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A METHOD OF CONSTRUCTING COMPATIBLE QUASI-UNIFORMITIES FOR AN ARBITRARY FRAME

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Abstract: Unlike a uniformity, a quasi-uniformity is not determined by its quasi-uniform covers. However, a classical construction, due to Fletcher, which assigns a transitive quasi-uniformity to each family of interior-preserving open covers, allows to describe all transitive quasi-uniformities on the topological spaces in terms of those families of covers.

In this paper we develop a pointfree generalization of this, which solves a problem posed by G. C. L. Brümmer, together with various examples and applications that illustrate its remarkable usefulness. By this construction, many kinds of interior-preserving open covers (e.g. locally finite, open spectrum, well-monotone) induce compatible quasi-uniformities on an arbitrary frame.

Keywords: frame, biframe, Skula biframe, Weil entourage, quasi-uniform frame, transitive quasi-uniformity, Fletcher construction, interior-preserving cover, locally finite cover, well-monotone cover, open spectrum.
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Introduction

Transitive quasi-uniform spaces form an important subcategory of the category of quasi-uniform spaces and uniformly continuous maps and they play a rôle almost as general as that of quasi-uniform spaces in the study of topological properties. The most striking aspect of transitive quasi-uniformities is that they can all be obtained by the Fletcher construction [8] (see also [9]) considering the interior-preserving open covers of their associated topological spaces:

Let $(X, \mathcal{T})$ be a topological space and let $\mathfrak{A}$ be a collection of interior-preserving open covers $\mathcal{A}$ of X such that $\bigcup \mathfrak{A}$ is a subbase for $\mathcal{T}$. For any $\mathcal{A}$ set

$$R_{\mathcal{A}} = \bigcap \{E_A \mid A \in \mathcal{A}\}$$

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where E_A stands for

$$\{(x,y) \in X \times X \mid x \in A \Rightarrow y \in A\}.$$

Then the collection $\{R_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}\}$ is a subbase for a *compatible* transitive quasi-uniformity $\mathcal{E}_{\mathfrak{A}}$ on X (that is, $(X, \mathcal{E}_{\mathfrak{A}})$ is a transitive quasi-uniform space inducing as first topology the given topology $\mathcal{T}$).

Corresponding to each kind of interior-preserving open cover we get well-known quasi-uniformities. For example, if $\mathfrak{A}$ is the collection of all finite (resp. point-finite, locally finite, interior-preserving, open spectra, well-monotone) open covers of X, then $\mathcal{E}_{\mathfrak{A}}$ is the Pervin (resp. point-finite, locally finite, fine transitive, semicontinuous, well-monotone) quasi-uniformity of X. Moreover, this construction gives all transitive quasi-uniformities on X [4].

The purpose of this paper is to extend these considerations to pointfree topology, solving Problem 3 of [5], posed by G.C.L. Brümmer at the topology conference in Ankara, summer 2001. This extension is by no means immediate in that a number of technical difficulties, which have no place in the spatial case (by the very nature of entourages as reflexive relations on a set X and quasi-uniformities as filters of entourages on X), have now to be surpassed.

In the classical theory, the construction relies on the fact that every open subspace of a space X has a complement. To get the localic counterpart of this construction, we turn to the frame of congruences (more precisely, the congruence biframe) which provides the right tools for the translation of the topological properties pertaining to the Fletcher construction. The topological intuition behind our arguments concerning congruences can be easily traced back by the correspondence between sublocales and congruences.

The paper is organized as follows:

In Sections 1 and 2, we recall the specific notions and facts which will be used later on. After that, we discuss the concept of interior-preserving cover (Section 3) and the covers which induce entourages in our construction (Section 4). In Section 5, we present a technical result on the construction of general transitive frame quasi-uniformities that has specific relevance in the construction of compatible quasi-uniformities that we describe in Section 6, the main goal of this paper. Next, in Section 7, we present the fundamental result that the construction of Section 6 accounts for all transitive compatible quasi-uniformities, as well as a result that has the noteworthy consequence of making the construction in question functorial, a fact which will be the subject of a forthcoming paper. In the final section we show the effectiveness of our construction with a brief survey of various examples and applications. The examples essentially cover all the facts concerning the Fletcher construction from [9] which make pointfree sense.

In a sequel [7] to this paper, we show that the general method introduced here gives precisely all functorial transitive quasi-uniformities on frames. This extends the results of Brümmer [4] on functorial transitive quasi-uniformities on the topological spaces.

1. Background

In order to fix terminology let us recall a few concepts.

1.1. Frames and biframes. A *frame* (also *locale*) is a complete lattice satisfying the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. A *frame homomorphism* $f : L \to M$ is a map between frames which preserves finite meets (including the top element 1) and arbitrary joins (including the bottom element 0). The corresponding category will be denoted by Frm. A *cover* A of L is a subset $A \subseteq L$ such that $\bigvee A = 1$.

If L is a frame and $x \in L$ then

$$x^* := \bigvee \{a \in L \mid a \wedge x = 0\}$$

is the *pseudocomplement* of x. Obviously, if $x \vee x^* = 1$, x is complemented and we denote the complement x^* by $\neg x$. Note that, in any frame, the first De Morgan law

$$(\bigvee_{i \in I} x_i)^* = \bigwedge_{i \in I} x_i^*$$

holds but for infima we have only the trivial inequality

$$\bigvee_{i \in I} x_i^* \leq (\bigwedge_{i \in I} x_i)^*.$$

Recall also that a *biframe* is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, which together generate L_0. A *biframe homomorphism,* $f : (L_0, L_1, L_2) \longrightarrow (M_0, M_1, M_2)$, is a frame homomorphism

$f : L_0 \longrightarrow M_0$ which maps L_i into $M_i\,(i = 1, 2)$ and BiFrm denotes the resulting category.

Further, a biframe (L_0, L_1, L_2) is *strictly zero-dimensional* [2] if it satisfies the following condition or its counterpart with L_1 and L_2 reversed: each $x \in L_1$ is complemented in L_0, with complement in L_2, and L_2 is generated by these complements. Along this paper, when referring to a strictly zero-dimensional biframe, we always assume that it satisfies the condition above, not its counterpart with L_1 and L_2 reversed.

For general facts concerning frames we refer to Johnstone [13] or Vickers [19]. Additional information concerning biframes may be found in [2] and [3].

1.2. Weil entourages. For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented, as usual (cf. [13]), as the subset of $\mathcal{D}(L \times L)$ consisting of all *C-ideals*, that is, of those sets A which satisfy

$$\{x\} \times S \subseteq A \;\Rightarrow\; (x, \bigvee S) \in A$$

and

$$S \times \{y\} \subseteq A \;\Rightarrow\; (\bigvee S, y) \in A.$$

Since the premise is trivially satisfied if $S = \emptyset$, each C-ideal A contains $\mathbf{O} := \{(0, a), (a, 0) \mid a \in L\}$, and $\mathbf{O}$ is the zero of $L \oplus L$. Obviously, each $x \oplus y = {\downarrow}(x, y) \cup \mathbf{O}$ is a C-ideal and for each C-ideal A one has

$$A = \bigvee\{x \oplus y \mid x \oplus y \leq A\} = \bigvee\{x \oplus y \mid (x, y) \in A\}.$$

The coproduct injections $u_i^L : L \to L \oplus L$ are defined by $u_1^L(x) = x \oplus 1$ and $u_2^L(x) = 1 \oplus x$ so that $x \oplus y = u_1^L(x) \wedge u_2^L(y)$.

For any frame homomorphism $h : L \longrightarrow M$, the definition of coproduct ensures us the existence (and uniqueness) of a frame homomorphism $h \oplus h : L \oplus L \longrightarrow M \oplus M$ such that $(h \oplus h) \cdot u_i^L = u_i^M \cdot h$ $(i = 1, 2)$.

A *Weil entourage* [15] on L is just an element E of $L \oplus L$ for which

$$\bigvee\{x \in L \mid (x, x) \in E\} = 1.$$

The collection $WEnt(L)$ of all Weil entourages of L with the inclusion is a partially ordered set with finitary meets (including a unit $1 = L \oplus L$).

If E and F are elements of $L \oplus L$ then

$$E \circ F := \bigvee \{x \oplus y \mid \exists z \in L \setminus \{0\} : (x, z) \in E,\ (z, y) \in F\}.$$

Note that $E \subseteq E \circ E$ for every Weil entourage E. A Weil entourage E is called

- *transitive* if $E \circ E = E$;
- *finite* if there exists a finite cover $x_1, \ldots, x_n$ of L such that $\bigvee_{i=1}^{n}(x_i \oplus x_i) \subseteq E$.

Let $\mathcal{E} \subseteq L \oplus L$ and x, $y \in L$. If

$$E \circ (x \oplus x) \subseteq y \oplus y \text{ for some } E \in \mathcal{E}, \tag{1.2.1}$$

we write $x \overset{\mathcal{E}}{\triangleleft} y$. When $\mathcal{E}$ is *symmetric* (that is, $E^{-1} \in \mathcal{E}$ whenever $E \in \mathcal{E}$) this is equivalent to

$$(x \oplus x) \circ E \subseteq y \oplus y \text{ for some } E \in \mathcal{E} \qquad \text{(cf. [15])}. \tag{1.2.2}$$

A set $\mathcal{E} \subseteq WEnt(L)$ is called *admissible* if, for every $x \in L$,

$$x = \bigvee \{y \in L \mid y \overset{\overline{\mathcal{E}}}{\triangleleft} x\},$$

where $\overline{\mathcal{E}} := \mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$.

1.3. Uniform and quasi-uniform frames. An admissible filter $\mathcal{E}$ of $WEnt(L)$ is a *(Weil) uniformity* on L if it satisfies the following conditions:

(U1) For each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.
(U2) For every $E \in \mathcal{E}, E^{-1} \in \mathcal{E}$.

Further, a *(Weil) uniform frame* is a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a uniformity on L. If $(L, \mathcal{E})$ and $(M, \mathcal{F})$ are uniform frames, $f : (L, \mathcal{E}) \to (M, \mathcal{F})$ is a *uniform homomorphism* if $f : L \to M$ is a frame homomorphism such that $(f \oplus f)(E) \in \mathcal{F}$, for all $E \in \mathcal{E}$. The resulting category is denoted by UFrm.

