On projective $\mathcal{E}$-generators and premonadic functors

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Abstract

It is shown that a well behaved category is premonadic over **Set** if it has a projective $\mathcal{E}$-generator, for a convenient class $\mathcal{E}$ of epimorphisms. A variety of examples is provided.

Introduction

A functor $U : \mathbf{A} \to \mathbf{Set}$ is said to be *premonadic* provided that it is a right-adjoint and $\mathbf{A}$ is equivalent to a full subcategory of the corresponding category of Eilenberg-Moore algebras. If $\mathbf{A}$ is a category with coequalizers, then the premonadicity of U means that $\mathbf{A}$ is equivalent to a reflective subcategory of the category of algebras.

Of course every monadic functor is premonadic. Concerning monadicity, it is known that a category $\mathbf{A}$ is monadic over **Set** if and only if $\mathbf{A}$ is an exact category, has a regular projective generator P and arbitrary copowers of P. The aim of this paper is to analyze what part of the above monadicity characterizing properties is relevant for a characterization of premonadicity over **Set**. In particular, we show that an enough well behaved category is premonadic over **Set** if it has a projective $\mathcal{E}$-generator P, for a convenient "coreflective" class $\mathcal{E}$ of $\mathbf{A}$-epimorphisms. A kind of partial converse of this is also achieved. A significant role is played by the stabilization of $\mathcal{E}$ which is shown to be often of the form $\mathrm{Proj}(P)$, for some object P, that is, it consists of exactly those morphisms to which P is projective.

1 Projective $\mathcal{E}$-generators

Definition 1.1 *A class $\mathcal{E}$ of epimorphisms of a category $\mathbf{A}$ (closed under the composition with isomorphisms) is said to be a coreflective class whenever, for each $B \in \mathbf{A}$, the embedding $\mathcal{E}(B) \to B \downarrow \mathbf{A}$, where $\mathcal{E}(B)$ denotes de subcategory of $B \downarrow \mathbf{A}$ whose objects are $\mathcal{E}$-morphisms, is a left adjoint; that is, each $\mathbf{A}$-morphism f has a factorization $m \cdot e$ with $e \in \mathcal{E}$, and such that if $m' \cdot e'$ is another such a factorization of f then there is a (unique) morphism t fulfilling the equalities $t \cdot e' = e$ and $m \cdot t = m'$. We say that $m \cdot e$ is the $\mathcal{E}$-factorization of f. (cf. [8], [13] and [4].)*

Remark 1.2 *If $\mathbf{A}$ has pushouts and $\mathcal{E}$ is a pushout-stable coreflective class, the following facts are easy consequences of the above definition:*

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1. *The "local coreflections" $\mathcal{E}(B) \to B \downarrow \mathbf{A}$ determine a "global coreflection" from $Mor(\mathbf{A})$ to $\mathcal{E}$, where $\mathcal{E}$ is regarded as a full subcategory of $Mor(\mathbf{A})$. (That is, using terminology of [8], $\mathbf{A}$ has a locally orthogonal $\mathcal{E}$-factorization.)*

2. *$\mathcal{E}$ determines a factorization system for morphisms if and only if it is closed under composition.*

We add another property which will play a role throughout:

Lemma 1.3 *If $\mathcal{E}$ is a pushout-stable coreflective class in a category with pushouts, then the following conditions are equivalent:*

(i) Any split epimorphism m which is part of an $\mathcal{E}$-factorization $m \cdot e$ is an isomorphism.

(ii) $\mathcal{E}$ is closed under the composition with split epimorphisms from the left.

(iii) $\mathcal{E}$ is closed under the composition with split epimorphisms (from the left and from the right).

Proof.. (i) $\Rightarrow$ (iii): Let $r \cdot s$ be defined with $s \in \mathcal{E}$ and r a split epi, and let $m \cdot e$ be the $\mathcal{E}$-factorization of $r \cdot s$. Then there is t such that $m \cdot t = r$ and, since r is a split epi, so is m, thus m is an isomorphism and $r \cdot s$ belongs to $\mathcal{E}$.

Take $r \cdot s$ with $r \in \mathcal{E}$ and s a split epi, let u be such that $s \cdot u = 1$, and let $m \cdot e$ be the $\mathcal{E}$-factorization of $r \cdot s$. From 1.2.1 and the equality $1 \cdot r = m \cdot e \cdot u$, we get a morphism t such that $mt = 1$ and $tr = eu$. Condition (i) ensures that m is an iso, and $r \cdot s \in \mathcal{E}$.

