Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 03–23

COVARIANCE OF THE LIMIT EMPIRICAL PROCESS UNDER ASSOCIATION: CONSISTENCY AND RATES FOR THE HISTOGRAM

CARLA HENRIQUES AND PAULO EDUARDO OLIVEIRA

Abstract: The empirical process induced by a sequence of associated random variables has for limit in distribution a centered Gaussian process with covariance function defined by an infinite sum of terms of the form $\varphi_k(s,t) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - F(s)F(t)$. We study the estimation of such series using the histogram estimator. Under a convenient decrease rate on the covariance structure of the variables we prove the strong consistency with rates, pointwise and uniformly, of the estimator of the covariance of the limit empirical process. We also study the estimation of the eigenvalues of the integral operator defined by this limit covariance function. The knowledge of these eigenvalues is relevant for the characterization of tail probabilities of some functionals of the empirical process. We approximate the eigenvalues by those of the integral operator defined by the estimator of the limit covariances and prove, under the same assumptions as for the estimation of this covariance, the strong consistency of such estimators, with rates.

Keywords: histogram estimation, association, empirical process, convergence rates.
AMS Subject Classification (2000): 62G05, 62G20, 62G30.

1. Introduction

The empirical process based on a sequence X_n, $n \geq 1$, of independent and identically distributed random variables converges in distribution, as it is well known, to a centered Gaussian process with covariance function $\Gamma(s,t) = F(s \wedge t) - F(s)F(t)$, where F is the common distribution function of the variables X_n. For dependent sequences, under convenient control on the dependence structure, the limit empirical process is also a centered Gaussian process but with a more complex covariance function, which reflects the presence of dependence between the original random variables. This covariance function envolves the unknown terms $\varphi_k(s,t) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - F(s)F(t)$ and is given by

$$\Gamma(s,t) = F(s \wedge t) - F(s)F(t) + \sum_{k=1}^{\infty} \Big(\varphi_k(s,t) + \varphi_k(t,s) \Big) . \tag{1.1}$$

Both authors partially supported by Centro de Matemática da Universidade de Coimbra, Fundação para a Ciência e Tecnologia (FCT) and POCTI.

Having in mind the characterization of tail probabilities of some functionals of the empirical process, such as the Cramér-von Mises test statistic, we need to estimate the covariance function $\Gamma(s,t)$ given by (1.1) and its eigenvalues. In this article we will suppose that the random variables are associated, as introduced by Esary, Proschan and Walkup [4]: the random variables X_n, $n \geq 1$, are said to be associated if

$$\mathrm{Cov}\Big(f\left(X_1,...,X_n\right),\, g\left(X_1,...,X_n\right)\Big) \geq 0\,,$$

for any $n \in \mathbb{N}$ and any real-valued coordinatewise increasing functions f and g for which the covariance above exists.

The estimation of the covariance (1.1) and its eigenvalues has been studied by Franche [5], using histograms, but restricted to the case of uniform $[0,1]$ random variables. In this reference it was proved the mean square convergence of the estimator of $\Gamma(s,t)$, as well as the mean square convergence of the estimates for the eigenvalues of the integral operator with kernel $\Gamma(s,t)$. Rates of convergence were also provided. For general random variables and using kernel type estimators, this estimation has also been studied by Azevedo and Oliveira [1], proving strong consistency of the kernel estimator of $\Gamma(s,t)$ and again the mean square convergence for the estimation of the eigenvalues. In the present article we will consider an histogram type estimator of $\Gamma(s,t)$, for which we will establish pointwise and also uniform strong convergence, providing convergence rates. Also, we will prove the strong consistency of the estimates for the eigenvalues of the integral operator with kernel $\Gamma(s,t)$, again providing rates for this convergence.

2. Definitions and assumptions

Let X_n, $n \geq 1$, be a sequence of random variables. Throughout this paper we will always assume that this sequence satisfies the following assumption:

(A1) X_n, $n \geq 1$, is an associated and strictly stationary sequence of random variables, having a common distribution function F and a density function bounded by B_0; let $B_1 = 2\max(2/\pi^2,\, 45B_0)$.

If the variables satisfy (A1), we may apply Lemma 2.6 in Roussas [16] to obtain

$$\mathrm{Cov}\Big(I_{(-\infty,s]}(X_i), I_{(-\infty,t]}(X_j)\Big) \leq B_1\,\mathrm{Cov}^{1/3}\left(X_i, X_j\right), \qquad s,t \in \mathbb{R}, \qquad (2.1)$$

an inequality that will be used throughout the article to control the covariances between the terms summed in the estimator.

Given the strictly stationarity of the sequence X_n, $n \geq 1$, we denote the distribution function of the random vector (X_1, X_{k+1}) by F_k. The estimation of $F_k(s,t)$ was considered in Henriques and Oliveira [6], using the histogram type estimator given by

$$\widehat{F}_{k,n}(s,t) = \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{(-\infty,s]}(X_i) I_{(-\infty,t]}(X_{i+k}) \right) . \tag{2.2}$$

For this estimator, the strong consistency, the uniform strong consistency and the asymptotic normality follows if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}^{1/3}\left(X_1, X_j\right) = 0\,,$$

(actually the asymptotic normality follows under a slightly milder condition). However, in Henriques and Oliveira [6] no convergence rates were obtained. Later, Henriques and Oliveira [7] proved an exponential inequality from which a convergence rate for the estimator $\widehat{F}_{k,n}$ of F_k was derived. This later reference also considered the estimation of the infinite sum $\sum_{k=1}^{\infty} \varphi_k(s,t)$, proving strong consistency, but now without rates.

In Section 4 of the present article we reformulate and extend some results of Henriques and Oliveira [7] that will be used, in Section 5, to establish the uniform strong consistency of the estimator for $\Gamma(s,t)$, giving rates of convergence. Finally, in Section 6, we prove the almost sure convergence of the estimates for the eigenvalues of the integral operator with kernel $\Gamma(s,t)$.

The estimator used for the terms $\varphi_k(s,t)$ is given by

$$\widehat{\varphi}_{k,n}(s,t) = \widehat{F}_{k,n}(s,t) - \widehat{F}_n(s)\widehat{F}_n(t)\,, \tag{2.3}$$

where $\widehat{F}_n$ is the empirical distribution function defined, as usual, as $\widehat{F}_n(s) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,s]}(X_i)$. The covariance function $\Gamma(s,t)$ is estimated by

$$\widehat{\Gamma}_n(s,t) = \widehat{F}_n(s \wedge t) - \widehat{F}_n(s)\widehat{F}_n(t) + \sum_{k=1}^{q_n} \Big(\widehat{\varphi}_{k,n}(s,t) + \widehat{\varphi}_{k,n}(t,s) \Big), \tag{2.4}$$

where $q_n \longrightarrow +\infty$ in a way to be precised later.

3. Notation and preliminary results

We now introduce some lemmas that will be used while proving the theorems of the next section.

Lemma 3.1 (Devroye [2]). *Let X be a centered random variable. If there exist $a, b \in I\!R$ such that* $\mathrm{P}(a \leq X \leq b) = 1$*, then, for every $\lambda > 0$,*

$$I\!E(e^{\lambda X}) \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right).$$

The next lemma provide the tool used to prove the exponential inequality which is the basis of the consistency results and the characterizations of minimal convergence rates proved in the article. It appears under the present form in Dewan and Prakasa Rao [3] and is a version for generating functions of Newman's [14] inequality.

