EXPONENTIAL RATES FOR KERNEL DENSITY ESTIMATION UNDER ASSOCIATION

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ABSTRACT. The estimation of density based on positive dependent samples has been studied recently with consistency and asymptotic normality results being obtained. In what concerns the characterization on decrease rates the results have been scarce. The article proves an exponential decrease rate for the kernel estimator of the density with an uniform version, over compact sets. The conditions assumed impose convenient decrease rates on the covariance structure of the sample. Some examples supposing exponential but also polynomial decrease rates on the covariances that fulfill our assumptions are presented in the last section.

1. Introduction

Estimation of the density of random variables has been a classical statistical problem. Results establishing the properties of the proposed estimators were, naturally, first derived based on independent data. This independence assumption was eventually replaced by some kind of control on the dependence structure of the sample upon which the estimation is carried. Typically, control of the dependence structure was achieved through some mixing conditions. There were various estimation methods proposed, among which we will be interested in the nonparametric kernel estimator. For this type of estimator and for strong mixing samples, the asymptotic properties are well established, including convergence rates. For an account of results and literature we refer the reader to Bosq [4]. The dependence structure that will be considered in this article is association, a concept introduced by Esary, Proschan and Walkup [12] that has attracted some attention from statisticians during recent years. The consistency of the kernel estimator under associated sampling was proved by Bagai and Prakasa Rao [2] which returned to the problem in Bagai and Prakasa Rao [3] proving a uniform consistency result. In these references no convergence rates were obtained. Independently, Roussas [15] also proved a consistency result for the kernel estimator under associated sampling but giving also a convergence rate characterization. Namely, in Roussas [15], it was proved a uniform polynomial convergence rate under some regularity conditions. An extension of the consistency of the estimator, without rates, was proved in Oliveira [13] without absolute continuity of the joint distributions. Estimation of distribution functions under associated sampling, a similar problem to the one described above, has been studied, using kernels, by Cai and Roussas [5, 6] and Azevedo and Oliveira [1], proving consistency and asymptotic normality, and, using histograms, by Henriques and Oliveira [9, 10], proving consistency, asymptotic normality and some convergence rates.

The main result in this article is an exponential rate for the pointwise convergence and also for the uniform convergence on compacts, with an extra condition on the kernel function for the later case. This study was motivated by similar results announced by Dewan and Prakasa Rao [8]. However, the method of proof used by these authors forced them to suppose a quite fast convergence rate on the covariances of the variables. Unfortunately, for associated variables this rate is far to

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fast, so it is, in fact, unattainable. So, the characterization of an exponential rate for associated sampling remained open. The method of proof used in this article is inspired by the blocking technique used in Ioannides and Roussas [11] and the approximation to independence as used in Dewan and Prakasa Rao [8]. In the final section we will present some examples of covariance structures that fulfill the assumptions used in this article. Note that, in previous articles dealing with exponential inequalities for associated variables (Ioannides and Roussas [11], Henriques and Oliveira [10]) the examples exhibited always assumed a geometric decrease rate on the covariance structure. Moreover, in Henriques and Oliveira [10] it was even showed that, if the covariances decreased only at a polynomial rate the assumptions made there could not be fulfilled. Here, for the kernel estimator of the density we will provide an example of polynomial decrease rate on the covariance structure that still verifies the conditions under which the general results hold. The possibility of providing such an example is due to the fact that the bandwidth selection appears on the exponential rate in a convenient way.

2. **Definitions and assumptions**

Let $X_1, X_2, \ldots$ be random variables with the same distribution as X for which there exists a density function f. Let K be a fixed probability density and h_n a sequence on nonnegative real numbers converging to zero. The kernel estimator of the density function f is, as usual, defined as

$$\widehat{f}_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right),$$

which is well known to be asymptotically unbiased, if there exists a bounded and continuous version of the density. Moreover, the convergence of $\mathbb{E}[\widehat{f}_n(x)]$ to $f(x)$ is, under these assumptions on f, uniform on compact sets.

The random variables $X_1, X_2, \ldots$ will be supposed associated, which means, as defined in Esary, Proschan and Walkup [12], that for every $n \in \mathbb{N}$ and $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ coordinatewise increasing

$$\mathrm{Cov}\Big(f(X_1, \ldots, X_n),\, g(X_1, \ldots, X_n)\Big) \geq 0,$$

whenever this covariance exists.