(ii) $\Rightarrow$ (i): Let $m \cdot e$ be the $\mathcal{E}$-factorization of some f with m a split epi. Then $f \in \mathcal{E}$ and so f has an $\mathcal{E}$-factorization of the form $1 \cdot f$. Therefore, m is iso. $\square$

Definition 1.4 *(cf. [3]) An object P is an $\mathcal{E}$-generator of the category $\mathbf{A}$ with copowers of P if, for each $A \in \mathbf{A}$, the canonical morphism ε_A from the coproduct $\coprod_{\hom(P,A)} P$ to A belongs to $\mathcal{E}$.*

Assumptions 1.5 *From now on we assume that the category $\mathbf{A}$ has pullbacks and pushouts, and $\mathcal{E}$ is a pushout-stable coreflective class contained in $Epi(\mathbf{A})$ which is closed under the composition with split epimorphisms.*

A morphism is said to be *$\mathcal{E}$-stable* if its pullback along any morphism belongs to $\mathcal{E}$. The *stabilization of $\mathcal{E}$* is the class of all $\mathcal{E}$-stable morphisms; it is denoted by $\mathrm{St}(\mathcal{E})$ and it is clearly contained in $\mathcal{E}$.(cf. [4])

Lemma 1.6 *$\mathcal{E}$ and $St(\mathcal{E})$ are strongly right-cancellable.*

Proof.. Given $r \cdot s \in \mathcal{E}$, we want to show that $r \in \mathcal{E}$. Let $m \cdot e$ be the $\mathcal{E}$-factorization of r. Then the equality $1 \cdot (r \cdot s) = (m \cdot e) \cdot s$ determines, by 1.2.1, the existence of a morphism t such that $mt = 1$ and $t \cdot (r \cdot s) = e \cdot s$. Since m is a split epimorphism and $m \cdot e$ is an $\mathcal{E}$-factorization, then, by 1.3, m is an isomorphism; so $r \in \mathcal{E}$. The right-cancellability of $\mathrm{St}(\mathcal{E})$ follows easily from the right-cancellability of $\mathcal{E}$. $\square$

Definition 1.7 *An object P is said to be a projective $\mathcal{E}$-generator provided that it is an $\mathcal{E}$-generator which is St($\mathcal{E}$)-projective, that is, for each $f \in$ St($\mathcal{E}$), the function hom(P, f) is surjective.*

Notations 1.8 *$\mathbb{C}(P)$ denotes the colimit-closure of P in $\mathbf{A}$, that is, the smallest full subcategory of $\mathbf{A}$ containing P and closed under all colimits in $\mathbf{A}$.*

For A an $\mathbf{A}$-object, Proj(A) denotes the class of all $\mathbf{A}$-morphisms f such that A is f-projective. It is easily seen that Proj(A) is pullback-stable.

Assumptions 1.9 *In the following, besides the assumptions stated in 1.5, we also assume that $\mathbf{A}$ is cocomplete.*

We are going to make use of the following lemmas.

Lemma 1.10 *If P is a projective $\mathcal{E}$-generator, then St($\mathcal{E}$) = Proj(P).*

Proof. By the assumption on P, one of the inclusions is trivial. It remains to show that if P is $(f : X \to Y)$-projective then $f \in \mathrm{St}(\mathcal{E})$, i.e., any pullback of f along any morphism belongs to $\mathcal{E}$. Let $(\bar{f} : W \to Z, \bar{g} : W \to X)$ be the pullback of (f, g). Since P is f-projective, any coproduct of P is also f-projective; thus, from the pullback-stability of any class Proj(A), $\bar{f} \in \mathrm{Proj}(\coprod_{\mathrm{hom}(P,Z)} P)$. Let s be a morphism fulfilling $\bar{f} \cdot s = \varepsilon_Z$ and let $m \cdot e$ be the $\mathcal{E}$-factorization of $\bar{f}$. We get the equality $1_Z \cdot \varepsilon_Z = m \cdot e \cdot s$, which, since $\varepsilon_Z \in \mathcal{E}$, implies the existence of some t such that $t \cdot \varepsilon_Z = e \cdot s$ and $m \cdot t = 1_Z$. Then, from 1.3, and in view of 1.5, m is an iso and $\bar{f} \in \mathcal{E}$. □

Lemma 1.11 *If $\mathbf{B}$ is an $\mathcal{E}$-coreflective subcategory*[1] *of $\mathbf{A}$, then it is (St($\mathcal{E}$) $\cap$ Mono($\mathbf{A}$))-coreflective.*