Lemma 3.2. *Let $X_1, X_2, ..., X_n$ be associated random variables that are bounded by a constant M. Then, for any $\theta > 0$,*

$$\left| I\!E\left(e^{\theta \sum_{i=1}^n X_i}\right) - \prod_{i=1}^n I\!E\left(e^{\theta X_i}\right) \right| \leq \theta^2 e^{n\theta M} \sum_{1 \leq i < j \leq n} \mathrm{Cov}(X_i, X_j).$$

We now start introducing the notation used throughout the article. First, let t_n be a sequence of positive integers such that $t_n \longrightarrow \infty$. For each $n \in I\!N$ and each $i = 1, \ldots, t_n$, put $x_{n,i} = Q(i/t_n)$, where Q is the quantile function of F.

In order to simplify the expressions that will follow, we define, for $n, k \in I\!N$ and fixed $s, t \in I\!R$,

$$W_{k,n} = I_{(-\infty,s]}(X_n) I_{(-\infty,t]}(X_{k+n}) - F_k(s,t).$$

Given the last definition we may write,

$$\widehat{F}_{k,n}(s,t) - F_k(s,t) = \frac{1}{n-k} \sum_{i=1}^{n-k} W_{k,i}. \tag{3.1}$$

Define also, for $n, k \in I\!N$,

$$D_{n,k} = \sup_{s,t \in I\!R} \left| \widehat{F}_{k,n}(s,t) - F_k(s,t) \right|,$$

and

$$D^*_{n,k} = \max_{i,j=1,\ldots,t_n} \left| \widehat{F}_{k,n}(x_{n,i}, x_{n,j}) - F_k(x_{n,i}, x_{n,j}) \right|.$$

The following lemma will be used to obtain an exponential inequality for $D_{n,k}$. The proof of this lemma is contained in the proof of Theorem 2 of Henriques and Oliveira [6].

Lemma 3.3. *If the sequence X_n, $n \geq 1$, satisfies (A1), then, for each $n \in I\!N$ and each $k \in I\!N$,*

$$D_{n,k} \leq D^*_{n,k} + \frac{2}{t_n} \qquad a.s..$$

Now we introduce some additional notation to be used in the sequel. Let a_n and p_n be two sequences of positive integers such that $a_n \longrightarrow +\infty$ and $a_n < p_n \leq n - a_n$. For each $n \in I\!N$ and each $k = 1, \ldots, a_n$, let $r_{k,n}$ be the largest integer less or equal to $\frac{n-k}{2p_n}$. We will consider a partition of the set of indexes $\{1, \ldots, n-k\}$ into $2r_{k,n}+1$ subsets, each of which containing p_n elements except the last one, that will have $n-k-2r_{k,n} < 2p_n$ elements. We will suppose that, for each $k \in \{0, 1, ..., a_n\}$, $r_{k,n} \longrightarrow \infty$, so that, $\frac{n-k}{2r_{k,n}p_n} \longrightarrow 1$. Note also that the set $\{1, \ldots, n-1\}$ has more $a_n - 1$ elements than $\{1, \ldots, n-a_n\}$. As $a_n < p_n$, this means that, for each $k \in \{0, 1, .., a_n\}$, we will have for $\{1, \ldots, n-1\}$ at most two more sets in the partition than for $\{1, \ldots, n-k\}$, that is, we have $r_{k,n} = r_{1,n}$ or $r_{k,n} = r_{1,n} - 1$.

Now define the sets $O_i = \{2(i-1)p_n + 1, ..., (2i-1)p_n\}$, $E_i = \{(2i-1)p_n + 1, ..., 2ip_n\}$, for each $i = 1, \ldots, r_{k,n}$, and $R = \{2r_{k,n}p_n + 1, ..., n-k\}$. These sets of indexes will be used to decompose the sum $\widehat{F}_{k,n}(s,t) - F_k(s,t)$ into blocks in a way similar to what was done in Ioannides and Roussas [9]. This technique has been used by the authors to prove exponential inequalities and, following from these, convergence rates for the histogram estimator for distribution functions of associated variables and for the kernel estimator of the density (Henriques and Oliveira [7, 8]).

For the purpose just mentioned define the random variables

$$U_{k,i} = \sum_{j \in O_i} W_{k,j}\,,\ V_{k,i} = \sum_{j \in E_i} W_{k,j}\,,\ i = 1, ...r_{k,n} \quad \text{and} \quad Z_{k,n} = \sum_{j \in R} W_{k,j}\,,$$

and set

$$\overline{U}_{k,n} = \frac{1}{n-k} \sum_{i=1}^{r_{k,n}} U_{k,i}\,, \quad \overline{V}_{k,n} = \frac{1}{n-k} \sum_{i=1}^{r_{k,n}} V_{k,i} \quad \text{and} \quad \overline{Z}_{k,n} = \frac{1}{n-k} Z_{k,n}\,.$$

With these definitions we have,

$$\widehat{F}_{k,n}(s,t) - F_k(s,t) = \overline{U}_{k,n} + \overline{V}_{k,n} + \overline{Z}_{k,n}\,. \tag{3.2}$$

The exponential inequality proved in Theorem 1 of Henriques and Oliveira [7] required the sequence $\mathrm{Cov}(X_1, X_{k+1})$ to be decreasing as k increases. This was so because of the technique to control the covariances between the blocks $U_{k,i}$, based on the method used in Ioannides and Roussas [9], where the same assumption is used. In Henriques and Oliveira [8] the computation of these covariances was carried in more detail and this assumption on the covariances could be dropped, without any consequence on the decreasing rate also assumed on the covariances (in fact, Ioannides and Roussas [9] mentioned that the assumption was used to avoid technical difficulties but did not give any indication about how their assumptions or their inequality should be modified). The following lemma gives the precise formulation of our result.

Lemma 3.4. *Under assumption* (A1)*, we have, for the variables* $U_{k,i}$*,* $i = 1, \ldots, r_{k,n}$*, defined earlier,*

$$\sum_{1 \le i < j \le r_{k,n}} \mathrm{Cov}(U_{k,i}, U_{k,j}) \le 4 B_1 r_{k,n} p_n \sum_{l=p_n-k}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_l)\,,$$

where B_1 *was defined in* (A1)*, and analogously for the variables* $V_{k,i}$*,* $i = 1, \ldots, r_{k,n}$*.*

Proof: The assumption of stationarity enables us to write

$$\sum_{1 \le i < j \le r_{k,n}} \mathrm{Cov}(U_{k,i}, U_{k,j}) =$$
$$= \sum_{i=1}^{r_{k,n}-1} \sum_{j=i+1}^{r_{k,n}} \mathrm{Cov}(U_{k,i}, U_{k,j}) = \sum_{j=1}^{r_{k,n}-1} (r_{k,n} - j)\, \mathrm{Cov}(U_{k,1}, U_{k,j+1})\,.$$