By just dropping the symmetry condition (U2) in the definition of uniform frame we get the category of *quasi-uniform frames*, denoted by QUFrm. With the lack of symmetry the equivalence between conditions (1.2.1) and (1.2.2) is no longer valid; whence, in the place of $\overset{\mathcal{E}}{\triangleleft}$ we have two partial orders

$$x \overset{\mathcal{E}}{\triangleleft}_1 y \equiv E \circ (x \oplus x) \subseteq y \oplus y, \text{ for some } E \in \mathcal{E},$$

$$x \overset{\mathcal{E}}{\triangleleft}_2 y \equiv (x \oplus x) \circ E \subseteq y \oplus y, \text{ for some } E \in \mathcal{E},$$

which in turn, lead to the following subframes of L:

$$\mathcal{L}_1(\mathcal{E}) := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_1 x\}\Big\},$$

$$\mathcal{L}_2(\mathcal{E}) := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_2 x\}\Big\}.$$

It is worth pointing that then, for each $x \in L$,

$$x \in \mathcal{L}_1(\mathcal{E}) \Leftrightarrow x = \bigvee\{y \in \mathcal{L}_1(\mathcal{E}) \mid y \overset{\mathcal{E}}{\lhd}_1 x\} \tag{1.3.1}$$

and

$$x \in \mathcal{L}_2(\mathcal{E}) \Leftrightarrow x = \bigvee\{y \in \mathcal{L}_2(\mathcal{E}) \mid y \overset{\mathcal{E}}{\lhd}_2 x\} \text{ [15].} \tag{1.3.2}$$

Further, the admissibility condition is equivalent to saying that the triple

$$(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$$

is a biframe [16]. This is the pointfree expression of the classical fact that each quasi-uniform space $(X, \mathcal{E})$ induces a bitopological structure $(\mathcal{T}_1(\mathcal{E}), \mathcal{T}_2(\mathcal{E}))$ on X.

Regarding quasi-uniform frames, we shall need the following notions: a quasi-uniform frame $(L, \mathcal{E})$ is called

- *transitive* if $\mathcal{E}$ has a base consisting of transitive entourages;
- *totally bounded* if $\mathcal{E}$ has a base of finite entourages.

For more information on transitive quasi-uniformities and totally bounded quasi-uniformities we refer to [12] and [11], respectively.

Throughout this paper, L always represents a frame.

2. Tools

In this section we collect the facts needed later on concerning the notions which play a particularly significant rôle in our construction: the *strong inclusion* $(\overset{\mathcal{E}}{\lhd}_1, \overset{\mathcal{E}}{\lhd}_1)$ induced by a quasi-uniformity $\mathcal{E}$ and the *congruence lattice* $\mathfrak{C}L$.

2.1. Let $(L, \mathcal{E})$ be a quasi-uniform frame. The pair $(\overset{\mathcal{E}}{\lhd}_1, \overset{\mathcal{E}}{\lhd}_2)$ is a strong inclusion [18] on the biframe $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$. In particular, this means that $x \overset{\mathcal{E}}{\lhd}_i y$ implies $x \prec_i y$, that is, the existence of $z \in \mathcal{L}_j(\mathcal{E})$, $j \neq i$, such that $z \wedge x = 0$ and $z \vee y = 1$.

We should also note that $\overset{\mathcal{E}}{\lhd}_1$ and $\overset{\mathcal{E}}{\lhd}_2$ may be characterized in the following way [16]:

- $x \overset{\mathcal{E}}{\lhd_1} y$ if and only if there exists $E \in \mathcal{E}$ such that

$$st_1(x, E) := \bigvee\{a \in L \mid (a, b) \in E, b \wedge x \neq 0\} \leq y; \qquad (2.1.1)$$

- $x \overset{\mathcal{E}}{\lhd_2} y$ if and only if there exists $E \in \mathcal{E}$ such that

$$st_2(x, E) := \bigvee\{b \in L \mid (a, b) \in E, a \wedge x \neq 0\} \leq y. \qquad (2.1.2)$$

The elements $st_i(x, E)$ $(i = 1, 2)$ satisfy the following properties, for every $x, y \in L$ and every $E, F \in L \oplus L$ [15]:

(S1) $x \leq y \Rightarrow st_i(x, E) \leq st_i(y, E)$;
(S2) For every Weil entourage E, $x \leq st_1(x, E) \wedge st_2(x, E)$;
(S3) $st_i(x, E \cap F) \leq st_i(x, E) \wedge st_i(x, F)$;
(S4) $st_1(st_1(x, E), F) \leq st_1(x, F \circ E)$ and $st_2(st_2(x, E), F) \leq st_2(x, E \circ F)$;
(S5) $st_i(\bigvee_{j \in J} x_j, E) = \bigvee_{j \in J} st_i(x_j, E)$;
(S6) If $(x, y) \in E$ then $(x, st_2(y, E))$ and $(st_1(x, E), y)$ belong to $E \circ E$;

2.2. The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. A good presentation of the congruence frame is given by Frith [10]. Here, we shall need the following properties:

(1) For any $x \in L$, $\nabla_x = \{(a, b) \mid a \vee x = b \vee x\}$ is the least congruence containing $(0, x)$; $\Delta_x = \{(a, b) \mid a \wedge x = b \wedge x\}$ is the least congruence containing $(1, x)$. The ∇_x are called *closed* and the Δ_x *open*.
(2) Each ∇_x is complemented in $\mathfrak{C}L$ with complement Δ_x.
(3) $\nabla L = \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. Since $\theta = \bigvee\{\nabla_y \wedge \Delta_x \mid (x, y) \in \theta, x \leq y\}$, for every $\theta \in \mathfrak{C}L$, the triple $(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe (usually referred to as the *Skula biframe* [10]). Clearly, this is a strictly zero-dimensional biframe.
(4) The map $x \mapsto \nabla_x$ is a frame isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

For any $\theta \in \mathfrak{C}L$, the *interior* of θ, denoted by $int(\theta)$, is given by

$$\bigwedge\{\Delta_x \mid \theta \leq \Delta_x\} = \Delta_{\bigvee\{x \in L \mid \theta \wedge \nabla_x = 0\}}.$$

Obviously, $int(\nabla_x) = \Delta_{x^*}$ and $int(\bigvee \Delta_{x_i}) = \Delta_{\bigwedge x_i}$.

3. Interior-preserving covers

3.1. Fletcher's construction starts from a collection $\mathcal{A}$ of interior-preserving open covers of the given space X. The immediate translation of the classical notion of interior-preserving open covers of a space into the pointfree setting says that an open cover $\{\Delta_a \mid a \in A\}$ of a locale L (i.e., $\bigwedge_{a\in A} \Delta_a = 0$, which means, internally in L, that A is a cover of L) is *interior-preserving* if, for each $B \subseteq A$,

$$\bigvee_{b\in B} \Delta_b = \Delta_{\bigwedge B}. \tag{3.1.1}$$

More generally, we say that a subset A of L is *interior-preserving* if condition (3.1.1) holds for any $B \subseteq A$.

Classically, De Morgan laws imply immediately that an open cover $\mathcal{A}$ of a topological space X is interior-preserving if and only if $\{X \setminus A \mid A \in \mathcal{A}\}$ is a closure-preserving closed co-cover of X. This equivalence is no longer true in the pointfree setting. Indeed, a closed co-cover $\{\nabla_a \mid a \in A\}$ (internally in L, this means that A is again a cover of L, since $\nabla_{\bigvee A} = \bigvee_{a\in A} \nabla_a = 1$) is *closure-preserving* if,

$$\text{for each } B \subseteq A, \quad \bigwedge_{b\in B} \nabla_b = \nabla_{\bigwedge B}. \tag{3.1.2}$$

Now, the first De Morgan law implies immediately that (3.1.1) implies (3.1.2) but the converse is not true.

We say that a cover A of L is *weakly interior-preserving* if it satisfies condition (3.1.2). We note that subsets A of L satisfying (3.1.2) were already studied by Chen [6] (under the name "*conservative subsets*"). In ([6], Lemma 2.3) Chen characterizes them by the condition

$$\text{for each } B \subseteq A, \quad x \vee \bigwedge B = \bigwedge\{x \vee b \mid b \in B\} \text{ for all } x \in L.$$

As we shall see later on, condition (3.1.2) is the right condition we need to impose on the covers in order to fulfill our construction of a quasi-uniformity compatible with the given frame L.

In what follows we illustrate the scope of these notions by a number of important examples which we shall refer to later on as *guiding examples*.

3.2. Locally finite covers. Recall that a set $A \subseteq L$ is said to be *locally finite* [6] if there is a cover C of L such that $A_c := \{a \in A \mid c \wedge a \neq 0\}$ is finite for each $c \in C$. Such a cover C is said to *finitize* A.

Proposition. *Every locally finite set is interior-preserving.*

Proof: Let A be a locally finite set of L and let C be the corresponding cover that finitizes it. Then A is interior-preserving because, for every infinite $B \subseteq A$, $\bigvee_{b\in B} \Delta_b = 1$. Indeed:

Since B is infinite, for every $c \in C$ there exists $b \in B$ such that $b \wedge c = 0$, that is, $\Delta_{b\wedge c} = 1$. Thus $\bigvee_{b\in B} \Delta_{b\wedge c} = 1$ for every $c \in C$. Equivalently, $\bigvee_{b\in B} \Delta_b \geq \nabla_c$ for every $c \in C$. Hence $\bigvee_{b\in B} \Delta_b \geq \bigvee_{c\in C} \nabla_c = 1$. ■

3.3. Open spectra. We say that a cover $A = \{a_n \mid n \in \mathbb{Z}\}$ of L is an *open spectrum* if $a_n \leq a_{n+1}$, for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$ (which implies, in particular, that $\bigwedge_{n\in\mathbb{Z}} a_n = 0$).

Proposition. *Every open spectrum is interior-preserving.*

Proof: It suffices to show that

$$\bigvee_{n\in S} \Delta_{a_n} \geq \Delta_{\bigwedge_{n\in S} a_n} \text{ for every } S \subseteq \mathbb{Z}.$$

Let $S \subseteq \mathbb{Z}$. If S has a least element m then, obviously, $\bigvee_{n\in S} \Delta_{a_n} = \Delta_{a_m} = \Delta_{\bigwedge_{n\in S} a_n}$. Otherwise, $\bigvee_{n\in S} \Delta_{a_n} = \bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$. ■

3.4. Well-monotone covers. We say that a cover A of L is a *well-monotone cover* if it is well-ordered by the partial order $\leq$ of L. So A is a chain in L, satisfying the descending chain condition. The following is obvious:

Proposition. *Every well-monotone cover is interior-preserving.* ■

4. Fletcher covers

4.1. Our first aim is to cast the basic (transitive) entourages in Fletcher construction into pointfree form. As in initial step towards this, let

$$E_a = (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a)$$

for any $a \in L$. This is clearly a transitive Weil entourage of $\mathfrak{C}L$. It is also worth pointing that E_a is simply $(\nabla_a \oplus 1) \cup (1 \oplus \Delta_a)$, since this is already a C-ideal.