A technical problem arises when dealing with $\widehat{f}_n(x)$ for associated variables. In fact, association is only preserved under monotone transformations, which means that, in general, the variables $K\left(\frac{x-X_1}{h_n}\right), K\left(\frac{x-X_2}{h_n}\right), \ldots$ are not associated. This problem is resolved, as usual, by supposing the kernel K to be of bounded variation.

For easier future reference, we introduce now a set of assumptions.

(A1) $X_1, X_2, \ldots$ are strictly stationary and associated random variables with common bounded and continuous density function f; let $B_0 = \sup_{x \in \mathbb{R}} |f(x)|$;

(A2) The kernel function K is a probability density of bounded variation such that $\int K^2(u)\, du < \infty$; further, if $K = K_1 - K_2$ where K_1 and K_2 are nondecreasing functions, the derivatives K_1' and K_2' exist and are integrable.

Under (A1), we have that

$$\sup_{u,v \in \mathbb{R}} \left| F_{X_1,X_j}(u,v) - F_{X_1}(u) F_{X_1}(v) \right| \leq B_2 \,\mathrm{Cov}^{1/3}(X_1, X_j) \tag{2.1}$$

where F_{X_1,X_j} and F_{X_1} represent the distribution functions of (X_1, X_j) and X_1, respectively, and $B_2 = \max(4/\pi^2, 90B_0)$ (see Lemma 2.6 in Roussas [16] for details). This inequality provides an upper bound for the covariances between the variables $K_q\left(\frac{x-X_j}{h_n}\right)$, $q = 1, 2$, $j = 1, 2, \ldots$.

Lemma 2.1. *Suppose the variables $X_1, X_2, \dots$ satisfy* (A1) *and the kernel function satisfy* (A2). *Then*

$$\operatorname{Cov}\left(K_q\Big(\frac{x-X_1}{h_n}\Big), K_q\Big(\frac{x-X_j}{h_n}\Big)\right) \leq B_2 \operatorname{Cov}^{1/3}(X_1,X_j) \left(\int K_q'(u)\,du\right)^2, \quad q=1,2.$$

Proof. Just notice that

$$\operatorname{Cov}\left(K_q\Big(\frac{x-X_1}{h_n}\Big), K_q\Big(\frac{x-X_j}{h_n}\Big)\right) =$$

$$= \frac{1}{h_n^2}\int K_q'\Big(\frac{x-u}{h_n}\Big) K_q'\Big(\frac{x-v}{h_n}\Big) \left|F_{X_1,X_j}(u,v) - F_{X_1}(u)F_{X_1}(v)\right| \, dudv$$

and apply (2.1). □

The next result is the basis of the development used to establish the exponential inequality. It appears under the present form in Dewan and Parkasa Rao [8] and is a version for generating functions of Newman's [14] inequality for characteristic functions.

Lemma 2.2. *Let $X_1, \dots, X_n$ be associated random variables bounded by a constant M. Then, for every $\theta > 0$,*

$$\left| \mathbb{E}\left(e^{\theta\sum_{i=1}^n X_i}\right) - \prod_{i=1}^n \mathbb{E}\left(e^{\theta X_i}\right)\right| \leq \theta^2 e^{n\theta M} \sum_{1\leq i<j\leq n} \operatorname{Cov}(X_i,X_j).$$

This inequality for generating functions was used by Dewan and Prakasa Rao [8] to control the distance between the joint distribution of the variables and what one would find in case of independence. The way they manipulated the upper bound conducted them to the assumption $\frac{1}{n}\sum_{j=1}^n \operatorname{Cov}(X_1,X_j) = O(e^{-\theta n})$, $\theta > 3/2$. Now, if the random variables are associated, all the covariances $\operatorname{Cov}(X_1,X_n)$ are nonnegative so, at best, the sum defines a convergent series and the rate of convergence to zero of the expression considered by Dewan and Prakasa Rao [8] is n^{-1}. That is, the exponential decrease rate needed to control the difference between the joint distribution and the independent case is unattainable. This remark was at the origin of the present article which started as an effort to use the same type of independence approximation procedure. The control of this approximation is inspired in the blocking technique mentioned above, a quite different approach from what was used in Dewan and Prakasa Rao [8].