Using again the stationarity of the variables, we get

$$\begin{aligned}
\operatorname{Cov}(U_{k,1},U_{k,j+1}) &= \sum_{l\in O_1}\sum_{m\in O_{j+1}} \operatorname{Cov}(W_{k,l},W_{k,m}) = \\
&= \sum_{l=0}^{p_n-1}(p_n-l)\operatorname{Cov}(W_{k,1},W_{k,2jp_n+l+1}) + \\
&\qquad + \sum_{l=1}^{p_n-1}(p_n-l)\operatorname{Cov}(W_{k,l+1},W_{k,2jp_n+1}) \le \\
&\le p_n \sum_{l=(2j-1)p_n+2}^{(2j+1)p_n} \operatorname{Cov}(W_{k,1},W_{k,l}) \,.
\end{aligned}$$

We then have

$$\begin{aligned}
\sum_{1\le i<j\le r_{k,n}} & \operatorname{Cov}(U_{k,i},U_{k,j}) \le \\
&\le \sum_{j=1}^{r_{k,n}-1}(r_{k,n}-j)p_n \sum_{l=(2j-1)p_n+2}^{(2j+1)p_n} \operatorname{Cov}(W_{k,1},W_{k,l}) \qquad (3.3)\\
&\le r_{k,n}p_n \sum_{l=p_n+2}^{(2r_{k,n}-1)p_n} \operatorname{Cov}(W_{k,1},W_{k,l}) \,.
\end{aligned}$$

Now, using a classical inequality by Lebowitz [11] and (2.1), it follows

$$\begin{aligned}
\operatorname{Cov}(W_{k,1},W_{k,l}) \le B_1 \Big[& \operatorname{Cov}^{1/3}(X_1,X_l) + \operatorname{Cov}^{1/3}(X_1,X_{k+l}) + \\
& + \operatorname{Cov}^{1/3}(X_{k+1},X_l) + \operatorname{Cov}^{1/3}(X_{k+1},X_{k+l}) \Big] \,.
\end{aligned}$$

Inserting this into (3.3) we then obtain,

$$\sum_{1\leq i<j\leq r_{k,n}} \operatorname{Cov}(U_{k,i},U_{k,j}) \leq$$
$$\leq B_1 r_{k,n} p_n \sum_{l=p_n+2}^{\infty} \Big(2\operatorname{Cov}^{1/3}(X_1,X_l) +$$
$$+ \operatorname{Cov}^{1/3}(X_1,X_{k+l}) + \operatorname{Cov}^{1/3}(X_1,X_{l-k}) \Big) \leq$$
$$\leq 4B_1 r_{k,n} p_n \sum_{l=p_n-k}^{\infty} \operatorname{Cov}^{1/3}(X_1,X_l)\,,$$

since the covariances are non-negative due to association. ■

4. Exponential inequalities

We will now reformulate the exponential inequality contained in Henriques and Oliveira [7], establishing one without the assumption of $C(k) = \operatorname{Cov}(X_1, X_{k+1})$ being nonincreasing. Further, we will also obtain an uniform version of this exponential inequality as it has been done, in a different framework, in Henriques and Oliveira [8].

Lemma 4.1. *Let* $0<\varepsilon<1$*. Suppose that* (A1) *is satisfied and that there exists a constant* $C_1>0$ *such that*

$$\frac{1}{n-a_n}\exp\left(\frac{8r_{1,n}}{a_n}\right)\sum_{i=p_n-a_n}^{\infty} \operatorname{Cov}^{1/3}(X_1,X_i) \leq C_1\,. \tag{4.1}$$

Then, for each $k\in\{0,\ldots,a_n\}$*,*

$$\mathrm{P}\left(|\overline{U}_{k,n}| \geq \frac{\varepsilon}{a_n}\right) \leq 2(1+C_2)\exp\left(-2r_{k,n}\frac{\varepsilon^2}{a_n^2}\right),$$

where $C_2 = 8B_1C_1$*, and the same for the variables* $\overline{V}_{k,n}$*.*

Proof: Given $\lambda>0$, the Markov inequality yields

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{a_n}\right) \leq e^{-\lambda\frac{\varepsilon}{a_n}}\, I\!\!E\left(e^{\lambda\overline{U}_{k,n}}\right).$$

Note that, each one of the variables $U_{k,i}$ is bounded by p_n, so we may apply Lemma 3.2 to obtain

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{a_n}\right) \leq$$

$$\leq e^{-\frac{\lambda\varepsilon}{a_n}} \left(\prod_{i=1}^{r_{k,n}} I\!\!E\left(e^{\frac{\lambda}{n-k}U_{k,i}}\right) + \right.$$

$$\left. + \frac{\lambda^2}{(n-k)^2} \exp\left(\frac{r_{k,n}\,\lambda\, p_n}{n-k}\right) \sum_{1\leq i<j\leq r_{k,n}} \mathrm{Cov}(U_{k,i},U_{k,j}) \right).$$

Now apply successively Lemma 3.1 and Lemma 3.4 to get

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{a_n}\right) \leq$$

$$\leq e^{-\frac{\lambda\varepsilon}{a_n}} \left(\exp\left(\frac{4r_{k,n}p_n^2\lambda^2}{8(n-k)^2}\right) + \right.$$

$$\left. + \frac{\lambda^2}{(n-k)^2} \exp\left(\frac{r_{k,n}\,\lambda\, p_n}{n-k}\right) 4B_1 r_{k,n}\, p_n \sum_{l=p_n-k}^{\infty} \mathrm{Cov}^{1/3}(X_1,X_l) \right) \leq$$

$$\leq \exp\left(-\frac{\lambda\,\varepsilon}{a_n} + \frac{\lambda^2}{8r_{k,n}}\right) + 8B_1\, e^{-\frac{\lambda\varepsilon}{a_n}} \frac{e^{2\lambda}}{n-k} \sum_{l=p_n-k}^{\infty} \mathrm{Cov}^{1/3}(X_1,X_l),$$

since $2r_{k,n}p_n \leq n-k$ and $\lambda^2 \leq 4e^{\lambda}$.

Now, choose $\lambda = 4r_{k,n}\frac{\varepsilon}{a_n}$, which minimizes the first term of the last expression, and use (4.1) in the second term, taking into account that $r_{k,n} \leq r_{1,n}$, $\varepsilon < 1$ and $k \leq a_n$. It will follow that

$$\mathrm{P}\left(\overline{U}_{k,n} \geq \frac{\varepsilon}{a_n}\right) \leq$$

$$\leq \exp\left(-\frac{2r_{k,n}\varepsilon^2}{a_n^2}\right) + 8B_1C_1 \exp\left(-\frac{4r_{k,n}\varepsilon^2}{a_n^2}\right) \leq$$

$$\leq (1+8B_1C_1)\exp\left(-\frac{2r_{k,n}\varepsilon^2}{a_n^2}\right).$$

Using the same arguments we would obtain the same upper bound for $\mathrm{P}\left(-\overline{U}_{k,n} \geq \varepsilon/a_n\right)$, thus completing the proof. ∎

The next lemma deals with the sum in the last block, $\overline{Z}_{k,n}=\frac{1}{n-k}\sum_{j\in R}W_{k,j}$.