For any $A \subseteq L$ set

$$R_A = \bigcap \{E_a \mid a \in A\} \in \mathfrak{C}L \oplus \mathfrak{C}L.$$

In view of the following lemma, we shall consider only R_A with $A \in CovL$ (this is not a serious restriction since, for any $B \subseteq L$, $A = B \cup \{1\}$ is a cover and $R_B = R_A$).

Lemma. *If A and B are covers of L then $R_A \cap R_B = R_{A\wedge B}$.*

Proof: It is easy to see that, for every $A, B \subseteq L$,

$$(\bigcap_{a\in A} E_a) \cap (\bigcap_{b\in B} E_b) \subseteq \bigcap\{E_{a\wedge b} \mid a \in A, b \in B\},$$

since $\nabla_{a\wedge b} = \nabla_a \wedge \nabla_b$ and $\Delta_{a\wedge b} = \Delta_a \vee \Delta_b$.

For the reverse inclusion, consider $(\alpha, \beta) \in \bigcap\{E_{a\wedge b} \mid a \in A, b \in B\}$. Let $a \in A$. If $\alpha \leq \nabla_a$ then $(\alpha, \beta) \in E_a$. Otherwise, if $\alpha \not\leq \nabla_a$, then $\alpha \not\leq \nabla_{a\wedge b}$ so, necessarily, $\beta \leq \Delta_{a\wedge b}$, for every $b \in B$. Consequently, since B is a cover of L,

$$\beta \leq \bigwedge_{b\in B} \Delta_{a\wedge b} = \Delta_{\bigvee_{b\in B}(a\wedge b)} = \Delta_a.$$

Thus (α, β) also belongs to E_a in this case. Similarly, $(\alpha, \beta) \in E_b$ for every $b \in B$. Hence $(\alpha, \beta) \in R_A \cap R_B$. ■

Of course, R_A would be of little use for our purpose if it would not be a Weil entourage (which may happen, contrarily to the classical case, since infinite intersections of Weil entourages are not necessarily entourages).

For each cover A of L, let

$$d(A) = \bigvee\{(\bigwedge_{a\in A_1} \nabla_a) \wedge (\bigwedge_{a\in A_2} \Delta_a) \mid A_1 \cup A_2 = A\}.$$

Proposition. *Let A be a cover of L. The following assertions are equivalent:*

(i) *R_A is a Weil entourage of $\mathfrak{C}L$.*
(ii) *$d(A) = 1$.*
(iii) *There exists a cover C of L such that, for each $c \in C$, $\nabla_c = \bigvee_{i\in I_c} \theta_i^c$, where each pair (θ_i^c, θ_i^c) belongs to R_A.*

Proof: (i)⇔(ii): If R_A is a Weil entourage then $\bigvee\{\alpha \in \mathfrak{C}L \mid (\alpha, \alpha) \in R_A\} = 1$. But $(\alpha, \alpha) \in R_A$ means that there exists a partition $A_1 \cup A_2$ of A for which

$$\alpha \leq (\bigwedge_{a\in A_1} \nabla_a) \wedge (\bigwedge_{a\in A_2} \Delta_a).$$

Thus $d(A) = 1$.

The converse is obvious since

$$((\bigwedge_{a\in A_1} \nabla_a) \wedge (\bigwedge_{a\in A_2} \Delta_a), (\bigwedge_{a\in A_1} \nabla_a) \wedge (\bigwedge_{a\in A_2} \Delta_a)) \in R_A$$

whenever $A_1 \cup A_2 = A$.

(i)⇒(iii): Take just the cover A. For each $a \in A$,

$$\nabla_a = \nabla_a \wedge d(A) = \bigvee\{(\bigwedge_{b\in A_1} \nabla_b) \wedge (\bigwedge_{b\in A_2} \Delta_b) \mid A_1 \cup A_2 = A, a \in A_1\}.$$

Clearly,

$$((\bigwedge_{b\in A_1} \nabla_b) \wedge (\bigwedge_{b\in A_2} \Delta_b), (\bigwedge_{b\in A_1} \nabla_b) \wedge (\bigwedge_{b\in A_2} \Delta_b)) \in R_A.$$

(iii)⇒(i): It is obvious:

$$\bigvee_{(\theta,\theta)\in R_A} \theta \geq \bigvee_{c\in C} \bigvee_{i\in I_c} \theta_i^c = \bigvee_{c\in C} \nabla_c = 1.$$ ■

We say that a cover A of L is a *Fletcher cover* whenever it satisfies the equivalent conditions of the Proposition.

Remark. For each finite $A \subseteq L$, a straightforward proof by induction shows that $d(A) = 1$. Thus finite covers are examples of Fletcher covers.

Corollary. *If A is a Fletcher cover of L then R_A is a transitive Weil entourage of $\mathfrak{C}L$.*

Proof: It remains to check that $R_A \circ R_A \subseteq R_A$. So, consider $(\alpha, \beta), (\beta, \gamma) \in R_A$ with $\beta \neq 0$ and let $a \in A$. If $\alpha \not\leq \nabla_a$ then $\beta \leq \Delta_a$ and, consequently, $\beta \not\leq \nabla_a$, since $\beta \neq 0$. Therefore $\gamma \leq \Delta_a$ and $(\alpha, \gamma) \in R_A$. ■

In view of the Proposition, it is clear that Fletcher covers will play a central rôle in our context. We do not know whether (weakly) interior-preserving covers are Fletcher covers but we do know that, in each of our guiding examples, the covers are indeed Fletcher covers, as we shall show in the sequel.

4.2. Locally finite covers are Fletcher covers. Let A be a locally finite cover of L and let C be a cover that finitizes it.

Lemma. *For each $c \in C$, there exists $\{\theta_i \mid i \in I\} \subseteq \mathfrak{C}L$ such that $\nabla_c = \bigvee_{i\in I} \theta_i$ and $(\theta_i, \theta_i) \in R_A$ for every $i \in I$.*

Proof: Let $c \in C$ and consider $A_c = \{a_j \mid j \in J\}$. By Remark 4.1, since J is finite,

$$\bigvee_{J_1\cup J_2=J} ((\bigwedge_{j\in J_1} \nabla_{a_j}) \wedge (\bigwedge_{j\in J_2} \Delta_{a_j})) = 1.$$

Therefore

$$\nabla_c = \bigvee_{J_1 \cup J_2 = J} (\nabla_c \wedge (\bigwedge_{j\in J_1} \nabla_{a_j}) \wedge (\bigwedge_{j\in J_2} \Delta_{a_j})).$$

Denote the congruence $\nabla_c \wedge (\bigwedge_{j\in J_1} \nabla_{a_j}) \wedge (\bigwedge_{j\in J_2} \Delta_{a_j})$ by θ. It remains to check that $(\theta, \theta) \in R_A$, that is, $\theta \leq \nabla_a$ or $\theta \leq \Delta_a$ for every $a \in A$. Let $a \in A$. If $a \wedge c = 0$ then $\nabla_c \leq \Delta_a$ and, consequently, $\theta \leq \Delta_a$. Otherwise, $a = a_j$ for some $j \in J$. If $j \in J_1$ then $\theta \leq \nabla_a$. If $j \in J_2$, $\theta \leq \Delta_a$. ■

Then, by Proposition 4.1, we immediately get

Proposition. *Every locally finite cover is a Fletcher cover.* ■

4.3. Open spectra are Fletcher covers.

Proposition. *Let $A = \{a_n \mid n \in \mathbb{Z}\}$ be an open spectrum. Then*

$$d(A) = \bigvee_{n\in\mathbb{Z}} (\nabla_{a_n} \wedge \Delta_{a_{n-1}}) = 1.$$

Proof: Clearly,

$$\begin{aligned} 1 &= (\bigvee_{n\in\mathbb{Z}} \nabla_{a_n}) \wedge (\bigvee_{m\in\mathbb{Z}} \Delta_{a_m}) = \bigvee_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{Z}} (\nabla_{a_n} \wedge \Delta_{a_m}) = \\ &= \bigvee_{n\in\mathbb{Z}} \bigvee_{m<n} (\nabla_{a_n} \wedge \Delta_{a_m}). \end{aligned}$$

But, for $m = n - k$,

$$\nabla_{a_n} \wedge \Delta_{a_m} = (\nabla_{a_n} \wedge \Delta_{a_{n-1}}) \vee (\nabla_{a_{n-1}} \wedge \Delta_{a_{n-2}}) \vee \cdots \vee (\nabla_{a_{n-k+1}} \wedge \Delta_{a_m}).$$

Hence $1 = \bigvee_{n\in\mathbb{Z}} (\nabla_{a_n} \wedge \Delta_{a_{n-1}})$.

Let $\mathbb{Z}_1 \cup \mathbb{Z}_2$ be a partition of $\mathbb{Z}$. If $\mathbb{Z}_1$ has a least element m then $\bigwedge_{n\in\mathbb{Z}_1} \nabla_{a_n} = \nabla_{a_m}$ and $\bigwedge_{n\in\mathbb{Z}_2} \Delta_{a_n} \leq \Delta_{a_{m-1}}$ so $(\bigwedge_{n\in\mathbb{Z}_1} \nabla_{a_n}) \wedge (\bigwedge_{n\in\mathbb{Z}_2} \Delta_{a_n}) \leq \nabla_{a_m} \wedge \Delta_{a_{m-1}}$. Otherwise, $\bigwedge_{n\in\mathbb{Z}_1} \nabla_{a_n} = \bigwedge_{n\in\mathbb{Z}} \nabla_{a_n} = 0$ and $(\bigwedge_{n\in\mathbb{Z}_1} \nabla_{a_n}) \wedge (\bigwedge_{n\in\mathbb{Z}_2} \Delta_{a_n}) = 0$. In conclusion,

$$d(A) \leq \bigvee_{n\in\mathbb{Z}} (\nabla_{a_n} \wedge \Delta_{a_{n-1}}).$$

The reverse inequality is trivial (just take $\mathbb{Z}_1 = {\downarrow}(n-1)$ and $\mathbb{Z}_2 = {\uparrow}n$). ■

4.4. Well-monotone covers are Fletcher covers. Let γ be an ordinal and let $\{a_\alpha \mid \alpha \in \gamma\}$ be a well-monotone cover of L (that is, $a_\alpha \leq a_\beta$ for every $\alpha, \beta \in \gamma$, $\alpha < \beta$).