Before proceeding to some more precise notations connected with the method of proof, we quote a general lemma useful in course of proof.

Lemma 2.3 (Devroye [7])**.** *Let X be a centered random variable. If there exist $a, b \in \mathbb{R}$ such that* $\mathrm{P}(a \leq X \leq b) = 1$*, then, for every $\lambda > 0$,*

$$\mathbb{E}(e^{\lambda X}) \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right).$$

Now we introduce the notation that will be used in the sequel. Given (A2) define

$$\widehat{f}_{n,1}(x) = \frac{1}{nh_n}\sum_{j=1}^n K_1\Big(\frac{x-X_j}{h_n}\Big), \qquad \widehat{f}_{n,2}(x) = \frac{1}{nh_n}\sum_{j=1}^n K_2\Big(\frac{x-X_j}{h_n}\Big),$$

so that $\widehat{f}_n(x) = \widehat{f}_{n,1}(x) - \widehat{f}_{n,2}(x)$. For each $n \in \mathbb{N}$, $j = 1, \dots, n$, and $q = 1, 2$, let

$$T_{n,q,j} = \frac{1}{h_n}\left(K_q\Big(\frac{x-X_j}{h_n}\Big) - \mathbb{E}\, K_q\Big(\frac{x-X_j}{h_n}\Big)\right). \tag{2.2}$$

Note that these variables are associated if the $X_1, X_2, \dots$ are associated, as they are nonincreasing transformations of these variables.

Given a natural number $p < \frac{n}{2}$, let r be the greatest natural number less or equal to $\frac{n}{2p}$ and define, for each $j = 1, \ldots, 2r$, and $q = 1, 2$

$$Y_{n,q,j} = \sum_{l=(j-1)p+1}^{jp} T_{n,q,l}\,. \tag{2.3}$$

Note that, if the kernel K satisfies (A2), the functions K_1 and K_2 may be chosen bounded so that each variable $Y_{n,q,j}$ is bounded by $\frac{2p\|K_q\|_\infty}{h_n}$, where $\|\cdot\|_\infty$ represents the supremum norm.

Finally define, for $q = 1, 2$,

$$\begin{aligned} Z_{n,q}^{od} &= Y_{n,q,1} + Y_{n,q,3} + \cdots + Y_{n,q,2r-1}, \\ Z_{n,q}^{ev} &= Y_{n,q,2} + Y_{n,q,4} + \cdots + Y_{n,q,2r}. \end{aligned} \tag{2.4}$$

With these definitions, if $n = 2pr$ we have $\widehat{f}_{n,q}(x) - \mathbb{E}[\widehat{f}_{n,q}(x)] = \frac{1}{n}(Z_{n,q}^{ev} + Z_{n,q}^{od})$.

3. **Some preliminary results**

In this section we prove two preparatory lemmas that pave the way to the proof of the main result.

Lemma 3.1. *Let $X_1, X_2, \ldots$ be random variables and suppose that* (A2) *is satisfied. If $Y_{n,q,j}$, $q = 1, 2$, $j = 1, \ldots, 2r$ are defined by (2.3) then, for every $\lambda > 0$,*

$$\prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right) \leq \exp\left(\frac{\lambda^2 p \| K_q \|_\infty^2}{n h_n^2}\right), \quad q = 1, 2,$$

$$\prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j}}\right) \leq \exp\left(\frac{\lambda^2 p \| K_q \|_\infty^2}{n h_n^2}\right), \quad q = 1, 2.$$

Proof. As noted before, each variable $Y_{n,q,j}$ is bounded by $\frac{2p\|K_q\|_\infty}{h_n}$, so we may apply Lemma 2.3 to get the result. □

The next lemma provides the link towards the control of both terms $\mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right)$ and $\mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{ev}}\right)$.

Lemma 3.2. *Suppose* (A1) *and* (A2) *are satisfied. With the definitions made before, for every $\lambda > 0$,*

$$\left| \mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) - \prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right) \right| \leq \frac{\lambda^2}{2n} \exp\left(\frac{\lambda \| K_q \|_\infty}{h_n}\right) \sum_{j=p+2}^{(2r-1)p} \mathrm{Cov}(T_{n,q,1}, T_{n,q,j}), \quad q = 1, 2, \tag{3.1}$$

and analogously for the term corresponding to $Z_{n,q}^{ev}$.

