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REDUCTION OF JACOBI MANIFOLDS VIA DIRAC STRUCTURES THEORY

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ABSTRACT: We first recall some basic definitions and facts about Jacobi manifolds, generalized Lie bialgebroids, generalized Courant algebroids and Dirac structures. Then, we investigate some relations between Dirac subbundles L for the double $(A \oplus A^*, \phi + W)$ of a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ over M and the associated Dirac subbundles $\tilde{L}$ for the double $\tilde{A} \oplus \tilde{A}^*$ of the corresponding Lie bialgebroid $(\tilde{A}, \tilde{A}^*)$ over $M \times \mathbb{R}$. We establish an one-one correspondence between reducible Dirac structures for the generalized Lie bialgebroid of a Jacobi manifold (M, Λ, E) and Jacobi quotient manifolds of M. We study Jacobi reductions from the point of view of Dirac structures theory and we present some examples and applications.

KEYWORDS: Dirac structures, generalized Lie bialgebroids, generalized Courant algebroids, Jacobi manifolds, reduction.
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1. Introduction

The concept of a *Dirac structure* on a differentiable manifold M was introduced by T. Courant and A. Weinstein in [2] and developed by T. Courant in [3]. Its principal aim is to present a unified framework for the study of pre-symplectic forms, Poisson structures and foliations. More specifically, a *Dirac structure* on M is a subbundle $L \subset TM \oplus T^*M$ that is maximally isotropic with respect to the canonical symmetric bilinear form on $TM \oplus T^*M$ and satisfies a certain integrability condition. In order to formulate this integrability condition, T. Courant defines a bilinear, skew-symmetric, bracket operation on the space $\Gamma(TM \oplus T^*M)$ of smooth sections of $TM \oplus T^*M$ which does not satisfy the Jacobi identity. The nature of this bracket was clarified by Z.-J. Liu, A. Weinstein and P. Xu in [21] by introducing the structure of a *Courant algebroid* on a vector bundle $E \to M$ over M and by extending the notion of a Dirac structure to the subbundles $L \subset E$. The most important example of Courant algebroid is the direct sum $A \oplus A^*$ of a Lie bialgebroid ([25]).

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Alan Weinstein and its collaborators have studied several problems of Poisson geometry via Dirac structures theory. In [22], Z.-J. Liu et al. establish an one-one correspondence between Dirac structures on the double $TM \oplus T^*M$ of the triangular Lie bialgebroid (TM, T^*M, Λ) defined on a Poisson manifold (M, Λ) and Poisson structures on quotient manifolds of M. Using this correspondence and the results concerning the pull-backs Dirac structures under Lie algebroid morphisms, Z.-J. Liu constructs in [23] the Poisson reduction in terms of Dirac structures.

On the other hand, it is well known that the notion of *Jacobi manifold*, i.e. a differentiable manifold M endowed with a bivector field Λ and a vector field E satisfying an integrability condition, introduced by A. Lichnerowicz in [20], is a rich geometrical notion that generalizes the Poisson, symplectic, contact and co-symplectic manifolds. Thus, it is natural to research a simple interpretation of Jacobi manifolds by means of Dirac structures. A first approach of this problem is presented in [35] by A. Wade. Taking into account that to any Jacobi structure (Λ, E) on M is canonically associated a generalized Lie bialgebroid structure on $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ ([12]), she considers the Whitney sum $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, introduces the notion of $\mathcal{E}^1(M)$-*Dirac structures* by extending the Courant's bracket to the space $\Gamma(\mathcal{E}^1(M))$ of smooth sections of $\mathcal{E}^1(M)$ and shows that the graph of the vector bundle morphism $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ is a Dirac subbundle of $\mathcal{E}^1(M)$. But the extended bracket does not endow $\mathcal{E}^1(M)$ with a Courant algebroid structure. A second approach of the problem is the one proposed by the second author and J. Clemente-Gallardo in the recent paper [33]. They introduce the notions of *generalized Courant algebroid* (which is equivalent to the notion of *Courant-Jacobi algebroid* independently defined by J. Grabowski and G. Marmo in [10]) and of *Dirac structure for a generalized Courant algebroid* and give several connections between Dirac structure for generalized Courant algebroids and Jacobi manifolds. We note that the construction of [33] includes as particular case the one of Wade and that the main example of generalized Courant algebroid over M is the direct sum of a generalized Lie bialgebroid over M.