Proof. Let $s_X : S(X) \to X$ be a coreflection of X in $\mathbf{B}$ (with S the coreflector functor). By hypothesis, s_X lies in $\mathcal{E}$; in order to show that it belongs to $\mathrm{St}(\mathcal{E})$, let $(\bar{s} : W \to Y, \bar{g} : W \to S(X))$ be the pullback of (s_X, g), for some morphism g. To conclude that $\bar{s} \in \mathcal{E}$, let $m \cdot e$ be the $\mathcal{E}$-factorization of $\bar{s}$. Since $s_X \cdot S(g) = g \cdot s_Y$, there is a unique morphism v such that $\bar{s} \cdot v = s_Y$ and $\bar{g} \cdot v = S(g)$. Then we have $1_Y \cdot s_Y = m \cdot e \cdot v$; since $m \cdot e$ is an $\mathcal{E}$-factorization and $s_Y \in \mathcal{E}$, there is a morphism t such $m \cdot t = 1_Y$, and, taking into account 1.3 and 1.5, we get that $\bar{s} \in \mathcal{E}$. It remains to show that s_X is a monomorphism: Let $a, b : Y \to S(X)$ be such that $s_X \cdot a = s_X \cdot b$; then the equality $s_X \cdot a \cdot s_Y = s_X \cdot b \cdot s_Y$ implies that $a \cdot s_Y = b \cdot s_Y$, and thus, since $\mathcal{E} \subseteq \mathrm{Epi}(\mathbf{A})$, $a = b$. □

Let us recall that, for $\mathcal{E}$ a class containing all isomorphisms and closed under the composition with isomorphisms, $\mathbf{A}$ is said to be $\mathcal{E}$-cocomplete provided that every pushout of any $\mathcal{E}$-morphism exists and belongs to $\mathcal{E}$ and any family of morphisms of $\mathcal{E}$ has a cointersection belonging to $\mathcal{E}$. The $\mathcal{E}$-cocompletness of $\mathbf{A}$ implies that $\mathcal{E} \subseteq \mathrm{Epi}(\mathbf{A})$ and that $\mathcal{E}$ is a coreflective class (see [12]).

Proposition 1.12 *Let $\mathbf{A}$ be $\mathcal{E}$-cocomplete and let P be a projective $\mathcal{E}$-generator of $\mathbf{A}$. Then $\mathbb{C}(P)$ consists of all $\mathbf{A}$-objects A such that the function hom(f, A) is bijective for each $f \in$ St($\mathcal{E}$) $\cap$ Mono($\mathbf{A}$). Furthermore $\mathbb{C}(P)$ is the smallest $\mathcal{E}$-correflective subcategory of $\mathbf{A}$.*

[1]By subcategory we mean a full subcategory.

Proof. Since $\mathbf{A}$ is $\mathcal{E}$-cocomplete, P is an $\mathcal{E}$-generator and $\mathbb{C}(P)$ is closed under colimits in $\mathbf{A}$, a slight generalization of the dual of the Special Adjoint Functor Theorem ensures that $\mathbb{C}(P)$ is coreflective in $\mathbf{A}$. Moreover it is well-known that, then, it coincides with the co-orthogonal closure of P in $\mathbf{A}$, that is, $\mathbb{C}(P)$ consists of all those $\mathbf{A}$-objects A such that for any morphism f, $\hom(A, f)$ is a bijection whenever $\hom(P, f)$ is so. Consequently, in order to conclude that $\mathbb{C}(P)$ is the subcategory of $\mathbf{A}$ of those objects A such that the function $\hom(f, A)$ is bijective for each $f \in \mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$, it suffices to show that

$$\mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A}) = \{f \in \mathrm{Mor}(\mathbf{A}) \,|\, \hom(P, f) \text{ is an iso}\}. \tag{1}$$

From Lemma 1.10, the inclusion "$\subseteq$" is trivial. Let $\hom(P, f)$ be an iso. Then, again by 1.10, f belongs to $\mathrm{St}(\mathcal{E})$. In order to conclude that f is a mono, let $a, b : S \to X$ be morphisms such that $fa = fb$. Then for any $t : P \to S$, we have $fat = fbt$, what implies, since $\hom(P, f)$ is an iso, that $at = bt$. Thus $a = b$, because P is a generator.