Proof. As $Z_{n,q}^{od} = \sum_{j=1}^{r} Y_{n,q,2j-1}$, we may apply Lemma 2.2 to find

$$\left| \mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) - \prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right) \right| \leq \frac{\lambda^2}{n^2} \exp\left(\frac{\lambda}{n}\, r\, \frac{2p\| K_q \|_\infty}{h_n}\right) \sum_{1 \leq j < j' \leq r} \mathrm{Cov}(Y_{n,q,2j-1}, Y_{n,q,2j'-1}).$$

Now, as $2pr \le n$, the exponential is bounded above by $\exp\left(\frac{\lambda\|K_q\|_\infty}{h_n}\right)$. As for the term with the covariances, we first use the stationarity to rewrite it as

$$\sum_{1\le j<j'\le r} \operatorname{Cov}(Y_{n,q,2j-1}, Y_{n,q,2j'-1}) = \sum_{j=1}^{r-1} (r-j) \operatorname{Cov}(Y_{n,q,1}, Y_{n,q,2j+1}). \tag{3.2}$$

Using again the stationarity of the variables to develop the covariances in the right of this last expression, we find

$$\begin{aligned} &\operatorname{Cov}(Y_{n,q,1}, Y_{n,q,2j+1}) = \\ &\quad = \sum_{l=0}^{p-1} (p-l) \operatorname{Cov}(T_{n,q,1}, T_{n,q,2jp+l+1}) + \sum_{l=1}^{p-1} (p-l) \operatorname{Cov}(T_{n,q,l+1}, T_{n,q,2jp+1}) \le \\ &\quad \le p \sum_{l=(2j-1)p+2}^{(2j+1)p} \operatorname{Cov}(T_{n,q,1}, T_{n,q,l}). \end{aligned}$$

Inserting this into (3.2), we find

$$\sum_{1\le j<j'\le r} \operatorname{Cov}(Y_{n,q,2j-1}, Y_{n,q,2j'-1}) \le \sum_{j=1}^{r-1} rp \sum_{l=(2j-1)p+2}^{(2j+1)p} \operatorname{Cov}(T_{n,q,1}, T_{n,q,l}) \le rp \sum_{l=p+2}^{(2r-1)p} \operatorname{Cov}(T_{n,q,1}, T_{n,q,l})$$

due to the nonnegativity of all the covariances, as the variables $T_{n,q,j}$, $j \ge 1$, are associated. To conclude the proof, just remind again that $\frac{rp}{n} \le \frac{1}{2}$. □

4. Main results

We are now in position to prove the exponential rate for the estimator. On the sequel we let the integers p and r referred above depend on n, thus obtaining sequences p_n and r_n such that $\frac{n}{2p_n r_n} \longrightarrow 1$. We will need to choose these sequences conveniently to prove our results.

Lemma 4.1. *Suppose* (A1) *and* (A2) *are satisfied. Further assume that*

$$\frac{nh_n^4}{p_n^2} \exp\left(\frac{nh_n}{p_n}\right) \sum_{j=p_n+2}^{\infty} \operatorname{Cov}(T_{n,q,1}, T_{n,q,j}) \le C_0 < \infty. \tag{4.1}$$

Then, for every $\varepsilon \in \Big(0, \min(\| K_1 \|_\infty, \| K_1 \|_\infty^2, \| K_2 \|_\infty, \| K_2 \|_\infty^2)\Big)$,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,q}^{od}\right| > \varepsilon\right) \le (1+C_0)\exp\left(-\frac{\varepsilon^2 nh_n^2}{4p_n\| K_q \|_\infty}\right), \quad q=1,2,$$

and analogously for $Z_{n,q}^{ev}$.