In the present work, by using the results mentioned above, we establish a reduction theorem of Jacobi manifolds (Theorem 6.2). It is well known that there are already several geometric and algebraic treatments of the Jacobi reduction problem (see, for instance, [30], [31], [29], [11]). But, it is an original goal of the Dirac structures theory to describe Jacobi reduction and to construct a more general framework for the study of the related problems concerning the projection of Jacobi structures and the existence of Jacobi structures on certain submanifolds of Jacobi manifolds. Precisely, on the way to our principal result, we construct an one to one correspondence between Dirac subbundles, satisfying a certain regularity condition, of the double $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, where M is a Jacobi manifold, and quotient Jacobi manifolds of M (Theorem 5.12). Also, the Reduction Theorem 6.2 allows us to state new sufficient conditions under which a submanifold N of (M, Λ, E) inherits a Jacobi structure, that include as particular cases the results presented in [14], [4].

The paper is organized as follows. In sections 2 and 3 we recall some basic definitions and results concerning, respectively, Jacobi structures, generalized Lie bialgebroids and Dirac structures for generalized Courant algebroids. In section 4 we investigate relations between Dirac structures for a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ and the associated structures of the Lie bialgebroid $(\tilde{A}, \tilde{A}^*)$, with $\tilde{A} = A \times \mathbb{R}$ and $\tilde{A}^* = A^* \times \mathbb{R}$, which are useful throughout the paper. In section 5 we establish a correspondence between Dirac structures and quotient Jacobi manifolds (Theorem 5.12). Using this correspondence and the results for the pull-backs Dirac structures by Lie algebroid morphisms, we prove, in section 6, a Jacobi reduction theorem (Theorem 6.2) which differs at same points from that proved in [30] and independently in [29] and give us a less strict reduction condition. Finally, in section 7 we present some applications and examples.

Notation : In this paper, M is a C^∞-differentiable manifold of finite dimension. We denote by TM and T^*M, respectively, the tangent and cotangent bundles over M, $C^\infty(M, \mathbb{R})$ the space of all real C^∞-differentiable functions on M, $\Omega^k(M)$ the space of all differentiable k-forms on M and $\mathcal{X}(M)$ the space of all differentiable vector fields on M. Also, we denote by δ the usual differential operator on the graded space $\Omega(M) = \oplus_{k\in\mathbb{Z}}\Omega^k(M)$. For the Schouten bracket and the interior product of a form with a multi-vector field, we use the convention of sign indicated by Koszul [19], (see also [27]).

2. Jacobi structures and Generalized Lie bialgebroids

A *Jacobi manifold* is a differentiable manifold M equipped with a bivector field Λ and a vector field E such that the Schouten brackets of Λ with itself and of E with Λ are, respectively,

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda] = 0,$$

([20]). In this case, the pair (Λ, E) defines on $C^\infty(M, \mathbb{R})$ the internal composition law :

$$\{f, g\}_{(\Lambda,E)} = \Lambda(\delta f, \delta g) + \langle f\delta g - g\delta f, E\rangle, \;\; f, g \in C^\infty(M, \mathbb{R}), \tag{1}$$

which endows $C^\infty(M, \mathbb{R})$ with a local Lie algebra structure [17], [20], (or with a Jacobi algebra structure in the terminology of J. Grabowski et al., [8], [10]). Also, if $\psi \in C^\infty(M, \mathbb{R})$ is a function that never vanishes on M and $\{\,,\,\}^\psi : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ the new bilinear and skew-symmetric internal composition law on $C^\infty(M, \mathbb{R})$ given, for each pair $(f, g) \in C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R})$, by

$$\{f, g\}^\psi := \frac{1}{\psi}\{\psi f, \psi g\}_{(\Lambda,E)},$$

then $\{\,,\,\}^\psi$ endows $C^\infty(M, \mathbb{R})$ with a new Jacobi bracket that defines a new Jacobi structure (Λ^ψ, E^ψ) on M, which is said to be *ψ-conformal* to the initially given one. The structures (Λ, E) and (Λ^ψ, E^ψ) are said to be *conformally equivalent* and we have :

$$\Lambda^\psi = \psi\Lambda \;\; \text{and} \;\; E^\psi = \Lambda^\#(\delta\psi) + \psi E.$$