The coreflections into $\mathbb{C}(P)$ belong to $\mathrm{St}(\mathcal{E}) \subseteq \mathcal{E}$, because P is r_X-projective for each coreflection r_X. In order to show that $\mathbb{C}(P)$ is the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$, let $\mathbf{B}$ be another $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$. Let $X \in \mathbb{C}(P)$ and let $s_X : S(X) \to X$ be the coreflection of X in $\mathbf{B}$. By Lemma 1.11, s_X belongs to $\mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$. Therefore, using (1), we get a morphism $t : X \to S(X)$ such that $s_X \cdot t = 1_X$, and thus s_X is an isomorphism. □

Remark 1.13 *In the above proof, the only role of the $\mathcal{E}$-cocompleteness of $\mathbf{A}$ is to assure that $\mathbb{C}(P)$ is refletive in $\mathbf{A}$. So, in Proposition 1.12, we can replace "$\mathbf{A}$ is $\mathcal{E}$-cocomplete" by "$\mathbb{C}(P)$ is coreflective".*

We have just conclude that the existence of a projective $\mathcal{E}$-generator P gives a characterization of the stabilization of $\mathcal{E}$, $\mathrm{St}(\mathcal{E})$ =Proj(P), and, in case $\mathbf{A}$ is $\mathcal{E}$-cocomplete, it guarantees that P "generates" the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$. In the following we give several examples of this situation.

Examples 1.14

1. *For any monadic category $\mathbf{A}$ over $\mathbf{Set}$ and $\mathcal{E} = RegEpi(\mathbf{A})$, let $P = F\{*\}$, where F is the corresponding left adjoint. Then P is an $\mathcal{E}$-generator such that $St(\mathcal{E}) = \mathcal{E} = Proj(P)$, and $\mathbf{A} = \mathbb{C}(\mathbf{P})$.*

   We point out that, under the conditions of Proposition 1.12, $\mathbf{A}$ and $\mathbb{C}(P)$ are identical whenever $St(\mathcal{E}) \cap Mono(\mathbf{A}) = Iso(\mathbf{A})$, since $\{$ coreflections of $\mathbf{A}$ in $\mathbb{C}(P)$ $\} \subseteq St(\mathcal{E}) \cap Mono(\mathbf{A})$.

2. *For $\mathbf{A} = \mathbf{Cat}$, $\mathcal{E} = ExtrEpi(\mathbf{A})$, and $\mathbf{2} = \{\mathbf{0} \to \mathbf{1}\}$ the category given by the ordered set 2, we have that $\mathbf{2}$ is an $\mathcal{E}$-generator, $St(\mathcal{E}) = Proj(\mathbf{2})$, and $\mathcal{E}$ does not coincides with its stabilization (cf. [6]). The equality $\mathbf{A} = \mathbb{C}(\mathbf{P})$ also occurs.*

3. *Let $\mathbf{PreOrd}$ be the category whose objects are pairs (X, R_X) with X a set and R_X a preorder (i.e., a reflexive and transitive binary relation) on X, and whose morphisms are preorder-preserving maps. For $\mathcal{E} = \{$regular epimorphisms$\} = \{$extremal epimorphisms$\}$, the object P consisting of the set $\{0, 1\}$ with $(0, 1)$ as*

the only non-trivial relation pair, is a projective $\mathcal{E}$-generator, in particular St($\mathcal{E}$) = Proj(P), although St($\mathcal{E}$) $\neq \mathcal{E}$. The subcategory $\mathbb{C}(P)$ coincides with **PreOrd**.

In the following examples, it occurs the dual situation. That is, P is an injective $\mathcal{M}$-cogenerator of **A**, *with* **A** *and $\mathcal{M}$ fulfilling the dual conditions of 1.9. A morphism belongs to St($\mathcal{M}$) whenever its pushout along any morphism lies in $\mathcal{M}$. It holds the equality St($\mathcal{M}$)=Inj(P), and the limit closure of P in* **A**, *$\mathbb{L}(P)$, is the smallest $\mathcal{M}$-reflective subcategory of* **A**.

4. *In the category* **Set**, *the pushout-stable class $\mathcal{M}$ of monomorphisms coincides with Inj(P) for P the $\mathcal{M}$-cogenerator set $\{0,1\}$, and $\mathbb{L}(P) =$* **Set**.

5. *For* **A** *the category* **Top** *of topological spaces and continuous maps, let $\mathcal{M} = \{$embeddings$\}$, and let P be the topological space $\{0,1,2\}$ whose only non trivial open is $\{0\}$. Then P is an $\mathcal{M}$-cogenerator and Inj(P)=St($\mathcal{M}$)=$\mathcal{M}$. The subcategory $\mathbb{L}(P)$ is the whole category* **Top**.