Remark. If γ is a successor ordinal $\delta+1$ then, obviously, $a_\delta = \bigvee_{\alpha<\gamma} a_\alpha = 1$, that is, $1 = a_\delta \in A$.

Lemma. *Let $\delta < \gamma$, $\delta_1 \cup \delta_2 = \delta$. Then $\gamma \setminus \delta = \delta_1' \cup \delta_2'$ for some δ_1', δ_2' satisfying*

$$(\bigwedge_{\alpha\in\delta_1} \nabla_{a_\alpha}) \wedge (\bigwedge_{\alpha\in\delta_2} \Delta_{a_\alpha}) \leq (\bigwedge_{\alpha\in(\delta_1\cup\delta_1')} \nabla_{a_\alpha}) \wedge (\bigwedge_{\alpha\in(\delta_2\cup\delta_2')} \Delta_{a_\alpha}).$$

Proof: If $\delta_2 = 0$ then $(\bigwedge_{\alpha\in\delta_1} \nabla_{a_\alpha}) \wedge (\bigwedge_{\alpha\in\delta_2} \Delta_{a_\alpha}) = \bigwedge_{\alpha\in\delta} \nabla_{a_\alpha} = \nabla_{a_0} = \bigwedge_{\alpha\in\gamma} \nabla_{a_\alpha}$. So, in this case, take $\delta_1' = \gamma \setminus \delta_1$ and $\delta_2' = 0$.

If $\delta_2 \neq 0$ consider $a_{\beta_1} = \bigwedge_{\alpha\in\delta_1} a_\alpha$ and take $\delta_1' = \{\alpha \in \gamma \setminus \delta \mid \alpha \geq \beta\}$ and $\delta_2' = (\gamma \setminus \delta) \setminus \delta'$. Then $\bigwedge_{\alpha\in(\delta_1\cup\delta_1')} \nabla_{a_\alpha} = \nabla_{a_{\beta_1}} = \bigwedge_{\alpha\in\delta_1} \nabla_{a_\alpha}$. On the other hand, for each $\alpha \in \delta_2'$, $\alpha < \beta_1$ so $a_\alpha \leq a_{\beta_1}$ (and, moreover, $a_\alpha \leq a_{\beta_1-1}$ in case β_1 is a successor). Thus, for each $\alpha \in \delta_2'$,

$$\bigwedge_{\beta\in\delta_2} \Delta_{a_\beta} \leq \bigwedge_{\beta<\beta_1} \Delta_{a_\beta} = \begin{cases} \Delta_{a_{\beta_1-1}} \leq \Delta_{a_\alpha} & \text{if } \beta_1 \text{ is a successor} \\ \Delta_{a_{\beta_1}} \leq \Delta_{a_\alpha} & \text{if } \beta_1 \text{ is a limit.} \end{cases}$$

Hence $\bigwedge_{\beta\in\delta_2} \Delta_{a_\beta} \leq \bigwedge_{\alpha\in(\delta_2\cup\delta_2')} \Delta_{a_\alpha}$. ■

Finally, a transfinite induction gives

Proposition. *Each well-monotone cover is a Fletcher cover.*

Proof: We already know (Remark 4.1) that, if γ is finite, $d(A) = 1$.

If $\gamma = \delta + 1$ is a successor ordinal,

$$d(A) = \bigvee_{\gamma_1\cup\gamma_2=\gamma} ((\bigwedge_{\alpha\in\gamma_1} \nabla_{a_\alpha}) \wedge (\bigwedge_{\alpha\in\gamma_2} \Delta_{a_\alpha})) = \bigvee_{\delta_1\cup\delta_2=\delta} ((\bigwedge_{\alpha\in\delta_1} \nabla_{a_\alpha}) \wedge \nabla_{a_\delta} \wedge (\bigwedge_{\alpha\in\delta_2} \Delta_{a_\alpha})),$$

since $\delta \in \gamma_2$ implies $\bigwedge_{\alpha\in\gamma_2} \Delta_{a_\alpha} = 0$, by the Remark above. By induction, also $d(A) = 1$ in this case.

Finally, if $\gamma = \bigcup_{\delta<\gamma} \delta$ is a limit ordinal, by the Lemma we have

$$d(A) \geq \bigvee_{\delta\in\gamma} (\bigvee_{\delta_1\cup\delta_2=\delta} ((\bigwedge_{\alpha\in\delta_1} \nabla_{a_\alpha}) \wedge (\bigwedge_{\alpha\in\delta_2} \Delta_{a_\alpha}))) = 1.$$ ■

5. An important general procedure on the construction of transitive frame quasi-uniformities

5.1. In what follows let $(L_0, L_1, L_2) \in \mathsf{BiFrm}$ and let L_2' be a subframe of L_0 contained in L_2. Assume also that $\mathcal{S}$ is a family of transitive Weil entourages of L_0 which generates a filter $\mathcal{E}$ that satisfies the following conditions:

(Q1) For every $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F^2 \subseteq E$;
(Q2) $\mathcal{L}_1(\mathcal{E}) = L_1$;
(Q3) $\mathcal{L}_2(\mathcal{E}) = L_2'$.

If $L_2' = L_2$, this means that $\mathcal{E}$ is a quasi-uniformity on L_0. Otherwise, if $L_2' \subset L_2$, it need not be a quasi-uniformity; however, there is an easy way of obtaining a quasi-uniform frame by modifying $\mathcal{E}$ (and L_0):

Let L_0' denote the subframe of L_0 generated by $L_1 \vee L_2'$. Clearly, (L_0', L_1, L_2') is a biframe. Let

$$\mathcal{S}' = \{E \cap (L_0' \times L_0') \mid E \in \mathcal{S}\}.$$

Clearly, each $E' = E \cap (L_0' \times L_0')$ is a C-ideal of L_0. If $\mathcal{S}' \subseteq WEnt(L_0')$, denote by $\mathcal{E}'$ the corresponding filter of $WEnt(L_0')$.

Lemma. (a) *For every $x \in L_0'$ and every transitive $E \in \mathcal{E}$ such that $E' \in WEnt(L_0')$, $st_i(x, E) = st_i(x, E')$ $(i = 1, 2)$.*

(b) *If $\mathcal{S}' \subseteq WEnt(L_0')$ then, for every $x, y \in L_0'$, $x \overset{\mathcal{E}}{\triangleleft_i} y$ if and only if $x \overset{\mathcal{E}'}{\triangleleft_i} y$ $(i = 1, 2)$.*

Proof: (a) Consider $x \in L_0'$ and a transitive $E \in \mathcal{E}$ such that $E' = E \cap (L_0' \times L_0') \in WEnt(L_0)$. Obviously, $st_i(x, E') \leq st_i(x, E)$. Let us prove that $st_1(x, E) \leq st_1(x, E')$ (the case $i = 2$ may be shown in a similar way).

Since $E' \in WEnt(L_0')$,

$$x = \bigvee\{x \wedge a \mid x \wedge a \neq 0, (a, a) \in E'\}.$$

Therefore, by property (S5) of 2.1, we may write

$$st_1(x, E) = \bigvee\{st_1(x \wedge a, E) \mid x \wedge a \neq 0, (a, a) \in E'\}. \tag{5.1.1}$$

But, by (S6), $(x \wedge a, x \wedge a) \in E$ implies $(st_1(x \wedge a, E), x \wedge a) \in E \circ E = E$. On the other hand, again by the transitivity of E and by (S4),

$$st_1(st_1(x \wedge a, E), E) \leq st_1(x \wedge a, E).$$

Therefore $st_1(x\wedge a, E) \overset{\mathcal{E}}{\lhd_1} st_1(x\wedge a, E)$ and, consequently, $st_1(x\wedge a, E) \in \mathcal{L}_1(\mathcal{E}) = L_1 \subseteq L_0'$. Hence $(st_1(x \wedge a, E), x \wedge a) \in E'$, from which it follows that $st_1(x \wedge a, E) \leq st_1(x \wedge a, E')$. Consequently, (5.1.1) gives

$$st_1(x, E) \leq \bigvee\{st_1(x \wedge a, E') \mid x \wedge a \neq 0, (a, a) \in E'\} = st_1(x, E').$$

(b) It follows immediately from (a). ■

Theorem. *If* $\mathcal{S}' \subseteq WEnt(L_0')$ *then* $(L_0', \mathcal{E}')$ *is a quasi-uniform frame whose underlying biframe is* (L_0', L_1, L_2').

Proof: By hypothesis, every $E \cap (L_0' \times L_0')$, with E in $\mathcal{S}$, is a Weil entourage of L_0' so $\mathcal{E}'$ is filter of $WEnt(L_0')$. In order to prove that $(L_0', \mathcal{E}')$ is a quasi-uniform frame it remains to check that:

(1) For each $E' \in \mathcal{E}'$ there exists $F' \in \mathcal{E}'$ such that $F' \circ F' \subseteq E'$;
(2) $(L_0', \mathcal{L}_1(\mathcal{E}'), \mathcal{L}_2(\mathcal{E}')) \in \mathsf{Bifrm}$.

(1) It is easy: indeed, let $E' \in \mathcal{E}'$ and consider $E_1', \ldots, E_n' \in \mathcal{S}'$ such that $E_1' \cap \cdots \cap E_n' \subseteq E'$. Then, for each i, $E_i' = E_i \cap (L_0' \times L_0')$ for some $E_i \in \mathcal{S}$. But $E_1 \cap \cdots \cap E_n \in \mathcal{S}$. Therefore, by condition (Q1), there exists $F \in \mathcal{E}$ satisfying $F \circ F \subseteq E_1 \cap \cdots \cap E_n$. Then, obviously, for $F' = F \cap (L_0' \times L_0')$, we have $F' \circ F' \subseteq E_1' \cap \cdots \cap E_n' \subseteq E'$.
(2) In order to show (2) it suffices to prove that
 (2a) $\mathcal{L}_1(\mathcal{E}') = L_1$;
 (2b) $\mathcal{L}_2(\mathcal{E}') = L_2'$.
 (2a) By (Q2), it suffices to check that $\mathcal{L}_1(\mathcal{E}') = \mathcal{L}_1(\mathcal{E})$. If $x \in \mathcal{L}_1(\mathcal{E})$ then, by (1.3.1),

$$x = \bigvee\{y \in \mathcal{L}_1(\mathcal{E}) \mid y \overset{\mathcal{E}}{\lhd_1} x\} \leq \bigvee\{y \in L_0' \mid y \overset{\mathcal{E}}{\lhd_1} x\} \leq x.$$

 Hence, by the Lemma, $x = \bigvee\{y \in L_0' \mid y \overset{\mathcal{E}'}{\lhd_1} x\}$.
 The reverse inclusion follows from assertion (b) in the Lemma.
 (2b) It can be proved in a similar way, using (Q3). ■

6. The construction

6.1. We are finally in conditions to present our pointfree version of Fletcher's construction.

Let L be a frame. We say that a quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ (more generally, on a subframe of $\mathfrak{C}L$) is *compatible with* L whenever the "first topology" $\mathcal{L}_1(\mathcal{E})$ coincides with the frame $\nabla L \cong L$. It is our goal in this section to construct, for an arbitrary frame L, quasi-uniformities compatible with L.

