrary monad, aiming to prove that open and proper surjections are effective descent in $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$. Before doing this, we must of course define what open and proper means in our context. Looking first to topological spaces, we recall that a continuous map $f : X \to Y$ is proper if and only if, for each ultrafilter $\mathfrak{x}$ on X and each $y \in Y$ with $f(\mathfrak{x}) \to y$, there exists $x \in X$ such that $f(x) = y$ and $\mathfrak{x} \to x$; dually, f is open if and only if, for each $x \in X$ and each $\mathfrak{y} \in Y$, there exists an ultrafilter $\mathfrak{x}$ on X such that $f(\mathfrak{x}) = \mathfrak{y}$ and $\mathfrak{x} \to x$ ([1, 6], see also [2]).

$$\begin{array}{ccc} X & & \mathfrak{x} & \dashrightarrow & x \\ {\scriptstyle f}\downarrow & \text{proper:} & \downarrow & & \vdots \\ Y & & f(\mathfrak{x}) & \longrightarrow & y \end{array} \qquad\qquad \begin{array}{ccc} X & & \mathfrak{x} & \dashrightarrow & x \\ {\scriptstyle f}\downarrow & \text{open:} & \vdots & & \downarrow \\ Y & & \mathfrak{y} & \longrightarrow & f(x) \end{array}$$

Sobral showed that every open map in **Top** is effective descent [13]. This result can also be deduced from Moerdijk's axioms [10] as well as the fact that every proper map is effective descent in **Top** [11].

The characterizations above lead naturally to

Definition 3.1. A lax homomorphism $f : (X, a) \to (Y, b)$ is called *proper (open)* if the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow\!\!\!\!\!\!+ & & \downarrow\!\!\!\!\!\!+{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes (satisfies the Beck-Chevalley Property).

Explicitely, a morphism $f : (X, a) \to (Y, b)$ is proper if

$$b(Tf(\mathfrak{x}), y) = \bigvee_{x \in f^{-1}(y)} a(\mathfrak{x}, x),$$

for each $\mathfrak{x} \in TX$ and $y \in Y$; and open if

$$b(\mathfrak{y}, f(x)) = \bigvee_{\mathfrak{x} \in Tf^{-1}(\mathfrak{y})} a(\mathfrak{x}, x),$$

for each $\mathfrak{y} \in TY$ and $x \in X$. From both equations above we obtain the equality

$$\forall \mathfrak{y} \in TY\ \forall y \in Y \quad b(\mathfrak{y}, y) = \bigvee_{\substack{\mathfrak{x} \in Tf^{-1}(\mathfrak{y}) \\ x \in f^{-1}(y)}} a(\mathfrak{x}, x)$$

in case f is surjective, hence open and proper surjections are regular epimorphisms in $\mathrm{Alg}(T, e; \mathbf{V})$.

Lemma 3.2. *Assume that V is a Heyting algebra. Then the class of proper morphisms is pullback stable. If, in addition, T has (BCP), then also the class of open morphisms is pullback stable.*

Proof. Let

$$\begin{array}{ccc} (X \times_Z Y, d) & \xrightarrow{\pi_f} & (Y, b) \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Z, c) \end{array}$$

be a pullback in $\mathrm{Alg}(T, e; \mathbf{V})$. Assume first that f is proper and let $y \in Y$ and $\mathfrak{w} \in T(X \times_Z Y)$. It holds

$$\begin{aligned} b(T\pi_f(\mathfrak{w}), y) &\leq c(T(g \cdot \pi_f)(\mathfrak{w}), g(y)) \\ &= c(Tf(T\pi_g(\mathfrak{w})), g(y)) \\ &= \bigvee_{x : f(x) = g(y)} a(T\pi_g(\mathfrak{w}), x) \qquad \text{(f is proper)} \end{aligned}$$

and therefore

$$\begin{aligned} \bigvee_{x : f(x) = g(y)} d(\mathfrak{w}, (x, y)) &= \bigvee_{x : f(x) = g(y)} a(T\pi_g(\mathfrak{w}), x) \wedge b(T\pi_f(\mathfrak{w}), y) \\ &= \left(\bigvee_{x : f(x) = g(y)} a(T\pi_g(\mathfrak{w}), x) \right) \wedge b(T\pi_f(\mathfrak{w}), y) \quad (\mathbf{V} \text{ is l.c.c.}) \\ &= b(T\pi_f(\mathfrak{w}), y). \end{aligned}$$