The equivalence class of the Jacobi structures on M that are conformally equivalent to a given Jacobi structure is called *a conformal Jacobi structure on M*.

Let (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) be two Jacobi manifolds and $\Psi : M_1 \to M_2$ a differentiable map. If Λ_1 and E_1 are projectable by Ψ on M_2 and their projections are, respectively, Λ_2 and E_2, i.e $\Psi_*\Lambda_1 = \Lambda_2$ and $\Psi_*E_1 = E_2$, then $\Psi : M_1 \to M_2$ is said to be a *Jacobi morphism* or a *a Jacobi map.* When $\Psi : M_1 \to M_2$ is a diffeomorphism, the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are said to be *equivalent.* If there exists $\psi \in C^\infty(M_1, \mathbb{R})$ that never vanishes on M_1 such that $\Psi : (M_1, \Lambda_1^\psi, E_1^\psi) \to (M_2, \Lambda_2, E_2)$ is a Jacobi map, then $\Psi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ is called a *ψ-conformal Jacobi map.*

A *Lie algebroid* over a smooth manifold M is a vector bundle $A \to M$ with a Lie algebra structure $[\,,\,]$ on the space $\Gamma(A)$ of the global cross sections of $A \to M$ and a bundle map $a : A \to TM$, called the *anchor map*, such that

(1) the homomorphism $a : (\Gamma(A), [\,,\,]) \to (\mathcal{X}(M), [\,,\,])$, induced by the anchor map, is a Lie algebra homomorphism ;
(2) for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$,

$$[X, fY] = f[X, Y] + (a(X)f)Y.$$

We denote a Lie algebroid over M by the triple $(A, [\,,\,], a)$. For more details see, for example, [24], [1] and [27].

A trivial example of a Lie algebroid over a differentiable manifold M is $(TM, [\,,\,], Id)$, i.e. the tangent bundle TM of M equipped with the usual Lie bracket of vector fields on M and the identity map $Id : TM \to TM$ as anchor map. Another example of Lie algebroid over M is $(TM \times \mathbb{R}, [\,,\,], \pi)$, where, for all $(X, f), (Y, g) \in \Gamma(TM \times \mathbb{R}) \cong \mathcal{X}(M) \times C^\infty(M, \mathbb{R})$,

$$[(X, f), (Y, g)] = ([X, Y],\ X \cdot g - Y \cdot f), \tag{2}$$

and $\pi : TM \times \mathbb{R} \to TM$ is the projection on the first factor.

The *Lie algebroid of a Jacobi manifold* is defined in [16] as follows. We consider :

i) on the space $\Gamma(T^*M \times \mathbb{R}) \cong \Omega^1(M) \times C^\infty(M, \mathbb{R})$, the Lie algebra bracket $[\,,\,]_{(\Lambda,E)}$ given, for all $(\alpha, f), (\beta, g) \in \Gamma(T^*M \times \mathbb{R})$, by

$$[(\alpha, f), (\beta, g)]_{(\Lambda,E)} := (\gamma, h), \tag{3}$$

where

$$\gamma := \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta),$$

$$h := -\Lambda(\alpha, \beta) + \Lambda(\alpha, \delta g) - \Lambda(\beta, \delta f) + \langle f\delta g - g\delta f, E\rangle,$$

and

ii) the vector bundle morphism $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ defined, for any $(\alpha, f) \in \Gamma(T^*M \times \mathbb{R})$, by

$$(\Lambda, E)^\#((\alpha, f)) = (\Lambda^\#(\alpha) + fE,\ -\langle \alpha, E\rangle).$$

Then the triple $(T^*M \times \mathbb{R}, [\,,\,]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\#)$ is a Lie algebroid over M.