6. *For the category* $\mathbf{Top_0}$ *of T_0-topological spaces, $\mathcal{M} = \{$embeddings$\}$, the Sierpiński space S is an $\mathcal{M}$-cogenerator which fulfils Inj(S)=St($\mathcal{M}$)=$\mathcal{M}$. Here $\mathbb{L}(S)$ is the subcategory of sober spaces.*

7. *If* **A** *is the subcategory of* **Top** *of all 0-dimensional spaces, and $\mathcal{M}$ consists of all embeddings, then $\mathcal{M} \neq$ St($\mathcal{M}$), but again St($\mathcal{M}$) = Inj($\mathcal{P}$), where P is the space $\{0,1,2\}$ whose topology has as only non trivial opens $\{0\}$ and $\{1,2\}$. (The morphisms of St($\mathcal{M}$) are just those embeddings $m : X \to Y$ such that for each clopen set G of X there is some clopen H in Y such that $G = m^{-1}(H)$ (cf. [10]).) We have that $\mathbb{L}(P) =$* **A** *(by 1.5.7 of [10]). A similar situation happens for the category of 0-dimensional Hausdorff spaces and $\mathcal{M}$ the class of embeddings, which again is not pushout-stable, if we choose P as being the space $\{0,1\}$ with the discrete topology.*

8. *For* **Tych** *the category of Tychonoff spaces, the class $\mathcal{M}$ of embeddings is not stable under pushouts, and St($\mathcal{M}$) = Inj($\mathcal{I}$), where I is the unit interval, with the euclidean topology. The St($\mathcal{M}$)-morphisms are just the C^*-embeddings (cf. [10]) and $\mathbb{L}(I)$ is the subcategory of compact Hausdorff spaces.*

9. *For* $\mathbf{Vec_K}$ *the category of linear spaces and linear maps over the field $\mathbb{K}$, the class $\mathcal{M}$ of all monomorphisms is stable under pushouts and $\mathbb{K}$ is an injective $\mathcal{M}$-cogenerator.*

10. *In the category* **Ab** *of abelian groups and homomorphisms of group, let $\mathcal{M}$ be the class of all monomorphisms. Then $P = \coprod_{n\in\mathbf{N}_0} \mathbb{Q}/n\mathbb{Z}$ (where $\mathbb{Q}$ and $\mathbb{Z}$ are the groups of rational numbers and of integers numbers, respectively) is an $\mathcal{M}$-cogenerator*[2]*, and it holds that St($\mathcal{M}$) = $\mathcal{M}$ = Inj($\mathcal{P}$). The limit closure $\mathbb{L}(P)$ is*

[2] For each $X \in$ **Ab**, and each element $x \neq 0$ of X, let $f : \mathbb{Z}/n\mathbb{Z} \to X$ be the monomorphism determined by $f(1 + n\mathbb{Z}) = x$, where $n = 0$ if x is torsion-free, otherwise n is the minimum positive integer such that $nx = 0$. Then there is some $\bar{i} : X \to \mathbb{Q}/n\mathbb{Z}$ such that $\bar{i} \cdot f = i$, where i is the inclusion of $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Q}/n\mathbb{Z}$, and $\bar{i}(x) \neq 0$. So the quotients $\mathbb{Q}/n\mathbb{Z}$ with $n = 0, 1, 2, ...$, distinguish points in any abelian group, from what follows that the divisible abelian group $P = \coprod_{n\in\mathbf{N}_0} \mathbb{Q}/n\mathbb{Z}$ is an $\mathcal{M}$-cogenerator of **Ab**.

the whole category $\mathbf{Ab}$*, taking into account the dual of 1.12 and that in* $\mathbf{Ab}$ *any monomorphism is regular.*

11. *In the category of torsion-free abelian groups and homomorphisms of group, for* $\mathcal{M}$ *the class of all monomorphisms, the group of rational numbers* $\mathbb{Q}$ *is an* $\mathcal{M}$*-cogenerator and* $St(\mathcal{M}) = \mathcal{M} = Inj(\mathbb{Q})$. $\mathbb{L}(\mathbb{Q})$ *is the subcategory of torsion-free divisible abelian groups.*

One question arises: When is the stabilization of a corefletive class $\mathcal{E}$ of the form $\mathrm{Proj}(P)$, or, at least, when is it of the form $\mathrm{Proj}(\mathbf{B})$ for some subcategory $\mathbf{B}$ of $\mathbf{A}$? The following proposition gives a partial answer.

Let us recall that, if $\mathcal{F}$ is a class of morphisms of a category $\mathbf{A}$ containing all isomorphisms and closed under composition with isomorphisms, $\mathbf{A}$ is said to have *enough* $\mathcal{F}$*-projectives* provided that, for each $A \in \mathbf{A}$, there is an $\mathcal{F}$-morphism $f : B \to A$ with an $\mathcal{F}$-projective domain. An $\mathcal{F}$-morphism $f : B \to A$ is said to be $\mathcal{F}$*-coessential* whenever any composition $f \cdot g$ belongs to $\mathcal{F}$ only if $g \in \mathcal{F}$. We say that the category $\mathbf{A}$ has $\mathcal{F}$*-projective hulls* if, for each $\mathbf{A}$-object A there is some $\mathcal{F}$-coessential morphism $f : B \to A$ where B is $\mathcal{F}$-projective.