Lemma 4.2. *Let* $0<\varepsilon<1$. *Suppose that* (A1) *is satisfied and that* $a_n/r_{1,n}\longrightarrow 0$. *Then,*

$$\mathrm{P}\left(\left|\overline{Z}_{k,n}\right|\geq\frac{\varepsilon}{3a_n}\right)=0,$$

for each sufficiently large n *and each* $k\in\{0,\ldots,a_n\}$.

Proof: First note that, since $|W_{k,n}|\leq 1$ and the cardinal of R is less than $2p_n$, we have $\left|\overline{Z}_{k,n}\right|\leq\frac{2p_n}{n-k}$. Therefore,

$$\mathrm{P}\left(\left|\overline{Z}_{k,n}\right|\geq\frac{\varepsilon}{3a_n}\right)\leq\mathrm{P}\left(\frac{p_na_n}{n-k}\geq\frac{\varepsilon}{6}\right).$$

Now $\frac{p_na_n}{n-k}=\frac{p_nr_{1,n}}{n-k}\frac{a_n}{r_{1,n}}\longrightarrow 0$ as, according to the construction of the sequences, $\frac{p_nr_{1,n}}{n-k}\longrightarrow 1/2$, so the result follows. ∎

The following two theorems establish the exponential inequalities mentioned at the beginning of this section. This next theorem states a pointwise exponential inequality, extending the one proved in Theorem 1 in Henriques and Oliveira [7].

Theorem 4.3. *Let* $0<\varepsilon<1$. *Suppose* (A1) *is satisfied and that (4.1) holds for every sufficiently large* n. *Further assume that* $\lim_{n\to+\infty}\frac{a_n}{r_{1,n}}=0$. *Then, for every sufficiently large* n *and* $k\in\{0,\ldots,a_n\}$,

$$\mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t)-F_k(s,t)\right|>\frac{\varepsilon}{a_n}\right)\leq 4(1+C_2)\exp\left(-\frac{2}{9}\frac{r_{k,n}\,\varepsilon^2}{a_n^2}\right),$$

where C_2 *is defined in Lemma 4.1*

Proof: The result follows easily from Lemmas 4.1 and 4.2, using (3.1). ∎

Finally, we prove an uniform exponential inequality corresponding to the previous result.

Theorem 4.4. *Let* t_n *be a sequence of positive integres such that* $t_n\longrightarrow+\infty$ *and* $\frac{a_n}{t_n}\longrightarrow 0$. *Under the conditions of Theorem 4.3, we have, for every sufficiently large* n *and* $k\in\{0,\ldots,a_n\}$,

$$\mathrm{P}\left(\sup_{s,t,\in\mathbb{R}}\left|\widehat{F}_{k,n}(s,t)-F_k(s,t)\right|>\frac{\varepsilon}{a_n}\right)\leq 4(1+C_2)t_n^2\exp\left(-\frac{1}{18}\frac{r_{k,n}\,\varepsilon^2}{a_n^2}\right),$$

where C_2 *is defined in Lemma 4.1.*

Proof: Given $0<\varepsilon<1$, using Lemma 3.3, we obtain, for each $n\in I\!N$ and each $k\in\{0,\dots a_n\}$,

$$\mathrm{P}\left(D_{n,k}>\frac{\varepsilon}{a_n}\right)\leq \mathrm{P}\left(D^*_{n,k}+\frac{2}{t_n}>\frac{\varepsilon}{a_n}\right)\leq \mathrm{P}\left(D^*_{n,k}>\frac{\varepsilon}{2a_n}\right)+\mathrm{P}\left(\frac{2}{t_n}>\frac{\varepsilon}{2a_n}\right).$$

It follows from $\frac{a_n}{t_n}\longrightarrow 0$ that the second term of the last expression equals zero for n large enough. Then, for every sufficiently large n and $k\in\{0,\dots,a_n\}$, it holds

$$\begin{aligned}\mathrm{P}&\left(D_{n,k}>\frac{\varepsilon}{a_n}\right)\leq\\ &\leq\sum_{i,j=1,\dots,t_n}\mathrm{P}\left(\left|\widehat{F}_{k,n}(x_{n,i},x_{n,j})-F_k(x_{n,i},x_{n,j})\right|>\frac{\varepsilon}{2a_n}\right)\leq\\ &\leq t_n^2\max_{i,j=1,\dots,t_n}\mathrm{P}\left(\left|\widehat{F}_{k,n}(x_{n,i},x_{n,j})-F_k(x_{n,i},x_{n,j})\right|>\frac{\varepsilon}{2a_n}\right).\end{aligned}$$

Finally, applying Theorem 4.3 we obtain, for every sufficiently large n and $k\in\{0,\dots,a_n\}$,

$$\mathrm{P}\left(D_{n,k}>\frac{\varepsilon}{a_n}\right)\leq 4\,t_n^2\,(1+C_2)\exp\left(-\frac{1}{18}\frac{r_{k,n}\,\varepsilon^2}{a_n^2}\right).$$

■

5. Consistency and convergence rates

We are now in position to derive convergence rates for the almost sure convergence, in the pointwise as well as in the uniform sense, for the estimators $\widehat{F}_{k,n}$, $\widehat{\varphi}_{k,n}$ and $\widehat{\Gamma}_n$. The first two theorems of this section, Theorem 5.1 and Theorem 5.2, state the convergence rates for the estimators $\widehat{F}_{k,n}$ and $\widehat{\varphi}_{k,n}$, respectively. The consistency of the estimators for the infinite sum $\sum_{k=1}^{\infty}\varphi_k(s,t)$ and for $\Gamma(s,t)$ is established in Theorems 5.3 and 5.4, respectively. Finally, Theorems 5.6 and 5.7 give convergence rates for the last mentioned estimators.

Theorem 5.1. *Suppose* (A1) *is satisfied and that (4.1) holds for every sufficiently large n. Further assume that $\frac{a_n^2\ln n}{r_{1,n}}\longrightarrow 0$. Then,*

a) $a_n\left(\widehat{F}_{k,n}(s,t)-F_k(s,t)\right)\longrightarrow 0$ *a.s., for each $k\in\{0,\dots,a_n\}$ and $s,t\in I\!R$;*

b) $a_n \sup_{s,t\in\mathbb{R}} \left|\widehat{F}_{k,n}(s,t) - F_k(s,t)\right| \longrightarrow 0$ *a.s., for each* $k \in \{0,\ldots,a_n\}$.

Proof: Let $0 < \varepsilon < 1$. First note that all the conditions of Theorem 4.3 are satisfied, so we may apply it to obtain, for every sufficiently large n and $k \in \{0,\ldots,a_n\}$,

$$\begin{aligned} \mathrm{P}\left(a_n \left|\widehat{F}_{k,n}(s,t) - F_k(s,t)\right| > \varepsilon\right) &\le 4(1+C_2)\exp\left(-\tfrac{2}{9}\tfrac{r_{k,n}\varepsilon^2}{a_n^2}\right) \le \\ &\le 4(1+C_2)\exp\left(-\tfrac{2}{9}\tfrac{(r_{1,n}-1)\varepsilon^2}{a_n^2}\right), \end{aligned}$$

since $r_{k,n} \ge r_{1,n} - 1$. As $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$ it follows that this upper bound defines, for every $\varepsilon > 0$, a convergent series. Thus a) follows by the Borel-Cantelli Lemma.