For a Lie algebroid $(A, [\,,\,], a)$ over M, we denote by A^* its dual vector bundle over M and by $\bigwedge A^* = \oplus_{k\in\mathbb{Z}} \bigwedge^k A^*$ the graded exterior algebra of A^*. Sections of $\bigwedge A^*$ are called *A-differential forms* (or *A-forms*) on M. There exists a graded endomorphism of degree 1 of the exterior algebra of

A-forms, $d : \Gamma(\bigwedge A^*) \to \Gamma(\bigwedge A^*)$, called the *exterior derivative*, taking an A-k-form η to an A-$(k+1)$-form $d\eta$ such that, for all $X_1, \dots, X_{k+1} \in \Gamma(A)$,

$$\begin{aligned} d\eta(X_1, \dots, X_{k+1}) &= \\ &= \Sigma_{i=1}^{k+1}(-1)^{i+1}a(X_i)\cdot\eta(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) + \\ &+ \ \Sigma_{1\leq i<j\leq k+1}(-1)^{i+j}\eta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}). \end{aligned}$$

The Lie algebroid axioms of A imply that d is $C^\infty(M, \mathbb{R})$-multilinear and a superderivation of degree 1 and $d^2 = 0$. Also, we denote by $\bigwedge A = \oplus_{k\in\mathbb{Z}} \bigwedge^k A$ the graded exterior algebra of A whose sections are called *A-multivector fields*. The Lie bracket on $\Gamma(A)$ can be extended to the exterior algebra of A-multivector fields and the result is a graded Lie bracket $[\, ,]$, called the *Schouten bracket* of the Lie algebroid A. Details may be found, for instance, in [24], [18] and [1].

Let $(A, [\, ,], a)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ be an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients ([24], [12]), i.e. for any $X, Y \in \Gamma(A)$,

$$\langle \phi, [X, Y]\rangle = a(X)(\langle \phi, Y\rangle) - a(Y)(\langle \phi, X\rangle). \tag{4}$$

We modify the usual representation of the Lie algebra $(\Gamma(A), [\, ,])$ on the space $C^\infty(M, \mathbb{R})$ by defining a new representation $a^\phi : \Gamma(A) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ as

$$a^\phi(X, f) = a(X)f + \langle \phi, X\rangle f, \tag{5}$$

where $(X, f) \in \Gamma(A) \times C^\infty(M, \mathbb{R})$. The resulting cohomology operator $d^\phi : \Gamma(\bigwedge A^*) \to \Gamma(\bigwedge A^*)$ of the new cohomology complex is called the *ϕ-differential* of A and its expression in terms of d is

$$d^\phi\eta = d\eta + \phi \wedge \eta, \tag{6}$$

for $\eta \in \Gamma(\bigwedge^k A^*)$. The new cohomology operator d^ϕ allows us to define, in a natural way, the *ϕ-Lie derivative by* $X \in \Gamma(A)$, $\mathcal{L}^\phi_X : \Gamma(\bigwedge^k A^*) \to \Gamma(\bigwedge^k A^*)$, as the commutator of d^ϕ and of the contraction by X, i.e. $\mathcal{L}^\phi_X = d^\phi \circ i_X + i_X \circ d^\phi$. Its expression in terms of the usual Lie derivative $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is, for $\eta \in \Gamma(\bigwedge^k A^*)$,

$$\mathcal{L}^\phi_X\eta = \mathcal{L}_X\eta + \langle \phi, X\rangle\eta. \tag{7}$$

Using ϕ we can also modify the Schouten bracket $[\, ,]$ on $\Gamma(\bigwedge A)$ to the *ϕ-Schouten bracket* $[\, ,]^\phi$ on $\Gamma(\bigwedge A)$. It is defined, for all $P \in \Gamma(\bigwedge^p A)$ and

$Q \in \Gamma(\bigwedge^q A)$, by

$$[P, Q]^\phi = [P, Q] + (p-1)P \wedge (i_\phi Q) + (-1)^p (q-1)(i_\phi P) \wedge Q, \qquad (8)$$

where $i_\phi Q$ can be interpreted as the usual contraction of a multivector field with an 1-form. We remark that, when $p = q = 1$, $[P, Q]^\phi = [P, Q]$, i.e. the brackets $[\,,\,]^\phi$ and $[\,,\,]$ coincide on $\Gamma(A)$.