Proposition 1.15 *1. If* $\mathbf{A}$ *has enough* $St(\mathcal{E})$*-projectives, then* $St(\mathcal{E}) = Proj(\mathbf{B})$ *for some subcategory* $\mathbf{B}$ *of* $\mathbf{A}$*.*

2. If $St(\mathcal{E}) = Proj(\mathbf{B})$ *for some* $\mathcal{E}$*-coreflective subcategory* $\mathbf{B}$ *of* $\mathbf{A}$*, then* $\mathbf{B}$ *has* $St(\mathcal{E})$*-projective hulls.*

Proof. 1. Let $\mathbf{B}$ consist of all objects of $\mathbf{A}$ which are $\mathrm{St}(\mathcal{E})$-projective; clearly $\mathrm{St}(\mathcal{E}) \subseteq \mathrm{Proj}(\mathbf{B})$. In order to show the converse inclusion, let $f : A \to B$ belong to $\mathrm{Proj}(\mathbf{B})$, and let $(\bar{f} : D \to C, \bar{g} : D \to A)$ be the pullback of f and g, for some $g : C \to B$. Since $\mathbf{A}$ has enough $\mathrm{St}(\mathcal{E})$-projectives, there is some $\mathrm{St}(\mathcal{E})$-morphism $q : E \to C$ with $E \in \mathbf{B}$. It gives rise to the existence of a morphism $\bar{q}$ such that $g \cdot q = f \cdot \bar{q}$, and so, by the universality of the pullback, there is a morphism $t : E \to D$ such that $\bar{f} \cdot t = q$ and $\bar{g} \cdot t = \bar{q}$. Let $\bar{f} = m \cdot e$ be the $\mathcal{E}$-factorization of $\bar{f}$; then the equality $m \cdot e \cdot t = 1_C \cdot q$ guarantees the existence of a morphism d such that $m \cdot d = 1_C$, so, being a split epimorphism, m is an isomorphism, and thus $\bar{f}$ belongs to $\mathcal{E}$.

2. By Lemma 1.11, each coreflection $s_A : S(A) \to A$ into $\mathbf{B}$ belongs to $\mathrm{St}(\mathcal{E})$. It remains to show that s_A is a $\mathrm{St}(\mathcal{E})$-coessential morphism. Let g be such that $s_A \cdot g$ belongs to $\mathrm{St}(\mathcal{E})$, and let $\bar{g}$ be the pullback of g along any h. By 1.11, s_A is a monomorphism, then $\bar{g}$ is also the pullback of $s_A \cdot g$ along $s_A \cdot h$. Thus $\bar{g} \in \mathcal{E}$, and g belongs to $\mathrm{St}(\mathcal{E})$. □

2 Premonadicity

Definition 2.1 *We say that a class* $\mathcal{F}$ *of epimorphisms is saturated provided that, for each* $f \in \mathcal{F}$*, the coequalizer of the kernel pair of* f *belongs to* $\mathcal{F}$*.*

Lemma 2.2 *If* P *is a projective* $\mathcal{E}$*-generator, then the following two assertions are equivalent:*

1. $St(\mathcal{E})$ *is saturated.*

2. *For each $e \in St(\mathcal{E})$, the unique morphism d such that $d \cdot c = e$, for c the coequalizer of the kernel pair of e, is a monomorphism.*

Proof. By Lemma 1.10, $\mathrm{St}(\mathcal{E}) = \mathrm{Proj}(\mathcal{P})$. Let $\mathrm{St}(\mathcal{E})$ be saturated, let $e \in \mathrm{St}(\mathcal{E})$, let c be the coequalizer of the kernel pair (u, v) of e, and let d be the unique morphism d such that $d \cdot c = e$. Let a and b be morphisms such that $d \cdot a = d \cdot b$; in order to show that $a = b$, since P is a generator, we may assume without loss of generality that P is the domain of a and b. Consequently, since $c \in \mathrm{St}(\mathcal{E})$, there is $\bar{a}$ and $\bar{b}$ such that $c \cdot \bar{a} = a$ and $c \cdot \bar{b} = b$; then $e \cdot \bar{a} = d \cdot c \cdot \bar{a} = da = db = d \cdot c \cdot \bar{b} = e \cdot \bar{b}$. Since (u, v) is the kernel pair of e, this implies the existence of a unique morphism t such that $u \cdot t = \bar{a}$ and $v \cdot t = \bar{b}$. Therefore, $a = c \cdot \bar{a} = c \cdot u \cdot t = c \cdot v \cdot t = c \cdot \bar{b} = b$.