Before doing that, some technical results are required.

Lemma 1. *Let A be a Fletcher cover. Then, for every $a \in A$ and $B \subseteq A$, we have:*

(a) $st_1(\nabla_a, R_A) = \nabla_a$;
(b) $st_2(\Delta_a, R_A) = \Delta_a$;
(c) $st_2(\Delta_{\bigvee B}, R_A) = \Delta_{\bigvee B}$.

Proof: (a) The fact that $\nabla_a \leq st_1(\nabla_a, R_A)$ follows from property (S2) of 2.1. On the other hand, for every $(\alpha, \beta) \in R_A$ with $\beta \wedge \nabla_a \neq 0$, $\beta \not\leq \Delta_a$ thus $\alpha \leq \nabla_a$. Hence $st_1(\nabla_a, R_A) \leq \nabla_a$.
(b) Similar to the proof of (a).
(c) Let $(\alpha, \beta) \in R_A$ such that $\alpha \wedge \Delta_{\bigvee B} \neq 0$. Then, for each $b \in B$, $\alpha \wedge \Delta_b \neq 0$, that is, $\alpha \not\leq \nabla_b$. Consequently, $\beta \leq \Delta_b$ for every $b \in B$, that is, $\beta \leq \bigwedge_{\beta \in B} \Delta_b = \Delta_{\bigvee B}$. ■

Lemma 2. *Let $\theta \in \mathfrak{C}L$ and let A_i $(i = 1, 2, \ldots, n)$ be covers of L. Then:*

(a) $st_1(\theta, \bigcap_{i=1}^n R_{A_i}) = \bigvee\{\bigwedge_{i=1}^n \bigwedge_{a \in B_i} \nabla_a \mid B_i \subseteq A_i, (\bigwedge_{i=1}^n \bigwedge_{a \in A \setminus B_i} \Delta_a) \wedge \theta \neq 0\}$.
(b) $st_2(\theta, \bigcap_{i=1}^n R_{A_i}) = \bigvee\{\bigwedge_{i=1}^n \bigwedge_{a \in B_i} \Delta_a \mid B_i \subseteq A_i, (\bigwedge_{i=1}^n \bigwedge_{a \in A \setminus B_i} \nabla_a) \wedge \theta \neq 0\}$.

Proof: (a) Let $(\alpha, \beta) \in \bigcap_{i=1}^n R_{A_i}$ such that $\beta \wedge \theta \neq 0$. Then, for every i and every $a \in A_i$, $\alpha \leq \nabla_a$ or $\beta \leq \Delta_a$. Let $B_i = \{a \in A_i \mid \alpha \leq \nabla_a\}$. Then $\alpha \leq \bigwedge_{i=1}^n \bigwedge_{a \in B_i} \nabla_a$. On the other hand, for every i and every $a \in A \setminus B_i$, $\beta \leq \Delta_a$. Thus $\beta \leq \bigwedge_{i=1}^n \bigwedge_{a \in A \setminus B_i} \Delta_a$, which implies $\theta \wedge \bigwedge_{i=1}^n \bigwedge_{a \in A \setminus B_i} \Delta_a \geq \theta \wedge \beta \neq 0$. This shows that

$$st_1(\theta, \bigcap_{i=1}^n R_{A_i}) \leq \bigvee\{\bigwedge_{i=1}^n \bigwedge_{a \in B_i} \nabla_a \mid B_i \subseteq A_i, (\bigwedge_{i=1}^n \bigwedge_{a \in A \setminus B_i} \Delta_a) \wedge \theta \neq 0\}.$$

For the reverse inequality, it suffices to observe that

$$(\bigwedge_{i=1}^{n} \bigwedge_{a\in B_i} \nabla_a, \bigwedge_{i=1}^{n} \bigwedge_{a\in A\setminus B_i} \Delta_a) \in \bigcap_{i=1}^{n} R_{A_i}.$$

(b) Similar. ■

It follows immediately from Lemma 2 that

Proposition. *Let $\theta \in \mathfrak{C}L$ and let A_i $(i = 1, 2, \dots, n)$ be covers of L. Then:*

(a) $st_1(\theta, \bigcap_{i=1}^{n} R_{A_i}) \in \nabla L$ *whenever each A_i is weakly interior-preserving.*
(b) $st_2(\theta, \bigcap_{i=1}^{n} R_{A_i}) \in \Delta L$. ■

6.2. Let $\mathcal{A}$ be a collection of weakly interior-preserving Fletcher covers of a given frame L and consider the family $\mathcal{S}_{\mathcal{A}} = \{R_A \mid A \in \mathcal{A}\}$ of Weil entourages of $\mathfrak{C}L$. We denote by $\mathcal{E}_{\mathcal{A}}$ the filter of $WEnt(\mathfrak{C}L)$ generated by $\mathcal{S}_{\mathcal{A}}$. Recall that a *subbase* of a frame L is a subset $S \subseteq L$ such that

$$x = \bigvee\{s_1 \wedge \cdots \wedge s_n \mid n \in \mathbb{N}, s_i \in S, s_1 \wedge \cdots \wedge s_n \leq x\}$$

for every $x \in L$.

Lemma. *If $\bigcup \mathcal{A}$ is a subbase for L then $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}}) = \nabla L$.*

Proof: Let $x \in L$. By hypothesis, we may write $x = \bigvee_{i\in I}(a_1^i \wedge \cdots \wedge a_{n_i}^i)$ for some $a_j^i \in \bigcup \mathcal{A}$ $(i \in I, j \in \{1, \dots, n_i\})$. Then

$$\nabla_x = \bigvee_{i\in I}(\nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i}).$$

So, in order to show that $\nabla_x \in \mathcal{L}_1(\mathcal{E}_{\mathcal{A}})$ it suffices to check that, for every i,

$$\nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i} \overset{\mathcal{E}_{\mathcal{A}}}{\lhd_1} \nabla_x.$$

For each i take $\bigcap_{j=1}^{n_i} R_{A_j^i} \in \mathcal{E}_{\mathcal{A}}$, where $a_j^i \in A_j^i \in \mathcal{A}$. Then

$$\begin{aligned} st_1(\nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i}, \bigcap_{j=1}^{n_i} R_{A_j^i}) &\leq \bigwedge_{j=1}^{n_i} st_1(\nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i}, R_{A_j^i}) \\ &\leq \bigwedge_{j=1}^{n_i} st_1(\nabla_{a_j^i}, R_{A_j^i}). \end{aligned}$$

Now, by Lemma 6.1.1(a), it follows that

$$st_1(\nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i}, \bigcap_{j=1}^{n_i} R_{A_j^i}) \leq \nabla_{a_1^i} \wedge \cdots \wedge \nabla_{a_{n_i}^i} \leq \nabla_x.$$

Finally, let us prove the reverse inclusion $\mathcal{L}_1(\mathcal{E}_\mathcal{A}) \subseteq \nabla L$. Let $\theta \in \mathcal{L}_1(\mathcal{E}_\mathcal{A})$, i.e., $\theta = \bigvee\{\alpha \in \mathfrak{C}L \mid \alpha \overset{\mathcal{E}_\mathcal{A}}{\triangleleft_1} \theta\}$. We only need to show that, for each such α, there exists $\nabla_x \in \nabla L$ satisfying $\alpha \leq \nabla_x \overset{\mathcal{E}_\mathcal{A}}{\triangleleft_1} \theta$ (since, then, θ is the join of all those ∇_x, which belongs to ∇L). So, let $\alpha \overset{\mathcal{E}_\mathcal{A}}{\triangleleft_1} \theta$. This means that there are $A_1, \ldots, A_n \in \mathcal{A}$ such that $\alpha \leq st_1(\alpha, \bigcap_{i=1}^n R_{A_i}) \leq \theta$. Since each A_i is weakly interior-preserving, Proposition 6.1(a) ensures us that $st_1(\alpha, \bigcap_{i=1}^n R_{A_i}) \in \nabla L$, as required. ∎

In view of this lemma, $\mathcal{E}_\mathcal{A}$ appears to be a good candidate for the compatible quasi-uniformity that we are looking for. However, there is a slight problem with $\mathcal{E}_\mathcal{A}$: the triple $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_\mathcal{A}), \mathcal{L}_2(\mathcal{E}_\mathcal{A}))$ need not be a biframe; in other words, $(\mathcal{L}_1(\mathcal{E}_\mathcal{A}) \vee \mathcal{L}_2(\mathcal{E}_\mathcal{A}), \mathcal{L}_1(\mathcal{E}_\mathcal{A}), \mathcal{L}_2(\mathcal{E}_\mathcal{A}))$ may not be the Skula biframe (in fact, in spite of $\mathcal{L}_1(\mathcal{E}_\mathcal{A}) = \nabla L$, $\mathcal{L}_2(\mathcal{E}_\mathcal{A})$ may not coincide with ΔL).

So, one should expect, for the biframe structure (L_0, L_1, L_2) induced by our quasi-uniformity, $L_1 = \nabla L \cong L$ but the second part L_2 to be, in general, a subframe of ΔL (and, consequently, L_0 to be the corresponding subframe of $\mathfrak{C}L$ generated by $\nabla L \cup L_2$).