Now, choose $\alpha > 1$ and set $t_n = a_n^\alpha$, so that $\frac{a_n}{t_n} \longrightarrow 0$. Then, by Theorem 4.4, we get, for every sufficiently large n and $k \in \{0,\ldots,a_n\}$,

$$\mathrm{P}\left(a_n \sup_{s,t\in\mathbb{R}} \left|\widehat{F}_{k,n}(s,t) - F_k(s,t)\right| > \varepsilon\right) \le 4(1+C_2)a_n^{2\alpha}\exp\left(-\frac{1}{18}\frac{(r_{1,n}-1)\varepsilon^2}{a_n^2}\right),$$

and the assumption $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$ concludes the proof as for the first part. ∎

Note that under the conditions of Theorem 5.1, we obtain, setting $k = 0$ and $s = t$,

$$a_n(\widehat{F}_n(s) - F(s)) \longrightarrow 0 \qquad a.s. \tag{5.1}$$

and also,

$$a_n \sup_{s\in\mathbb{R}} \left|\widehat{F}_n(s) - F(s)\right| \longrightarrow 0 \qquad a.s.. \tag{5.2}$$

Moreover, since

$$\begin{aligned} &a_n\left(F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t)\right) = \\ &\quad = F(s)\,a_n\left(F(t) - \widehat{F}_n(t)\right) + \widehat{F}_n(t)\,a_n\left(F(s) - \widehat{F}_n(s)\right), \end{aligned}$$

Theorem 5.1 also implies that

$$a_n\left(F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t)\right) \longrightarrow 0 \qquad a.s. \tag{5.3}$$

and

$$a_n \sup_{s,t\in\mathbb{R}} \left|F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t)\right| \longrightarrow 0 \qquad a.s.. \tag{5.4}$$

Theorem 5.2. *Under the conditions of Theorem 5.1 it holds,*

a) $a_n\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right) \longrightarrow 0$ *a.s., for each* $k \in \{0,\ldots,a_n\}$ *and* $s,t\in I\!R$*;*

b) $a_n \sup_{s,t\in I\!R}\left|\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right| \longrightarrow 0$ *a.s., for each* $k\in\{0,\ldots,a_n\}$.

Proof: To prove this theorem it suffices to write

$$\begin{aligned} &a_n\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right)= \\ &\quad = a_n\left(\widehat{F}_{k,n}(s,t)-F_k(s,t)\right)+a_n\Big(F(s)F(t)-\widehat{F}_n(s)\widehat{F}_n(t)\Big). \end{aligned}$$

Then, apply Theorem 5.1 a) together with (5.3) to obtain a), and Theorem 5.1 b) together with (5.4) to find b). ■

Theorem 5.3. *Under the conditions of Theorem 5.1 it holds,*

a) $\sum_{k=1}^{a_n}\widehat{\varphi}_{k,n}(s,t)\longrightarrow \sum_{k=1}^{\infty}\varphi_k(s,t)$ *a.s., for each* $s,t\in I\!R$*;*

b) $\sup_{s,t\in I\!R}\left|\sum_{k=1}^{a_n}[\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)]\right|\longrightarrow 0$ *a.s.*.

Proof: We may decompose the difference as

$$\begin{aligned} &\left|\sum_{k=1}^{a_n}\widehat{\varphi}_{k,n}(s,t)-\sum_{k=1}^{\infty}\varphi_k(s,t)\right|\leq \\ &\quad\leq\left|\sum_{k=1}^{a_n}\left(\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)\right)\right|+\left|\sum_{k=a_n+1}^{\infty}\varphi_k(s,t)\right|\leq \\ &\quad\leq\left|\sum_{k=1}^{a_n}\left(\widehat{F}_{k,n}(s,t)-F_k(s,t)\right)\right|+a_n\left|F(s)F(t)-\widehat{F}_n(s)\widehat{F}_n(t)\right|+ \\ &\qquad\qquad+\left|\sum_{k=a_n+1}^{\infty}\varphi_k(s,t)\right|. \end{aligned}$$

As mentioned before (see (2.1)) $\varphi_k(s,t)=\mathrm{Cov}\left(I_{(-\infty,s]}(X_1),I_{(-\infty,t]}(X_{k+1})\right)\leq B_1\,\mathrm{Cov}^{1/3}\left(X_1,X_{k+1}\right)$, and (4.1) implies that $\sum_k \mathrm{Cov}^{1/3}\left(X_1,X_{k+1}\right)<\infty$, so the third term of the upper bound above converges to zero as $a_n\longrightarrow+\infty$. The almost sure convergence to zero of the second term is just (5.3). Finally, for the first term, using Theorem 4.3, we have, for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\mathrm{P}\left(\left|\sum_{k=1}^{a_n}\left(\widehat{F}_{k,n}(s,t)-F_k(s,t)\right)\right|>\varepsilon\right) &\leq \\
&\leq \sum_{k=1}^{a_n}\mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t)-F_k(s,t)\right|>\frac{\varepsilon}{a_n}\right) \leq \\
&\leq 4(1+C_2)a_n\exp\left(-\frac{2}{9}\frac{(r_{1,n}-1)\varepsilon^2}{a_n^2}\right).
\end{aligned}$$

Now the Borel-Cantelli Lemma justifies the almost sure convergence to zero of the first term.

For the second part of the theorem, since

$$\begin{aligned}
&\sup_{s,t\in I\!R}\left|\sum_{k=1}^{a_n}[\widehat{\varphi}_{k,n}(s,t)-\varphi_k(s,t)]\right| \leq \\
&\quad\leq \sup_{s,t\in I\!R}\left|\sum_{k=1}^{a_n}\left(\widehat{F}_{k,n}(s,t)-F_k(s,t)\right)\right| + a_n\sup_{s,t\in I\!R}\left|F(s)F(t)-\widehat{F}_n(s)\widehat{F}_n(t)\right|,
\end{aligned}$$

and, as the second term on the right-hand side above converges to zero according to (5.4), b) will follow if we prove that

$$\sup_{s,t\in I\!R}\left|\sum_{k=1}^{a_n}\left[\widehat{F}_{k,n}(s,t)-F_k(s,t)\right]\right| \longrightarrow 0 \qquad a.s.. \tag{5.5}$$

For this purpose, choose $\alpha > 1$ and $t_n = a_n^\alpha$. We have

$$\begin{aligned}
\mathrm{P}\left(\sup_{s,t\in I\!R}\left|\sum_{k=1}^{a_n}\left[\widehat{F}_{k,n}(s,t)-F_k(s,t)\right]\right|>\varepsilon\right) &\leq \\
&\leq \sum_{k=1}^{a_n}\mathrm{P}\left(\sup_{s,t\in I\!R}\left|\widehat{F}_{k,n}(s,t)-F_k(s,t)\right|>\frac{\varepsilon}{a_n}\right) \leq \\
&\leq 4(1+C_2)a_n^{2\alpha+1}\exp\left(-\frac{1}{18}\frac{(r_{1,n}-1)\varepsilon^2}{a_n^2}\right),
\end{aligned}$$

using Theorem 4.4.