Proof. Applying Markov's inequality we find that, for every $\lambda > 0$,

$$\begin{aligned} &\mathrm{P}\left(\frac{1}{n}\left|Z_{n,q}^{od}\right| > \varepsilon\right) \le \\ &\quad \le \mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) e^{-\lambda\varepsilon} \le \left(\left|\mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) - \prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right)\right| + \prod_{j=1}^{r} \left|\mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right)\right|\right) e^{-\lambda\varepsilon} \le \\ &\quad \le \exp\left(\frac{\lambda^2 p_n \| K_q \|_\infty^2}{nh_n^2} - \lambda\varepsilon\right) + e^{-\lambda\varepsilon}\left|\mathbb{E}\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) - \prod_{j=1}^{r} \mathbb{E}\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right)\right|, \end{aligned}$$

using Lemma 3.1. An optimization of the exponent in the first term of this last upper bound leads to $\lambda = \frac{\varepsilon n h_n^2}{2p_n \|K_q\|_\infty^2}$. From Lemma 3.2 it follows, using (4.1), that

$$\left| I\!E\left(e^{\frac{\lambda}{n} Z_{n,q}^{od}}\right) - \prod_{j=1}^{r} I\!E\left(e^{\frac{\lambda}{n} Y_{n,q,2j-1}}\right) \right| \leq C_0.$$

That is, we have derived the upper bound,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,q}^{od}\right| > \varepsilon\right) \leq \exp\left(-\frac{\varepsilon^2 n h_n^2}{4p_n \| K_q \|_\infty^2}\right) + C_0 \exp\left(-\frac{\varepsilon^2 n h_n^2}{2p_n \| K_q \|_\infty^2}\right) \leq (1+C_0) \exp\left(-\frac{\varepsilon^2 n h_n^2}{4p_n \| K_q \|_\infty^2}\right).$$

□

In order to state the main result we have to deal with the terms in $\widehat{f}_{n,q}(x)$ that are not in $\frac{1}{n}(Z_{n,q}^{od} + Z_{n,q}^{ev})$. But these are, as expected, negligible. For easier reference, define $R_{n,q} = \widehat{f}_{n,q}(x) - I\!E[\widehat{f}_{n,q}(x)] - \frac{1}{n}(Z_{n,q}^{od} + Z_{n,q}^{ev})$, $q = 1, 2$.

Lemma 4.2. *Suppose* (A1), (A2) *are satisfied and*

$$\frac{n h_n^2}{p_n} \longrightarrow +\infty. \tag{4.2}$$

Then, for n large enough and every $\varepsilon > 0$, $\mathrm{P}(|R_{n,q}| > \varepsilon) = 0$*, $q = 1, 2$.*

Proof. Write $R_{n,q} = \frac{1}{n}\sum_{j=2r_n p_n+1}^{n} T_{n,q,j}$. As the functions K_q, $q = 1, 2$, are bounded, it follows that $R_{n,q}$ is bounded by $2\frac{n-2p_n r_n}{n h_n} \| K_q \|_\infty \leq \frac{4p_n \|K_q\|_\infty}{n h_n}$ according to the construction of the sequences p_n and r_n. As $h_n \longrightarrow 0$ it follows from (4.2) that eventually this upper bound becomes less than ε, so the lemma follows. □

Now, the main work has been completed. It remains to collect the various partial results in order to obtain the exponential rate for the kernel estimator centered at its mean.

Theorem 4.3. *Suppose* (A1), (A2) *and (4.2) are satisfied and that*

$$\frac{n h_n^2}{p_n^2} \exp\left(\frac{n h_n}{p_n}\right) \sum_{j=p_n+2}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_j) \leq C_1 < \infty. \tag{4.3}$$

Then, for every $\varepsilon \in \left(0, 6\min(\| K_1 \|_\infty, \| K_1 \|_\infty^2, \| K_2 \|_\infty, \| K_2 \|_\infty^2)\right)$ *and n large enough,*

$$\mathrm{P}\left(\left|\widehat{f}_n(x) - I\!E[\widehat{f}_n(x)]\right| > \varepsilon\right) \leq D \exp\left(-\frac{\varepsilon^2 n h_n^2}{144 C p_n}\right), \tag{4.4}$$

where $C = \max(\| K_1 \|_\infty, \| K_2 \|_\infty)$, $D = 2\left[2 + B_2 C_1 \left(\left(\int K_1'(u)du\right)^2 + \left(\int K_1'(u)du\right)^2\right)\right]$.