Remarks. (1) This should not come as a surprise: in the one-sided approach to quasi-uniformities, where a quasi-uniformity is considered over a single underlying topology, one only cares about the first topology; more precisely, starting with a space $(X, \mathcal{T})$ and a collection $\mathcal{A}$ of "nice" covers, Fletcher constructed a quasi-uniformity $\mathcal{E}_\mathcal{A}$ by imposing only conditions on the first topology $\mathcal{T}_1(\mathcal{E}_\mathcal{A})$, which has to coincide with the given $\mathcal{T}$; then $\mathcal{T}_2(\mathcal{E}_\mathcal{A})$ and the corresponding bispace

$$(\mathcal{T}_1(\mathcal{E}_\mathcal{A}) \vee \mathcal{T}_2(\mathcal{E}_\mathcal{A}), \mathcal{T}_1(\mathcal{E}_\mathcal{A}), \mathcal{T}_2(\mathcal{E}_\mathcal{A}))$$

are automatically defined.
(2) On the other hand, we show in [7] that, by imposing the functoriality in our construction, the induced biframe $(\mathcal{L}_1(\mathcal{E}_\mathcal{A}) \vee \mathcal{L}_2(\mathcal{E}_\mathcal{A}), \mathcal{L}_1(\mathcal{E}_\mathcal{A}), \mathcal{L}_2(\mathcal{E}_\mathcal{A}))$ do coincide with the Skula biframe. This is the pointfree expression of the classical fact, due to Salbany [17], that for any functorial quasi-uniformity

F on the topological spaces, the join of the two topologies generated by the quasi-uniformity of $F(X,\mathcal{T})$ is precisely the Skula topology

$$\mathcal{T}(\mathcal{E}_{\mathcal{P}}(\mathcal{T}))\vee\mathcal{T}(\mathcal{E}_{\mathcal{P}}(\mathcal{T})^{-1}).$$

This is the reason why, in all our guiding examples, as we shall see in Section 8, $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}})=\Delta L$ and $(\mathfrak{C}L,\mathcal{L}_1(\mathcal{E}_{\mathcal{A}}),\mathcal{L}_2(\mathcal{E}_{\mathcal{A}}))$ is already the Skula biframe and we get a compatible quasi-uniformity on $\mathfrak{C}L$.

Here, in the general case, without assuming functoriality, we may have to go to a subframe of $\mathfrak{C}L$. Theorem 5.1 gives us the way to modify $\mathcal{E}_{\mathcal{A}}$ in order to get a compatible quasi-uniformity, as we shall see below.

6.3. Let $\mathcal{A}$ be a collection of weakly interior-preserving Fletcher covers of L such that $\bigcup\mathcal{A}$ is a subbase for L and let $\mathfrak{C}L'$ be the subframe of $\mathfrak{C}L$ generated by $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}})\cup\mathcal{L}_2(\mathcal{E}_{\mathcal{A}})=\nabla L\cup\mathcal{L}_2(\mathcal{E}_{\mathcal{A}})$. Note that, by Lemma 6.1.1(b),

$$\{\Delta_a\mid a\in\bigcup\mathcal{A}\}\subseteq\mathcal{L}_2(\mathcal{E}_{\mathcal{A}}),$$

and, therefore, $\mathfrak{C}L'$ contains ∇L and $\{\Delta_a\mid a\in\bigcup\mathcal{A}\}$. Then consider

$$R'_A=R_A\cap(\mathfrak{C}L'\times\mathfrak{C}L')$$

and

$$\mathcal{S}'_{\mathcal{A}}=\{R'_A\mid A\in\mathcal{A}\}.$$

Lemma. *Each R'_A is a Weil entourage of $\mathfrak{C}L'$.*

Proof: Let us denote by $\bigsqcup_{i\in I}\theta_i$ and $\bigsqcap_{i\in I}\theta_i$, respectively, the joins and meets in $\mathfrak{C}L'$. Of course, $\bigsqcup_{i\in I}\theta_i=\bigvee_{i\in I}\theta_i$ but, in general, $\bigsqcap_{i\in I}\theta_i\le\bigwedge_{i\in I}\theta_i$.

As for R_A (recall Proposition 4.1), $R'_A\in WEnt(\mathfrak{C}L')$ if and only if

$$\bigsqcup_{A_1\cup A_2=A}((\bigsqcap_{a\in A_1}\nabla_a)\sqcap(\bigsqcap_{a\in A_2}\Delta_a))=1.$$

Since each A is weakly interior-preserving, $\bigwedge_{a\in A_1}\nabla_a=\nabla_{\bigwedge A_1}\in\nabla L\subseteq\mathfrak{C}L'$. Therefore $\bigsqcap_{a\in A_1}\nabla_a=\bigwedge_{a\in A_1}\nabla_a$. On the other hand, $\bigsqcap_{a\in A_2}\Delta_a=\bigwedge_{a\in A_2}\Delta_a$ because, by Lemma 6.1.1(c), $\bigwedge_{a\in A_2}\Delta_a=\Delta_{\bigvee A_2}$ belongs to $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}})\subseteq\mathfrak{C}L'$.

In conclusion,

$$\bigsqcup_{A_1\cup A_2=A}((\bigsqcap_{a\in A_1}\nabla_a)\sqcap(\bigsqcap_{a\in A_2}\Delta_a))=\bigvee_{A_1\cup A_2=A}((\bigwedge_{a\in A_1}\nabla_a)\wedge(\bigwedge_{a\in A_2}\Delta_a))=d(A)=1.$$

■

Thus $\mathcal{S}'_{\mathcal{A}} \subseteq WEnt(\mathfrak{C}L')$. Let $\mathcal{E}'_{\mathcal{A}}$ denote the filter of $WEnt(\mathfrak{C}L')$ generated by $\mathcal{S}'_{\mathcal{A}}$.

Theorem. *Let $\mathcal{A}$ be a nonempty family of weakly interior-preserving Fletcher covers of L such that $\bigcup \mathcal{A}$ is a subbase for L. Then $\mathcal{E}'_{\mathcal{A}}$ is a transitive quasi-uniformity on $\mathfrak{C}L'$, compatible with L.*

Proof: We know already that $\mathcal{S}'_{\mathcal{A}} \subseteq WEnt(\mathfrak{C}L')$. So, we may apply Theorem 5.1 and conclude that $\mathcal{E}'_{\mathcal{A}}$ is a transitive quasi-uniformity on $\mathfrak{C}L'$, whose underlying biframe is $(\mathfrak{C}L', \nabla L, \mathcal{L}_2(\mathcal{E}_{\mathcal{A}}))$, after we check that $\mathcal{S}_{\mathcal{A}}$ satisfies conditions (Q1)-(Q3):

(Q1) For every $E \in \mathcal{E}_{\mathcal{A}}$ there exists $F \in \mathcal{E}_{\mathcal{A}}$ such that $F \circ F \subseteq E$;
(Q2) $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}}) = \nabla L$;
(Q3) $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}}) \subseteq \Delta L$.

Condition (Q1) is trivial because, for each $A \in \mathcal{A}$, R_A is transitive.

Condition (Q2) was already proved in Lemma 6.2.

(Q3): Let $\theta \in \mathcal{L}_2(\mathcal{E}_{\mathcal{A}})$, that is, $\theta = \bigvee\{\alpha \in \mathfrak{C}L \mid \alpha \overset{\mathcal{E}_{\mathcal{A}}}{\lhd_2} \theta\}$. It is evident that it suffices to show that, for every such α, there exists $\beta \in \Delta L$ satisfying $\alpha \leq \beta \overset{\mathcal{E}_{\mathcal{A}}}{\lhd_2} \theta$. The existence of such β is guaranteed by Proposition 6.1; indeed, take $A_1, \ldots, A_n \in \bigcup \mathcal{A}$ for which $\alpha \leq st_2(\alpha, \bigcap_{i=1}^n R_{A_i}) \leq \theta$. The element $st_2(\alpha, \bigcap_{i=1}^n R_{A_i})$ belongs to ΔL by Proposition 6.1(b). ■

We note that our construction could be performed in any strictly zero-dimensional biframe (L_0, L_1, L_2) satisfying $L_1 \overset{\varphi}{\cong} L$, instead of the Skula biframe. In that approach we have to take as "interior-preserving covers" all covers A of L such that, for every $B \subseteq A$:

(a) the meet of $\varphi[B]$ in L_0, $\bigwedge_{b\in B} \varphi(b)$, belongs to L_1 (this condition defines the "weakly interior-preserving covers" in this context);
(b) $\bigvee_{b\in B} \neg\varphi(b) = \neg \bigwedge_{b\in B} \varphi(b)$ (i.e., the second De Morgan law holds in L_0 inside each cover $\varphi[A]$).

7. The construction accounts for all transitive quasi-uniformities

7.1. Let $\mathcal{E}$ be a transitive quasi-uniformity on a subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$, compatible with L, and consider a transitive subbase $\mathcal{S}$ of $\mathcal{E}$. Since each $E \in \mathcal{S}$ is transitive,

$$st_i(\theta, E) \overset{\mathcal{E}}{\lhd_i} st_i(\theta, E) \text{ for every } \theta \in \mathfrak{C}L' \qquad (i = 1, 2).$$

Therefore, $st_1(\theta, E) \in \mathcal{L}_1(\mathcal{E}) = \nabla L$ and $st_2(\theta, E) \in \mathcal{L}_2(\mathcal{E})$. In particular, this implies that $st_1(\theta, E) = \nabla_{E[\theta]}$ for some element $E[\theta] \in L$. Set

$$CovE = \{E[\theta] \mid (\theta, \theta) \in E\}.$$

Proposition. *For each $E \in \mathcal{S}$, we have:*

(a) *$CovE$ is a weakly interior-preserving cover of L;*
(b) *$R_{CovE} = E$.*

Proof: (a) First, let us see that $CovE$ is a cover of L. Consider $(\theta, \theta) \in E$. Since $\theta = \bigvee\{\nabla_y \wedge \Delta_x \mid (x, y) \in \theta, x \leq y\}$,

$$\bigvee\{\nabla_y \wedge \Delta_x \mid (x, y) \in \theta, x \leq y, (\theta, \theta) \in E\} = 1. \tag{7.1.1}$$

Further, since each pair $(\nabla_y \wedge \Delta_x, \nabla_y \wedge \Delta_x)$ belongs to E,

$$\bigvee_{(\theta,\theta)\in E} E[\theta] \geq \bigvee\{E[\nabla_y \wedge \Delta_x] \mid (x, y) \in \theta, x \leq y, (\theta, \theta) \in E\}. \tag{7.1.2}$$

On the other hand, $\nabla_y \wedge \Delta_x \leq st_1(\nabla_y \wedge \Delta_x, E) = \nabla_{E[\nabla_y \wedge \Delta_x]}$. So, by (7.1.1),

$$\bigvee\{\nabla_{E[\nabla_y \wedge \Delta_x]} \mid (x, y) \in \theta, x \leq y, (\theta, \theta) \in E\} = 1,$$

that is,

$$\bigvee\{E[\nabla_y \wedge \Delta_x] \mid (x, y) \in \theta, x \leq y, (\theta, \theta) \in E\} = 1.$$

Hence, by (7.1.2), $\bigvee_{(\theta,\theta)\in E} E[\theta] = 1$.