Conversely, if d is a monomorphism, for each morphism f with domain P and codomain in the codomain of c, we have some morphism f' such that $e \cdot f' = d \cdot f$, because $e \in \mathrm{Proj}(P)$. Thus, $d \cdot c \cdot f' = d \cdot f$, and then $c \cdot f' = f$. $\square$

Examples 2.3 *1. If $\mathcal{E} \subseteq \{$regular epimorphisms$\}$, $St(\mathcal{E})$ is trivially saturated, since a regular epimorphism is the coequalizer of its kernel pair. The saturation of $St(\mathcal{E})$ is also clear when $\mathcal{E}$ is pullback stable and $\mathcal{E}$ contains all regular epimorphisms.*

2. *In all examples of 1.14, for the considered class $\mathcal{E}$ or $\mathcal{M}$, the corresponding stabilization is saturated, except in the second example. In fact, for the category* **Cat** *and $\mathcal{E} = \{$extremal epimorphisms$\}$, $St(\mathcal{E})$ is not saturated. To see that, consider the functor $F : A \to B$ where $A =$* **2** *and B is the category with a unique object and with an only non-identity morphism f such that $f \cdot f = f$. Then the coequalizer of the kernel pair of F is $G : A \to C$ where C is the category with a unique object and f^n, $n \in \mathbb{N}$, as morphisms different from the identity. Clearly the morphism G does not belong to $St(\mathcal{E})$ (see [6]).*

It is known that there exists a monadic functor $U : \mathbf{A} \to \mathbf{Set}$ (see [1] or [7]) iff

(i) $\mathbf{A}$ has finite limits;
(ii) $\mathbf{A}$ is exact;
(iii) $\mathbf{A}$ has a regular generator P;
(iv) P is projective;
(v) $\mathbf{A}$ has copowers of P.

The assumption (ii) ensures that $\mathrm{St}(\mathcal{E}) = \mathcal{E}$ for $\mathcal{E}$ the class of regular epimorphisms; thus (iii) and (iv) mean that P is a projective $\mathcal{E}$-generator in the sense of definition 1.7. Moreover, the monadicity of U, combined with the cocompletness of $\mathbf{A}$, implies that $\mathbf{A} = \mathbb{C}(\mathbf{P})$.

The next theorem, where projectivity has a relevant role, gives sufficient conditions for the premonadicity of a right-adjoint $U : \mathbf{A} \to \mathbf{Set}$. The assumed conditions are a generalization of (i)–(v) above.

Theorem 2.4 *Let P be a projective $\mathcal{E}$-generator of* **A** *and let $St(\mathcal{E})$ be saturated. Then, assuming that $\mathbb{C}(P)$ is coreflective in* **A***, the functor $hom(P, -) : \mathbb{C}(P) \to$* **Set** *is premonadic, and $\mathbb{C}(P)$ is equivalent to a reflective subcategory of the corresponding category of Eilenberg-Moore algebras.*

Proof. Since $\hom(P,-)$ is a right adjoint and $\mathbb{C}(P)$ has coequalizers, we know that the comparision functor is a right adjoint. In order to show that it is full and faithful, it suffices to prove that the co-units of the right adjoint $\hom(P,-) : \mathbb{C}(P) \to \mathbf{Set}$ are regular epimorphisms. For each $B \in \mathbb{C}(P)$, let us consider the corresponding co-unit

$$\varepsilon_B : \coprod_{\hom(P,B)} P \longrightarrow B\,.$$

Since P is ε_B-projective, ε_B belongs to $\mathrm{St}(\mathcal{E})$, by 1.10. Let (u,v) be the kernel pair of ε_B and let c be the coequalizer of u and v. Since $\mathrm{St}(\mathcal{E})$ is saturated, the morphism d such that $d \cdot c = \varepsilon_B$ is a monomorphism. But $\mathrm{St}(\mathcal{E})$ is strongly right-cancellable, by 1.6, so $d \in \mathrm{St}(\mathcal{E})$. Then $d \in \mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$ and, thus, by Proposition 1.12 (see also Remark 1.13), there is some morphism t such that $d \cdot t = 1_B$. Therefore d is an isomorphism and ε_B is a regular epimorphism. $\square$

Definition 2.5 *P is a projective dense $\mathcal{E}$-generator of $\mathbf{A}$ if it is a projective $\mathcal{E}$-generator and $\mathbb{C}(P) = \mathbf{A}$.*