Proof. We have

$$\begin{aligned}
\widehat{f}_n(x) - I\!E[\widehat{f}_n(x)] &= \\
&= \left(\widehat{f}_{n,1}(x) - I\!E[\widehat{f}_{n,1}(x)]\right) + \left(\widehat{f}_{n,2}(x) - I\!E[\widehat{f}_{n,2}(x)]\right) = \\
&= \frac{1}{n}\left(Z_{n,1}^{od} + Z_{n,1}^{ev}\right) + \frac{1}{n}\left(Z_{n,2}^{od} + Z_{n,2}^{ev}\right) + R_{n,1} + R_{n,2}.
\end{aligned}$$

According to Lemma 4.2, for n large enough, $\mathrm{P}\left(|R_{n,1}|>\frac{\varepsilon}{6}\right)=\mathrm{P}\left(|R_{n,2}|>\frac{\varepsilon}{6}\right)=0$, so we need to concentrate only on the first terms. In order to apply Lemma 4.1 we must check that (4.1) is verified. For this purpose notice that, according to Lemma 2.1,

$$\mathrm{Cov}(T_{n,q,1},T_{n,q,j})=\frac{1}{h_n^2}\,\mathrm{Cov}\left(K_q\Big(\frac{x-X_1}{h_n}\Big),K_q\Big(\frac{x-X_j}{h_n}\Big)\right)\leq\frac{1}{h_n^2}\,B_2\left(\int K_q'(u)du\right)^2\mathrm{Cov}^{1/3}(X_1,X_j).$$

As K_1' and K_2' are assumed integrable, it follows that (4.3) implies (4.1). Applying then Lemma 4.1, we find, for $q=1,2$,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,q}^{od}\right|>\frac{\varepsilon}{6}\right)\leq\left(1+B_2C_1\left(\int K_q'(u)du\right)^2\right)\exp\left(-\frac{\varepsilon^2nh_n^2}{144p_n\|\,K_q\,\|_\infty}\right),$$

and analogously for $Z_{n,q}^{ev}$, from where the result follows. □

Strengthening the conditions on the kernel we may use a decomposition of a compact interval to derive an uniform exponential rate for the centered estimator. Let, on what follows, $[a,b]$ be a fixed interval and decompose $[a,b]=\cup_{j=1}^{s_n}[z_{n,j}-t_n,z_{n,j}+t_n]$ into s_n intervals of length $2t_n$. Then, obviously,

$$\begin{aligned}\sup_{x\in[a,b]}&\left|\widehat{f}_n(x)-\mathbb{E}[\widehat{f}_n(x)]\right|\leq\\&\leq\max_{1\leq j\leq s_n}\left|\widehat{f}_n(z_{n,j})-\mathbb{E}[\widehat{f}_n(z_{n,j})]\right|+\\&+\max_{1\leq j\leq s_n}\sup_{x\in[z_{n,j}-t_n,z_{n,j}+t_n]}\left|\widehat{f}_n(x)-\widehat{f}_n(z_{n,j})-\mathbb{E}[\widehat{f}_n(x)-\widehat{f}_n(z_{n,j})]\right|.\end{aligned}$$

If we suppose the kernel K to be Lipschitzian, it follows that there exists a constant $\theta>0$ such that

$$\left|\widehat{f}_n(x)-\widehat{f}_n(z_{n,j})-\mathbb{E}[\widehat{f}_n(x)-\widehat{f}_n(z_{n,j})]\right|\leq 2\,\frac{\theta\,|x-z_{n,j}|}{h_n^2}\leq\frac{2\theta t_n}{h_n^2}.$$

A correct choice of the sequence of radii will verify $\frac{t_n}{h_n^2}\longrightarrow 0$. Supposing this condition satisfied it follows then that

$$\begin{aligned}\mathrm{P}&\left(\sup_{x\in[a,b]}\left|\widehat{f}_n(x)-\mathbb{E}[\widehat{f}_n(x)]\right|>\varepsilon\right)\\&\leq\mathrm{P}\left(\max_{1\leq j\leq s_n}\left|\widehat{f}_n(z_{n,j})-\mathbb{E}[\widehat{f}_n(z_{n,j})]\right|>\varepsilon-\frac{2\theta t_n}{h_n^2}\right)\leq\\&\leq s_n\max_{1\leq j\leq s_n}\mathrm{P}\left(\left|\widehat{f}_n(z_{n,j})-\mathbb{E}[\widehat{f}_n(z_{n,j})]\right|>\frac{\varepsilon}{2}\right).\end{aligned}$$