For a representation of the differential calculus using the ϕ-modified derivative, Lie derivative and Schouten bracket, see [12] and [9].

The notion of *generalized Lie bialgebroid* has been introduced by D. Iglesias and J.C. Marrero in [12] in such a way that a Jacobi manifold has a generalized Lie bialgebroid canonically associated and conversely. We consider a Lie algebroid $(A, [\,,\,], a)$ over M and an 1-cocycle $\phi \in \Gamma(A^*)$ and we assume that the dual vector bundle $A^* \to M$ admits a Lie algebroid structure $([\,,\,]_*, a_*)$ and that $W \in \Gamma(A)$ is an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients of $(A^*, [\,,\,]_*, a_*)$. Then, we say that :

Definition 2.1. *The pair $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M if, for all $X, Y \in \Gamma(A)$ and $P \in \Gamma(\bigwedge^p A)$, the following conditions hold :*

$$d_*^W [X, Y] = [d_*^W X, Y]^\phi + [X, d_*^W Y]^\phi, \qquad (9)$$

$$\mathcal{L}_{*\phi}^W P + \mathcal{L}_W^\phi P = 0, \qquad (10)$$

where d_*^W and $\mathcal{L}_*^W$ are, respectively, the W-differential and the W-Lie derivative of A^*.

An equivalent definition of this notion was presented in [9] by J. Grabowski and G. Marmo under the name of *Jacobi bialgebroid*. Precisely, they define that :

Definition 2.2. *The pair $((A, \phi), (A^*, W))$ is a Jacobi bialgebroid if for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$,*

$$d_*^W [P, Q]^\phi = [d_*^W P, Q]^\phi + (-1)^{p+1} [P, d_*^W Q]^\phi.$$

In the particular case where $\phi = 0$ and $W = 0$, we recover by the above two definitions, respectively, the notion of *Lie bialgebroid* introduced by K. Mackenzie and P. Xu in [25] and its equivalent definition given by Yv. Kosmann-Schwarzbach in [18].

Remark 2.3. The property of duality of a Lie bialgebroid is also verified in the case of a generalized Lie bialgebroid : i.e. if $((A,\phi),(A^*,W))$ is a generalized Lie bialgebroid, so is $((A^*,W),(A,\phi))$ (see [12], [9]). Consequently, conditions (9)-(10) of Definition 2.1 as well as of Definition 2.2 can be replaced by their dual versions.

The fundamental results of [12], which will be used in the sequel, are the following theorems.

Theorem 2.4. *Let (M,Λ,E) be a Jacobi manifold. Then the pair $((TM\times I\!R,[\,,\,],\pi,(0,1)),(T^*M\times I\!R,[\,,\,]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#},(-E,0)))$ is a generalized Lie bialgebroid.*

Theorem 2.5. *Let $((A,\phi),(A^*,W))$ be a generalized Lie bialgebroid over a differentiable manifold M. Then the bracket of functions $\{\,,\,\}_J : C^\infty(M,I\!R)\times C^\infty(M,I\!R)\to C^\infty(M,I\!R)$ given, for all $f,g\in C^\infty(M,I\!R)$, by*

$$\{f,g\}_J := \langle d^\phi f, d_*^W g\rangle, \tag{11}$$

defines a Jacobi structure on M.