Next we are going to prove that $CovE$ is weakly interior-preserving, that is,

$$\bigwedge_{\theta\in C} \nabla_{E[\theta]} \in \nabla L, \text{ for every } C \subseteq \{\theta \mid (\theta, \theta) \in E\}.$$

Since $\nabla L = \mathcal{L}_1(\mathcal{E})$, it suffices to show that $st_1(\bigwedge_{\theta\in C} \nabla_{E[\theta]}, E) \leq \bigwedge_{\theta\in C} \nabla_{E[\theta]}$:

Let $(\alpha, \beta) \in E$ with $\beta \wedge \bigwedge_{\theta\in C} \nabla_{E[\theta]} \neq 0$. Then, for every $\theta \in C$, $\beta \wedge st_1(\theta, E) \neq 0$, which is easily seen to be equivalent to $st_2(\beta, E) \wedge \theta \neq 0$. But, by (S6), $(\alpha, st_2(\beta, E)) \in E^2 = E$, thus $\alpha \leq st_1(\theta, E)$.

(b) Let $(\alpha, \beta) \in R_{CovE} = \bigcap_{(\theta,\theta)\in E}(\nabla_{E[\theta]} \oplus 1) \cup (1 \oplus \Delta_{E[\theta]})$. Since

$$\beta = \bigvee\{\beta \wedge \theta \mid (\theta, \theta) \in E, \beta \wedge \theta \neq 0\},$$

checking that $(\alpha, \beta\wedge\theta) \in E$, for every such θ, is sufficient to conclude that $(\alpha, \beta) \in E$. So, consider $(\theta, \theta) \in E$ such that $\beta\wedge\theta \neq 0$. By hypothesis, $\alpha \leq \nabla_{E[\theta]} = st_1(\theta, E)$ or $\beta \leq \Delta_{E[\theta]} = \neg st_1(\theta, E)$. However, the latter is impossible because $\neg st_1(\theta, E) \leq \theta^*$ and $\beta\wedge\theta \neq 0$. Therefore, $\alpha \leq st_1(\theta, E)$ and then $(\alpha, \beta\wedge\theta) \leq (st_1(\theta, E), \theta) \in E^2 = E$.

To prove the reverse inclusion, consider $(\alpha, \beta) \in E$ and $(\theta, \theta) \in E$ for which $\beta \not\leq \Delta_{E[\theta]}$. Then $\beta\wedge\nabla_{E[\theta]} \neq 0$, that is, there exists $(\gamma_1, \gamma_2) \in E$ such that $\gamma_2\wedge\theta \neq 0$ and $\beta\wedge\gamma_1 \neq 0$. This implies $(\alpha, \gamma_2) \in E^2 = E$ and, consequently, $\alpha \leq st_1(\theta, E) = \nabla_{E[\theta]}$. ∎

It follows from (b) that $CovE$ is always a Fletcher cover of L.

7.2. Property (S3) asserts that $st_1(x, E\cap F) \leq st_1(x, E)\wedge st_1(x, F)$. However, if $(x, x) \in E$ and $(y, y) \in F$, with $x\wedge y \neq 0$, and E and F are transitive, the equality

$$st_1(x\wedge y, E\cap F) = st_1(x, E)\wedge st_1(y, F)$$

holds: indeed, if $(\alpha_1, \alpha_2) \in E$ is such that $\alpha_2\wedge x \neq 0$ and $(\beta_1, \beta_2) \in F$ is such that $\beta_2\wedge y \neq 0$, $(\alpha_1, x) \in E^2 = E$ and $(\beta_1, y) \in F^2 = F$ and, therefore, $(\alpha_1\wedge\beta_1, x\wedge y) \in E\cap F$, which proves the inequality $st_1(x\wedge y, E\cap F) \geq st_1(x, E)\wedge st_1(y, F)$ (the reverse one is trivial).

It follows, in particular, that, for every $(x, x) \in E\cap F$,

$$st_1(x, E\cap F) = st_1(x, E)\wedge st_1(x, F). \tag{7.2.1}$$

Proposition. *$\bigcup_{E\in\mathcal{S}} CovE$ is a subbase for L.*

Proof: For every $x \in L$, since $\nabla_x \in \nabla L = \mathcal{L}_1(\mathcal{E})$, we have $\nabla_x = \bigvee\{\nabla_y \mid \nabla_y \overset{\mathcal{E}}{\lhd_1} \nabla_x\}$. For each such y there exist $E_1, E_2, \dots, E_n \in \mathcal{S}$ satisfying

$$\nabla_y \leq st_1(\nabla_y, \bigcap_{i=1}^{n} E_i) \leq \nabla_x.$$

The fact that $\bigcap_{i=1}^{n} E_i$ is a Weil entourage implies that $\nabla_y = \bigvee\{\nabla_y\wedge\theta \mid (\theta, \theta) \in \bigcap_{i=1}^{n} E_i\}$. Thus, using (7.2.1),

$$st_1(\nabla_y, \bigcap_{i=1}^{n} E_i) = \bigvee\{\bigwedge_{i=1}^{n} st_1(\nabla_y\wedge\theta, E_i) \mid (\theta, \theta) \in \bigcap_{i=1}^{n} E_i\}$$

and we may conclude that

$$\nabla_x = \bigvee\{\bigwedge_{i=1}^{n} st_1(\alpha, E_i) \mid n \in \mathbb{N}, E_i \in \mathcal{S}, (\alpha,\alpha) \in E_i, \bigwedge_{i=1}^{n} st_1(\alpha, E_i) \leq \nabla_x\}$$

i.e.

$$x = \bigvee\{\bigwedge_{i=1}^{n} E_i[\alpha] \mid n \in \mathbb{N}, E_i \in \mathcal{S}, (\alpha,\alpha) \in E_i, \bigwedge_{i=1}^{n} E_i[\alpha] \leq x\},$$

where each $E_i[\alpha] \in CovE_i$. ■

7.3. Proposition 7.1.(a) may be improved with the help of the following lemma.

Lemma. *For every $E \in \mathcal{S}$ and every $(\theta,\theta) \in E$, $st_2(\Delta_{E[\theta]}, E) = \Delta_{E[\theta]}$.*

Proof: Consider $(\alpha,\beta) \in E$ with $\alpha \wedge \Delta_{E[\theta]} \neq 0$. By Proposition 7.1(b), $(\alpha,\beta) \in R_{CovE}$, so $\beta \leq \Delta_{E[\theta]}$. ■

In particular, this implies that $\Delta_{E[\theta]} \in \mathcal{L}_2(\mathcal{E})$. Moreover, by (S5),

$$st_2(\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]}, E) = \bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]} \quad \text{for every } F \subseteq E. \tag{7.3.1}$$

Proposition. *For each $E \in \mathcal{S}$, $CovE$ is interior-preserving.*

Proof: We need to prove that

$$\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]} = \Delta_{\bigwedge_{(\theta,\theta)\in F} E[\theta]} \quad \text{for every } F \subseteq E.$$

By (7.3.1),

$$\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]} \overset{\mathcal{E}}{\triangleleft}_2 \bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]}.$$

Then (recall 2.1)

$$\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]} \prec_2 \bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]}$$

or, equivalently, $\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]}$ is complemented. On the other hand, since $CovE$ is weakly interior-preserving, $\bigwedge_{(\theta,\theta)\in F} \nabla_{E[\theta]}$ is also complemented. Thus $\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]}$ and $\bigwedge_{(\theta,\theta)\in F} \nabla_{E[\theta]}$ are complemented to each other and, in conclusion,

$$\bigvee_{(\theta,\theta)\in F} \Delta_{E[\theta]} = \neg(\bigwedge_{(\theta,\theta)\in F} \nabla_{E[\theta]}) = \neg(\nabla_{\bigwedge_{(\theta,\theta)\in F} E[\theta]}) = \Delta_{\bigwedge_{(\theta,\theta)\in F} E[\theta]}.$$ ■

7.4. We say that a set $\mathcal{A}$ of covers of L *induces* a quasi-uniformity $\mathcal{E}$ if $\{R_A \mid A \in \mathcal{A}\}$ is a subbase for $\mathcal{E}$. Now we are able to establish

Theorem 1. *Every compatible transitive quasi-uniformity on a subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$ is induced by a set $\mathcal{A}$ of interior-preserving Fletcher covers of L such that $\bigcup \mathcal{A}$ is a subbase for L.*

Proof: Let $\mathcal{S}$ be any subbase of transitive entourages for the quasi-uniformity and take $\mathcal{A} = \{CovE \mid E \in \mathcal{S}\}$. It follows immediately from Propositions 7.1, 7.2 and 7.3 that $\mathcal{A}$ has the required properties. ■

Remarks. (1) If we start with a set $\mathcal{A}$ of weakly interior-preserving covers of L, Theorem 6.3 gives us a compatible quasi-uniformity $\mathcal{E}_{\mathcal{A}}$ on some subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$. Then, by Theorem 1 above, $\mathcal{B} = \{CovR_A \mid A \in \mathcal{A}\}$ is a set of interior-preserving covers of L inducing the same quasi-uniformity as the given $\mathcal{A}$.
(2) Many different $\mathcal{A}$ may induce the same quasi-uniformity. The result below gives us the construction for the largest $\mathcal{A}$ that induces $\mathcal{E}$. This construction is very useful in the functorial study of transitive quasi-uniformities that we pursue in the forthcoming paper [7].

Theorem 2. *Let $\mathcal{E}$ be a compatible transitive quasi-uniformity on a subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$ and let*

$$\mathcal{A} = \{A \mid A \in CovL \text{ and } R_A \in \mathcal{E}\}.$$

Then:

(a) *$\mathcal{A}$ is the largest set of covers of L that induces $\mathcal{E}$;*
(b) *Every $A \in \mathcal{A}$ is a weakly interior-preserving Fletcher cover of L;*
(c) *$\bigcup \mathcal{A}$ is a base for L.*

Proof: (a) First note that $\{R_A \mid A \in \mathcal{A}\}$ is closed under finite intersections, by Lemma 4.1. For each $E \in \mathcal{E}$, there exists a transitive $F \in \mathcal{E}$ satisfying $F \subseteq E$. By Proposition 7.1(b), $R_{CovF} = F \subseteq E$. Since $CovF \in \mathcal{A}$, this implies that $\{R_A \mid A \in\}$ is a base for $\mathcal{E}$, so $\mathcal{A}$ induces $\mathcal{E}$.

It is clear that $\mathcal{A}$ is the largest such set of covers of L.

(b) Let $A \in \mathcal{A}$. Since each R_A belongs to $\mathcal{E}$, then it is a Weil entourage, that is, A is a Fletcher cover of L. Further, for each $B \subseteq A$,

$$st_1(\bigwedge_{a \in B} \nabla_a, R_A) \leq \bigwedge_{a \in B} \nabla_a$$

(by Lemma 6.1.1(a)) and, consequently, $\bigwedge_{a\in B}\nabla_a \in \mathcal{L}_1(\mathcal{E}) = \nabla L$ and A is weakly interior-preserving.