Thus, under the assumptions of Theorem 4.3, as the upper bound derived is independent of x, for K Lipschitzian and sequences s_n,t_n such that $b-a=2s_nt_n$ and $\frac{t_n}{h_n^2}\longrightarrow 0$, it follows

$$\mathrm{P}\left(\sup_{x\in[a,b]}\left|\widehat{f}_n(x)-\mathbb{E}[\widehat{f}_n(x)]\right|>\varepsilon\right)\leq Ds_n\exp\left(-\frac{\varepsilon^2nh_n^2}{576Cp_n}\right),\tag{4.5}$$

where C and D are defined in Theorem 4.3.

Theorem 4.4. *Suppose* (A1), (A2), *(4.3) are satisfied, the kernel K is Lipschtzian and $p_n=nh_n^3\longrightarrow+\infty$. Then, for every $\varepsilon\in\left(0,6\min(\|\,K_1\,\|_\infty,\|\,K_1\,\|_\infty^2,\|\,K_2\,\|_\infty,\|\,K_2\,\|_\infty^2)\right)$, n large enough* *and each interval* $[a,b]$*,*

$$\mathrm{P}\left(\sup_{x\in[a,b]}\left|\widehat{f}_n(x)-I\!E[\widehat{f}_n(x)]\right|>\varepsilon\right)\leq D\frac{b-a}{2}\,h_n^{-3}\exp\left(-\frac{\varepsilon^2}{576Ch_n}\right). \tag{4.6}$$

where C *and* D *are defined as in Theorem 4.3.*

Proof. Choose $t_n=h_n^3$ to which corresponds $s_n=\frac{b-a}{2h_n^3}$ and use (4.5). It is easy to check that (4.2) holds and that this choice for the sequence t_n verifies all the assumptions made. □

Note that there are other possible choices for the sequences. Making these explicit would mean some more precise expressions for h_n and p_n. These will be referred in the next section.

5. **Some examples**

In the preceding we derived some sufficient conditions in order to prove an exponential rate for the kernel estimator for the density. We will now verify that these conditions are not void by constructing examples of covariance structures and choices of the sequences h_n and p_n (which determines r_n) that verify the two assumptions that involve these quantities: (4.2) and (4.3).

Let us first suppose that the covariances $\mathrm{Cov}(X_1,X_n)$ decrease geometrically. This is the situation where examples have been given for the validity of some other exponential inequalities (see Ioannides and Roussas [11] and Henriques and Oliveira [10]). Suppose that $\mathrm{Cov}(X_1,X_n)=\rho^n$ for some $\rho\in(0,1)$. Then

$$\sum_{j=p_n+2}^{\infty}\mathrm{Cov}^{1/3}(X_1,X_j)=\frac{\rho^{\frac{p_n+2}{3}}}{1-\rho^{1/3}},$$

so that (4.3) becomes

$$\frac{nh_n^2}{p_n^2(1-\rho^{1/3})}\exp\left(\frac{nh_n}{p_n}+\frac{p_n+2}{3}\log\rho\right)\leq C_1. \tag{5.1}$$

Theorem 5.1. *Suppose* (A1), (A2), *(4.2) are satisfied and* $\mathrm{Cov}(X_1,X_n)=\rho^n$ *for some* $\rho\in(0,1)$*. If* $\sup_{n\in I\!N}\frac{nh_n}{p_n^2}\leq M<\infty$ *and* $\rho\in(0,e^{-3M})$*, then inequality (4.4) holds.*

Proof. The exponent in (5.1) should be bounded, which is equivalent to $\log\rho\leq\frac{3A}{p_n}-3\frac{nh_n}{p_n^2}$, for some $A\in I\!R$. As $p_n\longrightarrow+\infty$ and $\frac{nh_n}{p_n^2}$ is bounded, it is enough that $\log\rho\leq-3M$. Finally, note that $\frac{nh_n^2}{p_n^2}\leq Mh_n\longrightarrow0$, so it is bounded. □

It is straightforward to check that for some usual forms for h_n and p_n the assumptions of the previous theorem are achieved.