Again, arguing as in the corresponding part of the proof of the Theorem 5.1 b), (5.5) follows. ∎

We now prove the pointwise and uniform consistency of the estimator $\widehat{\Gamma}_n(s,t)$.

Theorem 5.4. *Under the conditions of Theorem 5.1, and putting $q_n = a_n$ in the definition of the estimator $\widehat{\Gamma}_n$, it holds,*

a) $\widehat{\Gamma}_n(s,t) \longrightarrow \Gamma(s,t)$ *a.s., for each* $s,t \in \mathbb{R}$;

b) $\sup_{s,t\in\mathbb{R}} \left|\widehat{\Gamma}_n(s,t) - \Gamma(s,t)\right| \longrightarrow 0$ *a.s.*.

Proof: To prove a) just write

$$\begin{aligned}
&\widehat{\Gamma}_n(s,t) - \Gamma(s,t) = \\
&\quad = \left[\widehat{F}_n(s\wedge t) - F(s\wedge t)\right] + \left[F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t)\right] + \\
&\quad + \left[\sum_{k=1}^{a_n} \widehat{\varphi}_{k,n}(s,t) - \sum_{k=1}^{\infty} \varphi_k(s,t)\right] + \left[\sum_{k=1}^{a_n} \widehat{\varphi}_{k,n}(t,s) - \sum_{k=1}^{\infty} \varphi_k(t,s)\right],
\end{aligned} \tag{5.6}$$

and apply (5.1), (5.3) and Theorem 5.3 a).

To prove the second part of the theorem write,

$$\begin{aligned}
&\sup_{s,t\in\mathbb{R}} \left|\widehat{\Gamma}_n(s,t) - \Gamma(s,t)\right| \leq \\
&\quad \leq \sup_{s,t\in\mathbb{R}} \left|\widehat{F}_n(s\wedge t) - F(s\wedge t)\right| + \sup_{s,t\in\mathbb{R}} \left|F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t)\right| + \\
&\quad + \sup_{s,t\in\mathbb{R}} \left|\sum_{k=1}^{a_n} [\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)]\right| + \sup_{s,t\in\mathbb{R}} \left|\sum_{k=a_n+1}^{\infty} \varphi_k(s,t)\right| + \\
&\quad + \sup_{s,t\in\mathbb{R}} \left|\sum_{k=1}^{a_n} [\widehat{\varphi}_{k,n}(t,s) - \varphi_k(t,s)]\right| + \sup_{s,t\in\mathbb{R}} \left|\sum_{k=a_n+1}^{\infty} \varphi_k(t,s)\right|.
\end{aligned} \tag{5.7}$$

The almost sure convergence to zero of the first and second terms of the right-hand side above follows, respectively, from (5.2) and (5.4). For the third and fifth terms, convergence to zero follows from Theorem 5.3 b). So, to establish the convergence in b) it remains to prove that

$$\sup_{s,t\in\mathbb{R}} \left|\sum_{k=a_n+1}^{\infty} \varphi_k(s,t)\right| \longrightarrow 0 \qquad a.s., \tag{5.8}$$

but this follows from the argument used in the proof of Theorem 5.3 a) as inequality (2.1) is uniform with respect to s and t. ■

It is not difficult to verify that if the covariances $\mathrm{Cov}(X_1, X_{k+1})$ decrease at a polynomial rate the conditions (4.1) and $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$ can not be fulfilled simultaneously, which prevents the use of the results of this section in this case. However, such limitation is not new. In fact, in previous articles establishing exponential inequalities for associated random variables, namely Ioannides and Roussas [9], Henriques and Oliveira [7, 8], the same problem arises when exponential inequalities are used to obtain rates for the almost sure convergence. For polynomial decrasing covariances Masry [12] proved recently some strong consistency results with rates of convergence. The method of proof is quite different as it is based on Rosenthal type inequalities bounding the p^{th} centered moment, with $p > 2$. Another result for polynomial decreasing covariances, also for the estimation of the density, was proved in Henriques and Oliveira [8] using the same method of approach as in this article, thus not requiring the existence of any moment of order greater than 2.

The next corollary shows that, if we assume a geometrical decrease rate of the covariances $\mathrm{Cov}(X_1, X_{k+1})$, it is possible to find sequences a_n and $r_{1,n}$ (which determines p_n), such that the conditions of the preceding theorems are satisfied.

Corollary 5.5. *Suppose* (A1) *is satisfied and* $\mathrm{Cov}(X_1, X_{k+1}) = a_0\, a^{-k}$*, for some* $a_0 > 0$ *and* $a > 1$*. Choose* $a_n = n^{\beta}$*, with* $\beta < 1/3$*, then (4.1) is satisfied and* $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$.

Proof: Choose $r_{1,n} = a_n^2 (\ln n)^{\gamma}$, for some $\gamma > 1$. Then obviously, $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$, so it remains to prove that (4.1) is also satisfied.

Note that, as, $\frac{n-1}{2r_{1,n}p_n} \longrightarrow 1$, we may write $p_n = \frac{n-1}{2x_n r_{1,n}}$, for some sequence $0 < x_n \longrightarrow 1$. In order to prove that (4.1) is verified we rewrite the left side of this inequality as

$$\begin{aligned} \frac{1}{n-a_n} \exp\left(\frac{8r_{1,n}}{a_n}\right) \sum_{i=p_n-a_n}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_i) = \\ = \frac{1}{n-a_n} \exp\left[8a_n (\ln n)^{\gamma}\right]\, a_0^{1/3} \frac{a^{-\frac{p_n-a_n}{3}}}{1-a^{-1/3}}\,. \end{aligned} \tag{5.9}$$

To establish the asymptotic behaviour of (5.9) we compare the two exponents in order to identify which one is dominant. That is, we find, up to the multiplication by a constant,

$$\frac{a_n(\ln n)^\gamma}{p_n - a_n} = \frac{2a_n x_n r_{1,n}(\ln n)^\gamma}{n-1-2a_n x_n r_{1,n}} = \frac{2a_n^3 x_n(\ln n)^{2\gamma}}{n-1-2a_n^3 x_n(\ln n)^\gamma} \longrightarrow 0\,,$$

taking into account that $a_n = n^\beta$, for some $\beta < 1/3$. This means that the right side of (5.9) converges to zero so, a fortiori, (4.1) is verified, at least for n large enough. ■

According to the last corollary, if the covariances $\mathrm{Cov}(X_1, X_{k+1})$ decrease at a geometrical rate, we have, for the almost sure convergence of the estimators $\widehat{F}_{k,n}$ and $\widehat{\varphi}_{k,n}$, a convergence rate of order at least $n^-\beta$, with $\beta < 1/3$.

Theorem 5.6. *Let b_n and q_n be two sequences of positive integers tending to infinity and such that $a_n = b_n\, q_n$. Suppose that*

$$b_n \sum_{k=q_n+1}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_{k+1}) \longrightarrow 0\,. \tag{5.10}$$

Further assume that (A1) *is satisfied, (4.1) holds for every $n \in I\!N$ large enough and $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$. Then,*

a) $b_n\left(\sum_{k=1}^{q_n} \widehat{\varphi}_{k,n}(s,t) - \sum_{k=1}^{\infty} \varphi_k(s,t)\right) \longrightarrow 0$ *a.s., for each $s, t \in I\!R$;*

b) $b_n \sup_{s,t\in I\!R} \left|\sum_{k=1}^{q_n} [\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)]\right| \longrightarrow 0$ *a.s.*.