(c) Since $\{CovR_A \mid A \in \mathcal{A}\} \subseteq \mathcal{A}$, it follows from Proposition 7.2 that $\bigcup\mathcal{A}$ is a subbase for L. But, by Lemma 4.1, $\bigcup\mathcal{A}$ is closed under finite meets (in fact, for every $a \in A \in \mathcal{A}$ and $b \in B \in \mathcal{B}$, $a \wedge b \in A \wedge B$ and $R_{A\wedge B} = R_A \cap R_B \in \mathcal{E}$ so $a \wedge b \in \bigcup\mathcal{A}$). Hence $\bigcup\mathcal{A}$ is a base for L. ∎

8. Examples and applications

In closing, we describe various examples and applications of the construction presented here.

8.1. Let $\mathcal{A}$ be one of the following collections of covers:

(1) finite covers;
(2) locally finite covers;
(3) well-ordered covers;
(4) interior-preserving Fletcher covers;
(5) open spectra.

In each case, since $\mathcal{A}$ contains all finite covers, we have $\mathcal{L}_2(\mathcal{E}_\mathcal{A}) = \Delta L$. Indeed: for each $x \in L$, consider $A_x = \{x, 1\} \in \mathcal{A}$, in cases (1)-(4), or, in case (5), $A_x = \{a_n \mid n \in \mathbb{Z}\}$ with $a_n = 0$ if $n < 0$, $a_0 = x$ and $a_n = 1$ if $n > 0$. Then, by Lemma 6.1.1(b), $st_2(\Delta_x, R_{A_x}) = \Delta_x$ and, consequently, $\Delta_x \in \mathcal{L}_2(\mathcal{E}_\mathcal{A})$.

Hence, in each case, $\mathcal{E}'_\mathcal{A} = \mathcal{E}_\mathcal{A}$ is a quasi-uniformity on $\mathfrak{C}L$, whose underlying biframe is the Skula biframe.

8.2. Quasi-uniformities $\mathcal{LF}$ and $\mathcal{W}$. In case (2) (resp. (3)) $\mathcal{E}_\mathcal{A}$ is called the *locally finite* (resp. *well-monotone*) *covering quasi-uniformity* and is denoted by $\mathcal{LF}$ (resp. $\mathcal{W}$).

8.3. The fine transitive quasi-uniformity $\mathcal{FT}$. Theorem 7.4.1 gives us immediately:

Corollary. *Let $\mathcal{A}$ be the collection of all interior-preserving Fletcher covers of L. Then $\mathcal{E}_\mathcal{A}$ is the finest transitive quasi-uniformity on $\mathfrak{C}L$ compatible with L.* ∎

The finest transitive quasi-uniformity for $\mathfrak{C}L$, whose existence is guaranteed by the corollary above, is denoted by $\mathcal{FT}$ and is called the *fine transitive quasi-uniformity*.

8.4. The Frith quasi-uniformity $\mathcal{F}$. Let (L_0, L_1, L_2) be a strictly zero-dimensional biframe. By Theorem 5.5 of [12], the family $\{(a\oplus 1)\cup(1\oplus\neg a) \mid a \in L_1\}$ is a subbase for a transitive, totally bounded, quasi-uniformity on L_0, called the *Frith quasi-uniformity* on L_0. Clearly, the Frith quasi-uniformity $\mathcal{F}$ on $\mathfrak{C}L$ can be obtained from the construction given in Theorem 6.3.

Proposition. *Let L be a frame and let $\mathcal{A}$ be the collection of all finite covers of L. Then $\mathcal{E}_{\mathcal{A}} = \mathcal{F}$.* ■

This is a totally bounded quasi-uniformity. Next result characterizes all $\mathcal{E}_{\mathcal{A}}$ that are totally bounded.

Theorem. *Let L be a frame and let $\mathcal{A}$ be a set of interior-preserving Fletcher covers such that $\bigcup\mathcal{A}$ is a subbase for L. Then $\mathcal{E}_{\mathcal{A}}$ is totally bounded if and only if each $A \in \mathcal{A}$ is finite.*

Proof: If $A \in \mathcal{A}$ is finite then R_A is a finite entourage. Indeed, if $A = \{a_1, \ldots, a_n\}$ then, clearly,

$$\bigvee\{(\bigwedge_{i\in I_1}\nabla_{a_i} \wedge \bigwedge_{i\in I_2}\Delta_{a_i}) \oplus (\bigwedge_{i\in I_1}\nabla_{a_i} \wedge \bigwedge_{i\in I_2}\Delta_{a_i}) \mid I_1 \cup I_2 = \{1, \ldots, n\}\} \subseteq R_A$$

and

$$\{\bigwedge_{i\in I_1}\nabla_{a_i} \wedge \bigwedge_{i\in I_2}\Delta_{a_i} \mid I_1 \cup I_2 = \{1, \ldots, n\}\}$$

is a finite cover of $\mathfrak{C}L$. Therefore, since $\{R_A \mid A \in \mathcal{A}\}$ is a subbase for $\mathcal{E}_{\mathcal{A}}$, this is a totally bounded quasi-uniformity whenever each $A \in \mathcal{A}$ is finite.

Conversely, let $\mathcal{E}_{\mathcal{A}}$ be totally bounded. This means that there exists a finite cover $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{C}L$ such that $\bigvee_{i=1}^{n}(\alpha_i \oplus \alpha_i) \subseteq R_A$. Let $a \in A$. By Lemma 6.1.1, $\nabla_a = st_1(\nabla_a, R_A)$, that is, $R_A[\nabla_a] = a$. Thus, by the claim below, A is contained in

$$\{R_A[\bigvee_{i\in I}\alpha_i] \mid I \subseteq \{1, \ldots, n\}\},$$

which is finite.

Claim. *Let $\theta \in \mathfrak{C}L$, $\theta \neq 0$, and let $I_\theta = \{i \in \{1, \ldots, n\} \mid \theta \wedge \alpha_i \neq 0\} \neq \emptyset$. Then $R_A[\theta] = R_A[\bigvee_{i\in I_\theta}\alpha_i]$.*

Proof of the claim. We need to show that $st_1(\theta, R_A) = st_1(\bigvee_{i\in I_\theta}\alpha_i, R_A)$. Let $(\alpha, \beta) \in R_A$ with $\beta \wedge \theta \neq 0$. Then, since $\{\alpha_1, \ldots, \alpha_n\}$ is a cover, there exists $i \in I_\theta$ such that $\beta \wedge \theta \wedge \alpha_i \neq 0$, from which it follows that $\alpha \leq st_1(\alpha_i, R_A) \leq$

$st_1(\bigvee_{i\in I_\theta}\alpha_i, R_A)$. For the reverse inequality, since each (α_i,α_i) belongs to R_A, $\alpha_i \le st_1(\theta, R_A)$ for every $i \in I_\theta$. Thus, for every $i\in I_\theta$,

$$st_1(\alpha_i, R_A) \le st_1(st_1(\theta, R_A), R_A) = st_1(\theta, R_A^2) = st_1(\theta, R_A). \qquad \blacksquare$$

Corollary. *Let L be a frame. Then $\mathcal{F} = \mathcal{FT}$ if and only if every weakly interior-preserving Fletcher cover of L is finite.* $\blacksquare$

8.5. The semi-continuous quasi-uniformity $\mathcal{SC}$. Let $\mathcal{L}(\mathbb{R})$ denote the *frame of reals* [1]. This frame carries several natural quasi-uniformities; one of such, that we denote by $\mathcal{Q}$, is generated by the entourages

$$Q_n = \bigvee\{(p,-)\oplus(-,q) \mid p,q\in\mathbb{Q}, 0<q-p<\frac{1}{n}\} \qquad (n\in\mathbb{N}).$$

The first subframe $\mathcal{L}_1(\mathcal{Q})$ is the "lower frame of reals" $\mathcal{L}_l(\mathbb{R})$, that is, the subframe of $\mathcal{L}(\mathbb{R})$ generated by elements $(p,-) = \bigvee\{(p,q)\mid q\in\mathbb{Q}\}$. In fact, the underlying biframe of $(\mathcal{L}(\mathbb{R}),\mathcal{Q})$ is the *biframe of reals* [14]

$$\mathcal{L}(\mathbb{R}_b) = (\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})).$$

Recall that a map $f:(X,\mathcal{T})\to\mathbb{R}$ is *lower semi-continuous* if $f:(X,\mathcal{T})\to(\mathbb{R},l)$ is continuous, where l denotes the lower topology $\{(a,\infty)\mid a\in\mathbb{R}\}$. This motivates us to adopt the following definition: a *lower semi-continuous real function* on a frame L is a frame homomorphism $\mathcal{L}_l(\mathbb{R})\to L$.

Let $\mathcal{SC}$ be the coarsest quasi-uniformity on $\mathfrak{C}L$ for which each lower semi-continuous real function $h:\mathcal{L}_l(\mathbb{R})\to L$ (more precisely, each $\mathcal{L}_l(\mathbb{R}) \xrightarrow{h} L \xrightarrow{\nabla_L} \nabla L$) extends uniquely to a continuous real function $\overline{h}:\mathcal{L}(\mathbb{R})\to\mathfrak{C}L$ that is a uniform homomorphism $\overline{h}:(\mathcal{L}(\mathbb{R}),\mathcal{Q})\to(\mathfrak{C}L,\mathcal{SC})$. (We omit the description of the basic entourages of $\mathcal{SC}$, which can be given in terms of the lower semi-continuous real functions h and $n\in\mathbb{N}$.)

$\mathcal{SC}$ is transitive and can be obtained by our construction of Theorem 6.3:

Theorem. *Let $\mathcal{A}$ be the collection of all open spectra in L. Then $\mathcal{E}_\mathcal{A} = \mathcal{SC}$.*

The details (which are rather long and technical) and some ramifications of this will appear elsewhere.

8.6. In conclusion, we have the table:

Quasi-uniformity	Subbase
$\mathcal{FT}$	$\{R_A \mid A$ interior-preserving Fletcher cover of $L\}$
$\mathcal{F}$	$\{R_A \mid A$ finite cover of $L\}$
$\mathcal{LF}$	$\{R_A \mid A$ locally finite cover of $L\}$
$\mathcal{W}$	$\{R_A \mid A$ cover of $L,$ well-ordered by $\leq\}$
$\mathcal{SC}$	$\{R_A \mid A$ open spectrum of $L\}$

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