Corollary 5.2. *The assumptions of Theorem 5.1 are satisfied in each of the following situations*

a) $p_n=n^\delta$, $h_n=n^{-\beta}$, $\beta\in(0,1/3)$, $\delta\in(\frac{1-\beta}{2},1-2\beta)$*;*

b) $p_n=h_n^{-\delta}$, $\delta>1$, $nh_n^{2+\delta}\longrightarrow+\infty$ *and* $\sup_{n\in I\!N}nh_n^{1+2\delta}<+\infty$*;*

c) $p_n=nh_n^\alpha$, $\alpha>2$, $\inf_{n\in I\!N}nh_n^{2\alpha-1}>0$*; in this case, if* $h_n=n^{-\beta}$ *we must require* $\beta\in(0,\frac{1}{2\alpha-1})$.

Notice that the last choice mentioned in Corollary 5.2 includes the case where a uniform exponential rate holds, as mentioned in Theorem 4.4. The first two choices indicated in this Corollary also enable the proof of an uniform exponential bound as it is easily verified from (4.5), the starting point for the result in Theorem 4.4. The exponential rates derived in each case mentioned in Corollary 5.2 are of order $n^{3\beta}\exp\left(-n^{1-(2\beta+\delta)}\right)$, $h_n^{-2\delta}\exp\left(-nh_n^{2+\delta}\right)$, taking $t_n=h_n^\delta$, and $h_n^{-3}\exp\left(-h_n^{-1}\right)$, respectively.

Suppose now that the covariances decrease at a polynomial rate, that is, $\mathrm{Cov}(X_1, X_n) = n^{-a}$, for some $a > 3$. Then

$$\sum_{j=p_n+2}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_j) \sim (p_n + 2)^{\frac{3-a}{3}}.$$

Inserting this into (4.3) we find a term that behaves like

$$\frac{nh_n^2}{p_n^2} \exp\left(\frac{nh_n}{p_n} - a_1 \log p_n\right),$$

where $a_1 = \frac{a-3}{3}$ as $\frac{\log(p_n+2)}{\log p_n} \sim 1$. If this term is to be bounded, we may have, for some $A > 0$,

$$h_n \le \frac{Ap_n}{n} + a_1 \frac{p_n}{n} \log p_n. \tag{5.2}$$

Theorem 5.3. *Suppose* (A1), (A2), *(4.2) are satisfied,* $\mathrm{Cov}(X_1, X_n) = n^{-a}$, *for some* $a > 3$, $\frac{p_n}{n} \longrightarrow 0$, *and* $\sup_{n\in\mathbb{N}} \frac{nh_n}{p_n \log p_n} \le M < \infty$. *Then, if* $a_1 > M$, *(4.4) holds.*

Proof. From the assumptions made it follows easily that $h_n \le M\frac{p_n}{n} \log p_n$, so (5.2) holds. On the other hand, under these assumptions $\frac{nh_n^2}{p_n^2} \le \frac{Ah_n}{p_n} + a_1 \frac{h_n}{p_n} \log p_n \longrightarrow 0$, so $\frac{nh_n^2}{p_n^2}$ is bounded. □

As usual, we may be more precise about the choices of the sequences. No comment about the verification will be included as it is a quite straightforward task.

Corollary 5.4. *Suppose* (A1), (A2) *are satisfied and* $\mathrm{Cov}(X_1, X_n) = n^{-a}$, *for some* $a > 3$. *Suppose that one of the following conditions are satisfied*

a) $p_n = n^\delta$, $\delta \in (0, 1)$, $n^{1-\delta} h_n^2 \longrightarrow +\infty$ *and* $a_1 > 0$ *is such that* $h_n < a_1 \delta n^{\delta-1} \log n$*;*
b) $p_n = nh_n^3$, $h_n > \frac{1}{(M \log n)^{1/2}}$ *and* $a_1 > M$.

Then (4.4) holds.

In this case of polynomial decrease of the covariances we may obtain an uniform exponential rate using the second choice of the sequences mentioned in the previous corollary. In fact, this choice verifies the assumptions of Theorem 4.4, so (4.6) holds.

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