Corollary 2.6 *If $\mathbf{A}$ has a projective dense $\mathcal{E}$-generator P and $St(\mathcal{E})$ is saturated then $hom(P,-) : \mathbf{A} \to \mathbf{Set}$ is premonadic.*

Examples 2.7 *All examples of 1.14 fulfil the conditions of the above theorem, except the second one. As seen in 2.3.2, in this case, the saturation of $St(\mathcal{E})$ fails. And, curiously, the corresponding functor into* **Set** *is not premonadic: It is easily seen that the corresponding co-units are not necessarily regular epimorphisms (cf. [6]).*

In particular, the functor $hom(0 \to 1, -) : \mathbf{PreOrd} \to \mathbf{Set}$ is premonadic but not monadic, since regular epimorphisms are not stable under pullbacks.

1.14 and 2.3 also lead to several examples of categories $\mathbf{A}$ for which $\mathbf{A}^{\mathrm{op}}$ is premonadic over **Set**.

The above results bring up the question of knowing if the premonadicity implies the existence of a projective $\mathcal{E}$-generator for some $\mathcal{E}$. The next proposition derives from the analysis of this subject, although it does not give a complete answer.

Proposition 2.8 *Let $\mathbf{A}$ be a cocomplete and cowellpowered category. If $U : \mathbf{A} \to \mathbf{Set}$ is a premonadic functor, then there is a coreflective class $\mathcal{E}$ and a dense $\mathcal{E}$-generator P of $\mathbf{A}$ such that $Proj(P) \subseteq St(\mathcal{E})$.*

Proof. Let $U : \mathbf{A} \to \mathbf{Set}$ be premonadic. It reflects isomorphisms because $\mathbf{A}$ is a full subcategory of the corresponding category of Eilenberg-Moore algebras, and every monadic functor over **Set** reflects isomorphisms. Let $P = F\{*\}$ where F is the left adjoint of U. Since $U \simeq \hom(P,-)$ reflects isomorphisms we have that the class of morphisms co-orthogonal to P is just $\mathrm{Iso}(\mathbf{A})$, thus $\mathbf{A}$ is the co-orthogonal closure of P. Under the assumptions on $\mathbf{A}$, $\mathbb{C}(P)$ is coreflective, so it coincides with its co-orthogonal closure, and thus $\mathbb{C}(P) = \mathbf{A}$, that is, P is a dense generator.

Since the co-units of $\hom(P,-)$ are epimorphic, and $f \in \mathrm{Proj}(P)$ if and only if $\hom(P,f)$ is surjective, it follows that $\mathrm{Proj}(P) \subseteq \mathrm{Epi}(\mathbf{A})$. Let $\mathcal{E}$ be the closure of $\mathrm{Proj}(P)$ under pushouts and cointersections in $\mathrm{Mor}(\mathbf{A})$. Then $\mathcal{E}$ is a subclass of $\mathrm{Epi}(\mathbf{A})$; we are going to see that it is coreflective, more than that, it fulfils the condition stated in 1.2.1. In order to conclude that, given a morphism f, consider all pairs (e_i, m_i), such that $f = m_i \cdot e_i$ and e_i is in $\mathcal{E}$. Let $(e, (t_i))$ be the cointersection of all these e_i's, whose existence is guaranteed by the fact that $\mathbf{A}$ is cocomplete and cowellpowered. The definition of cointersection gives a unique morphism m such that $m \cdot e = f$. This is an $\mathcal{E}$-factorization of f, that is, $(1, m) : e \to f$ is a coreflection of $f \in \mathrm{Mor}(\mathbf{A})$ into $\mathcal{E}$. In fact, let q be another morphism in $\mathcal{E}$ and let $(r, h) : q \to f$ be a morphism in the category $\mathrm{Mor}(A)$, that is, $f \cdot r = h \cdot q$. Form the pushout (q', r') of (q, r) and let d be the unique morphism which fulfil the equalities $f = d \cdot q'$ and $d \cdot r' = h$. Then, for some i_o, $q' = e_{i_o}$ and $d = m_{i_o}$. Thus $t_{i_o} \cdot r'$ is a morphism such that $(t_{i_o} \cdot r') \cdot q = e \cdot r$ and $m \cdot (t_{i_o} \cdot r') = h$.

Since $\mathrm{Proj}(P) \subseteq \mathcal{E}$, and for each $A \in \mathbf{A}$, the co-unit $\varepsilon_A \in \mathrm{Proj}(P)$, we conclude that P is a dense $\mathcal{E}$-generator. Moreover, $\mathrm{Proj}(P)$ is contained in $\mathrm{St}(\mathcal{E})$, because it is pullback stable. □

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