Proof: Let $\varepsilon \in (0,1)$. Proceeding as in the proof of Theorem 5.3 a), we find, using Theorem 4.3,

$$\begin{aligned} \mathrm{P}\left(\left|b_n \sum_{k=1}^{q_n}\left(\widehat{F}_{k,n}(s,t) - F_k(s,t)\right)\right| > \varepsilon\right) &\leq \\ &\leq \sum_{k=1}^{q_n} \mathrm{P}\left(\left|\widehat{F}_{k,n}(s,t) - F_k(s,t)\right| > \varepsilon/a_n\right) \\ &\leq 4(1+C_2) q_n \exp\left(-\frac{2}{9}\frac{(r_{1,n}-1)\varepsilon^2}{a_n^2}\right), \end{aligned}$$

so that, using the Borel-Cantelli Lemma, it follows that,

$$b_n \sum_{k=1}^{q_n}\left[\widehat{F}_{k,n}(s,t) - F_k(s,t)\right] \longrightarrow 0 \qquad a.s.. \tag{5.11}$$

From (2.1) and (5.10), we get

$$0 \leq b_n \sum_{k=q_n+1}^{\infty} \varphi_k(s,t) \leq b_n B_1 \sum_{k=q_n+1}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_{k+1}) \longrightarrow 0\,. \tag{5.12}$$

Now, since

$$b_n \left(\sum_{k=1}^{q_n} \widehat{\varphi}_{k,n}(s,t) - \sum_{k=1}^{\infty} \varphi_k(s,t) \right) = b_n \sum_{k=1}^{q_n} \left(\widehat{F}_{k,n}(s,t) - F_k(s,t) \right) +$$
$$+ b_n q_n \left[F(s)F(t) - \widehat{F}_n(s)\widehat{F}_n(t) \right] - b_n \sum_{k=q_n+1}^{\infty} \varphi_k(s,t),$$

the convergence in a) follows from (5.11), (5.3) and (5.12).

Note that to prove a), we followed essentially the same steps as in Theorem 5.3 a). Analogously, proceeding as in the proof of Theorem 5.3 b), but considering the summation of the terms $\widehat{\varphi}_{k,n}(s,t)$ up to q_n instead of a_n, and using $a_n = b_n\, q_n$, we obtain the convergence stated in b). ∎

Theorem 5.7. *Under the conditions of Theorem 5.6, it holds,*

a) $b_n \left(\widehat{\Gamma}_n(s,t) - \Gamma(s,t) \right) \longrightarrow 0$ *a.s., for each* $s,t \in I\!R$;

b) $b_n \sup_{s,t \in I\!R} \left| \widehat{\Gamma}_n(s,t) - \Gamma(s,t) \right| \longrightarrow 0$ *a.s.*.

Proof: The proof of a) follows easily if we decompose the difference as in (5.6) and then apply (5.1), (5.3) and Theorem 5.6 a).

For the proof of b), we first note that, by (2.1) and (5.10), we have

$$0 \leq b_n \sup_{s,t \in I\!R} \left| \sum_{k=q_n+1}^{\infty} \varphi_k(s,t) \right| \leq B_1 b_n \sum_{k=q_n+1}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_{k+1}) \longrightarrow 0\,. \tag{5.13}$$

Now, using (5.7), b) follows directly from (5.2), (5.4), Theorem 5.6 b) and (5.13). ∎

As before, we show that, for geometrically decreasing covariances it is possible to construct sequences that fulfill our assumptions.

Corollary 5.8. *Suppose* (A1) *is satisfied and* $\mathrm{Cov}(X_1, X_{k+1}) = a_0\, a^{-k}$, *for some* $a_0 > 0$ *and* $a > 1$. *Choose* $a_n = n^{\beta}$, *with* $\beta < \min\left\{\frac{1}{3}, \frac{\ln a}{3}\right\}$, $q_n = \ln n$ *and* $b_n = \frac{n^{\beta}}{\ln n}$, *then all the assumptions of Theorems 5.6 and 5.7 are verified.*

Proof: Choose $r_{1,n} = a_n^2(\ln n)^\gamma$, for some $\gamma > 1$. We have already proved in Corollary 5.5, that, with these choices for a_n and $r_{1,n}$, both conditions, $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$ and (4.1), are satisfied. It remains to prove that (5.10) is also verified. This is in fact true, since

$$b_n \sum_{k=q_n+1}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_{k+1}) =$$

$$= a_0^{1/3} b_n \sum_{k=q_n+1}^{\infty} a^{-k/3} = \frac{a_0^{1/3}}{1-a^{-1/3}}\, b_n\, a^{-\frac{q_n+1}{3}} = \frac{a_0^{1/3} a^{-1/3}}{1-a^{-1/3}} \frac{n^\beta}{\ln n} a^{-\frac{\ln n}{3}} \longrightarrow 0\,,$$

as we have assumed that $\beta < \frac{\ln a}{3}$. ■

We can then say that, if the covariances $\mathrm{Cov}(X_1, X_n)$ decrease at a geometrical rate, the estimators for the infinite sum $\sum_{k=1}^{\infty} \varphi_k(s,t)$ and for $\Gamma(s,t)$ converge almost surely at the rate of at least $\frac{\ln n}{n^\beta}$.

6. Estimation of the eigenvalues of the integral operator of kernel $\Gamma(s,t)$

Let K be a non null, symmetric function of $L^2([a,b]^2)$ and denote by $\mathbb{K}$ the integral operator with kernel K. Let $\rho(K)$ denote the spectral radius of the operator $\mathbb{K}$. Further, denote by

$$\lambda_1^+(K) \geq \lambda_2^+(K) \geq \ldots \geq \lambda_i^+(K) \geq \ldots \geq 0,$$

the nonnegative eigenvalues of $\mathbb{K}$; and

$$\lambda_1^-(K) \leq \lambda_2^-(K) \leq \ldots \leq \lambda_i^-(K) \leq \ldots \leq 0,$$

the nonpositive eigenvalues of $\mathbb{K}$; put $\lambda_r^+(K) = 0$ for every $r \geq r_0$, where r_0 is the largest integer such that $\lambda_r^+(K) \neq 0$, and analogously for the nonpositive eigenvalues.

We now present two auxiliary lemmas to be used in the proof of Theorem 6.3 below.

Lemma 6.1. *Let K_n, for each $n \in \mathbb{N}$, be a symmetric kernel of $L^2([a,b]^2)$. If*

$$\lim_{n\to+\infty} \sup_{s,t\in[a,b]} |K_n(s,t)| = 0,$$

then $\lim_{n\to+\infty} \rho(K_n) = 0$.

Proof: Indeed, the assumption made implies that $\lim_{n\to+\infty} \|\, K_n(s,t)\,\|_{L^2([a,b]^2)} = 0$. Now using classical results about operator norms we have

$$0 < \rho(K_n) = \|\,\mathbb{K}_n\,\| \leq \|\,K_n(s,t)\,\|_{L^2([a,b]^2)} \longrightarrow 0.$$

■

The next lemma is proved in Theorem 7 of Franche [5].