Corollary 2.6. *If $((A,\phi),(A^*,W)) = ((TM\times I\!R,[\,,\,],\pi,(0,1)),(T^*M\times I\!R,[\,,\,]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#},(-E,0)))$ is the generalized Lie bialgebroid associated to a Jacobi manifold (M,Λ,E), then*

$$\{f,g\}_J = \{f,g\}_{(\Lambda,E)}, \qquad \forall\, f,g\in C^\infty(M,I\!R). \tag{12}$$

Proof : Effectively, for all $f,g\in C^\infty(M,\mathbb{R})$,

$$\begin{aligned}\{f,g\}_J &= \langle d^{(0,1)}f, d_*^{(-E,0)}g\rangle = \\ &= \langle(\delta f,f),(-\Lambda^{\#}(\delta g)-gE,\langle\delta g,E\rangle)\rangle = \\ &= \Lambda(\delta f,\delta g)+\langle f\delta g-g\delta f,E\rangle = \\ &\overset{(1)}{=} \{f,g\}_{(\Lambda,E)}.\ \bullet\end{aligned}$$

An important class of generalized Lie bialgebroids is the one of *triangular generalized Lie bialgebroids* defined, also in [12] and [13], as follows :

Definition 2.7. *A generalized Lie bialgebroid $((A,\phi),(A^*,W))$ is said to be a triangular generalized Lie bialgebroid if there exists $P\in\Gamma(\bigwedge^2 A)$ such that $[P,P]^\phi=0$, the Lie bracket $[\,,\,]_*$ on $\Gamma(A^*)$ is the bracket*

$$[\alpha,\beta]_P = \mathcal{L}^\phi_{P\#(\alpha)}\beta-\mathcal{L}^\phi_{P\#(\beta)}\alpha-d^\phi(P(\alpha,\beta)), \qquad \forall\,\alpha,\beta\in\Gamma(A^*), \tag{13}$$

$a_* = a \circ P^{\#}$ *and* $W = -P^{\#}(\phi)$.

A characteristic example of triangular generalized Lie bialgebroid is the generalized Lie bialgebroid of a Jacobi manifold of Theorem 2.4, where $[(\Lambda, E),(\Lambda, E)]^{(0,1)} = 0$ holds.

Let us now recall how, given a Lie algebroid $(A, [\,,\,], a)$ over M, we can construct Lie algebroid structures on the vector bundle $\tilde{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of $\tilde{A} \to M \times \mathbb{R}$ can be identified with the set of the time-dependent sections of $A \to M$, i.e. for any $\tilde{X} \in \Gamma(\tilde{A})$ and $(x,t) \in M \times \mathbb{R}$, t being the canonical coordinate on $\mathbb{R}$, $\tilde{X}(x,t) = \tilde{X}_t(x)$, where $\tilde{X}_t \in \Gamma(A)$. This identification induces, in a natural manner, a Lie bracket $[\,,\,]\tilde{}$ on $\Gamma(\tilde{A})$:

$$[\tilde{X}, \tilde{Y}]\tilde{}(x,t) = [\tilde{X}_t, \tilde{Y}_t](x), \quad \tilde{X}, \tilde{Y} \in \Gamma(\tilde{A}), \;\; (x,t) \in M \times \mathbb{R},$$

and a bundle map $\tilde{a} : \tilde{A} \to T(M \times \mathbb{R})$, $\tilde{a}(\tilde{X})(x,t) = a(\tilde{X}_t)(x)$, which endow $\tilde{A}$ with a Lie algebroid structure over $M \times \mathbb{R}$. On the other hand, taking an 1-cocycle of A, $\phi \in \Gamma(A^*)$, we deform $([\,,\,]\tilde{}, \tilde{a})$ in two different ways and we obtain two new Lie algebroid structures on $\tilde{A}$, [12]. Precisely, for $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$, we set

$$\begin{aligned} [\tilde{X}, \tilde{Y}]\tilde{}^{\phi} &= [\tilde{X}, \tilde{Y}]\tilde{} + \langle \phi, \tilde{X}_t \rangle \frac{\partial \tilde{Y}}{\partial t} - \langle \phi, \tilde{Y}_t \rangle \frac{\partial \tilde{X}}{\partial t}, \\ \tilde{a}^{\phi}(\tilde{X}) &= \tilde{a}(\tilde{X}) + \langle \phi, \tilde{X}_t \rangle \frac{\partial}{\partial t}; \end{aligned} \tag{14}$$