If f is open, we obtain analogously, for any $(x, y) \in X \times_Z Y$ and $\mathfrak{y} \in TY$,

$$b(\mathfrak{y}, y) \leq \bigvee_{\mathfrak{x} : Tf(\mathfrak{x}) = Tg(\mathfrak{y})} a(\mathfrak{x}, x)$$

and

$$\begin{aligned}
\bigvee_{\mathfrak{w}:T\pi_f(\mathfrak{w})=\mathfrak{y}} d(\mathfrak{w},(x,y)) &= \bigvee_{\mathfrak{w}:T\pi_f(\mathfrak{w})=\mathfrak{y}} a(T\pi_g(\mathfrak{w}),x)\wedge b(\mathfrak{y},y)\\
&= \left(\bigvee_{\mathfrak{w}:T\pi_f(\mathfrak{w})=\mathfrak{y}} a(T\pi_g(\mathfrak{w}),x)\right)\wedge b(\mathfrak{y},y) \qquad (\mathbf{V}\text{ is l.c.c.})\\
&= \left(\bigvee_{\mathfrak{x}:Tf(\mathfrak{x})=Tg(\mathfrak{y})} a(\mathfrak{x},x)\right)\wedge b(\mathfrak{y},y) \qquad (T\text{ has (BCP)})\\
&= b(\mathfrak{y},y).
\end{aligned}$$

□

Lemma 3.3. *Assume that $T(f\cdot r)=Tf\cdot Tr$ for any map f. Then*

(1) *with $f:(X,a)\to(Y,b)$ also $Tf:(TX,Ta)\to(TY,Tb)$ is proper (open), and*
(2) *proper and open surjections preserve transitivity.*

Proof. The first statement of the lemma is clear. Regarding the second statement, we only present the proof for a proper map $f:(X,a)\to(Y,b)$; the "open case" is similar. Let $\mathfrak{Y}\in T^2Y$, $\mathfrak{y}\in TY$ and $y\in Y$ be given. Since T^2f is surjective, there exists $\mathfrak{X}\in T^2X$ with $T^2f(\mathfrak{X})=\mathfrak{Y}$. Hence

$$\begin{aligned}
Tb(\mathfrak{Y},\mathfrak{y})\otimes b(\mathfrak{y},y) &= b(T^2f(\mathfrak{X}),\mathfrak{y})\otimes b(\mathfrak{y},y)\\
&= \bigvee_{\mathfrak{x}:Tf(\mathfrak{x})=\mathfrak{y}} Ta(\mathfrak{X},\mathfrak{x})\otimes b(Tf(\mathfrak{x}),y) \qquad (Tf\text{ is proper})\\
&= \bigvee_{\mathfrak{x}:Tf(\mathfrak{x})=\mathfrak{y}}\ \bigvee_{x:f(x)=y} Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x) \qquad (f\text{ is proper})\\
&\le \bigvee_{\mathfrak{x}:Tf(\mathfrak{x})=\mathfrak{y}}\ \bigvee_{x:f(x)=y} a(m_X(\mathfrak{X}),x) \qquad (a\text{ is transitive})\\
&\le b(m_Y(\mathfrak{Y}),y).
\end{aligned}$$

□

Recall that, since T preserves the pseudo-involution $\_^\circ$, T preserves composition with maps on the right. Hence the hypothesis of the lemma above implies that T is functorial regarding composition with maps (on the left and on the right), as well as regarding composition with map transposes. In particular, T has (BCP).

Combining our results we obtain

Theorem 3.4. *Assume that $\mathbf{V}$ is a complete Heyting algebra, equipped with a tensor product so that $\mathbf{V}$ becomes a symmetric-monoidal closed category. Let T be a monad in* **Set** *lax-extended to* Mat($\mathbf{V}$) *and such that*

$$T(r^\circ)=(Tr)^\circ \quad\textit{and}\quad T(f\cdot r)=Tf\cdot Tr$$

for every r and every map f. Then open and proper surjections are effective descent morphisms in $\mathrm{Alg}(\mathsf{T};\mathbf{V})$.

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