Lemma 6.2. *Let K_1 and K_2 be two symmetric kernels of $L^2([a,b]^2)$. Then,*

$$\left|\lambda_i^+(K_1) - \lambda_i^+(K_2)\right| \leq \rho(K_1 - K_2)$$

and

$$\left|\lambda_i^-(K_1) - \lambda_i^-(K_2)\right| \leq \rho(K_1 - K_2)\,.$$

If the kernel K is continuous and non-negative definite, as is the case for covariance functions, it is well known that all the eigenvalues are nonnegative and we will denote them by $\lambda_i(K)$. To estimate the eigenvalues of the operator defined by the true covariance Γ we will use the eigenvalues of the operator defined by $\widehat{\Gamma}_n(s,t)$. The next theorem establishes the consistency of these estimators and describes the convergence rates.

Theorem 6.3. *Let b_n and q_n be two sequences of positive integers tending to infinity and such that $a_n = b_n\,q_n$. Suppose that*

$$b_n \sum_{k=q_n+1}^{\infty} \mathrm{Cov}^{1/3}(X_1, X_{k+1}) \longrightarrow 0.$$

Further assume that (A1) *is satisfied, (4.1) holds for every sufficiently large n and $\frac{a_n^2 \ln n}{r_{1,n}} \longrightarrow 0$. Then,*

$$b_n \left|\lambda_i(\widehat{\Gamma}_n) - \lambda_i(\Gamma)\right| \longrightarrow 0 \ \ a.s.\,.$$

Proof: By Theorem 5.7 b)

$$\sup_{s,t\in\mathbb{R}} b_n \left|\widehat{\Gamma}_n(s,t) - \Gamma(s,t)\right| \longrightarrow 0 \qquad a.s.\,. \tag{6.1}$$

Define $A = \left\{\omega : \sup_{s,t\in\mathbb{R}} b_n \left|\widehat{\Gamma}_n(\omega,s,t) - \Gamma(s,t)\right| \longrightarrow 0\right\}$. Then, for each $\omega \in A$, we have, using Lemma 6.1,

$$\rho\left(b_n\left[\widehat{\Gamma}_n(\omega,\cdot,\cdot) - \Gamma(\cdot,\cdot)\right]\right) \longrightarrow 0\,.$$

Applying now Lemma 6.2 we get

$$\begin{aligned} 0 < b_n \left| \lambda_i(\widehat{\Gamma}_n(\omega,\cdot,\cdot)) - \lambda_i(\Gamma(\cdot,\cdot)) \right| \leq \\ \leq b_n\, \rho\left(\widehat{\Gamma}_n(\omega,\cdot,\cdot) - \Gamma(\cdot,\cdot)\right) = \rho\left(b_n \left[\widehat{\Gamma}_n(\omega,\cdot,\cdot) - \Gamma(\cdot,\cdot)\right]\right) \longrightarrow 0\,. \end{aligned}$$

Thus $b_n \left| \lambda_i(\widehat{\Gamma}_n(\omega,\cdot,\cdot)) - \lambda_i(\Gamma(\cdot,\cdot)) \right| \longrightarrow 0$. We then conclude that

$$\mathrm{P}\left(\left\{\omega : b_n \left| \lambda_i(\widehat{\Gamma}_n(\omega,\cdot,\cdot)) - \lambda_i(\Gamma(\cdot,\cdot)) \right| \longrightarrow 0\right\}\right) = 1\,,$$

completing the proof. ■

Note that, as what regards the construction of the sequences a_n, q_n and b_n, the assumptions of Theorem 6.3 and Theorems 5.6 and 5.7 coincide. So, by Corollary 5.8, if we choose $a_n = n^\beta$, with $\beta < \min\left\{\frac{1}{3}, \frac{\ln a}{3}\right\}$, $q_n = \ln n$ and $b_n = \frac{n^\beta}{\ln n}$, it follows that the estimators considered in this article for the eigenvalues are consistent with convergence rate of order at least $\frac{\ln n}{n^\beta}$.

References

[1] C. Azevedo and P. E. Oliveira (2000). *Kernel-type estimation of bivariate distribution function for associated random variables.* In: *New Trends in Probability and Statistics*, Vol. 5, Proceedings of the 6^{th} Tartu Conference, VSP, 17–25.

[2] L. Devroye (1991). *Exponential inequalities in nonparametric estimation.* In: *Nonparametric Functional Estimation and Related Topics*, G. Roussas, ed., Kluwer Academic Publishers, Dordrecht, 31–44.

[3] I. Dewan and B. L. S. Prakasa Rao (1999). A general method of density estimation for associated random variables. *J. Nonparametr. Statist.* 10, 405–420.

[4] J. D. Esary, F. Proschan and D. W. Walkup (1967). Association of random variables, with applications. *Ann. Math. Statist.* 38, 1466–1474.

[5] A. Franche (2000). Estimation des valeurs propes de la covariance du processus empirique sous dépendence faible. Application à la statistique de test de Cramée-von Mises. Preprint, Univ. Scienc. Techn. Lille.

[6] C. Henriques and P. E. Oliveira(2003). Estimation of a two-dimensional distribution function under association. *J. Statist. Planning Inf.*. 113 , 137–150.

[7] C. Henriques and P. E. Oliveira (2002). Convergence rates for the estimation of two-dimensional distribution functions under association and estimation of the covariance of the limit empirical process. Preprint, Pré-Publicações do Departamento de Matemática da Universidade de Coimbra, 02-05.

[8] C. Henriques and P. E. Oliveira (2002). Exponential rates for kernel density estimation under association. Preprint, Pré-Publicações do Departamento de Matemática da Universidade de Coimbra, 02-23.

[9] D. A. Ioannides and G. G. Roussas (1998). Exponencial inequality for associated random variables. *Statist. Probab. Lett.* 42, 423–431.

[10] E. Kreyszig (1978). Introductory functional analysis with applications. Jonh Wiley & Sons. Inc., New York.

[11] J. Lebowitz (1972). Bounds on the correlations and analycity properties of ferromagnetic Ising spin systems. *Comm. Math. Phys.* 28, 313–321.

[12] E. Masry (2002). Mulitvariate probability density estimation for associated processes: strong consistency and rates. *Statist. Probab. Letters* 58, 205–219.

[13] C. M. Newman (1980). Normal fluctuations and the FKG inequalities. *Comm. Math Phys.* 74, 119–128.

[14] C. Newman (1984). *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in Y. L. Tong (ed) Inequalities in Statistics and Probability, IMS Lecture Notes - Monograph Series Vol. 5, 127–140.

[15] P. E. Oliveira (2003). An exponential inequality for associated variables. Preprint, Pré-Publicações do Departamento de Matemática da Universidade de Coimbra, 03-17.

[16] Roussas, G. G. (1995). Asymptotic normality of a smooth estimate of a random field distribution function under association. *Statist. Probab. Letters* 24, 77–90.

Carla Henriques
Escola Superior de Tecnologia de Viseu, 3504-510 Viseu, Portugal

Paulo Eduardo Oliveira
Dep. Matemática, Univ. Coimbra, Apartado 3008, 3001 - 454 Coimbra, Portugal