$$\begin{aligned} [\tilde{X}, \tilde{Y}]\hat{}^{\phi} &= e^{-t}([\tilde{X}, \tilde{Y}]\tilde{} + \langle \phi, \tilde{X}_t \rangle (\frac{\partial \tilde{Y}}{\partial t} - \tilde{Y}) - \langle \phi, \tilde{Y}_t \rangle (\frac{\partial \tilde{X}}{\partial t} - \tilde{X})), \\ \hat{a}^{\phi}(\tilde{X}) &= e^{-t}(\tilde{a}(\tilde{X}) + \langle \phi, \tilde{X}_t \rangle \frac{\partial}{\partial t}). \end{aligned} \tag{15}$$

We have :

Theorem 2.8 ([12])**.** *Let* $(A, [\,,\,], a)$ *be a Lie algebroid over* M *and* $\phi \in \Gamma(A^*)$ *be an 1-cocycle. Suppose that* $([\,,\,]_*, a_*)$ *is a Lie algebroid structure on* A^* *and* $W \in \Gamma(A)$ *is an 1-cocycle. Consider on* $\tilde{A} = A \times \mathbb{R}$ *and* $\tilde{A}^* = A^* \times \mathbb{R}$ *the Lie algebroid structures* $([\,,\,]\tilde{}^{\phi}, \tilde{a}^{\phi})$ *and* $([\,,\,]\hat{}^{W}_{*}, \hat{a}^{W}_{*})$*, respectively. Then* $(\tilde{A}, \tilde{A}^*)$ *is a Lie bialgebroid over* $M \times \mathbb{R}$ *if and only if* $((A, \phi), (A^*, W))$ *is a generalized Lie bialgebroid over* M*. The induced Poisson structure on* $M \times \mathbb{R}$ *is the Poissonization of the induced Jacobi structure on* M.

Moreover, it is well known that the image Ima of the anchor map a of a Lie algebroid $(A, [\,,\,], a)$ over M is an integrable distribution on M, (see [6] and [7]). Thus, Ima defines a singular foliation $\mathcal{F}_A$ of M, called the *Lie algebroid foliation of M associated with A* ([15]) or *the orbit foliation of the Lie algebroid A* ([7]).

The relation between the leaves of the Lie algebroid foliation $\mathcal{F}_{\tilde{A}}$ of $M \times \mathbb{R}$ associated with $(\tilde{A}, \widetilde{[\,,\,]}^{\phi}, \tilde{a}^{\phi})$ (given by (14)) and the leaves of the Lie algebroid foliation $\mathcal{F}_A$ of M associated with A was studied in [15] by D. Iglesias and J.C. Marrero. They have proved :

Theorem 2.9 ([15])**.** *Under the above considerations, suppose that $(x_0, t_0) \in M \times \mathbb{R}$ and that $\tilde{F}$ and F are the leaves of the Lie algebroid foliations $\mathcal{F}_{\tilde{A}}$ and $\mathcal{F}_A$ passing through $(x_0, t_0) \in M \times \mathbb{R}$ and $x_0 \in M$, respectively, and denote by A_{x_0} the fiber of A over x_0. Then*

(1) *if $\ker(a|_{A_{x_0}})$ is not contained in $\langle \phi(x_0) \rangle^{\circ}$, we have that $\tilde{F} = F \times \mathbb{R}$;*
(2) *if $\ker(a|_{A_{x_0}})$ is contained in $\langle \phi(x_0) \rangle^{\circ}$ and $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection onto the first factor, we have that $\pi_1(\tilde{F}) = F$ and that the map $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map.*

3. Generalized Courant algebroids and Dirac structures

The notion of *generalized Courant algebroid* has been introduced by the second author and J. Clemente-Gallardo in [33] and independently, under the name of *Courant-Jacobi algebroid*, by J. Grabowski and G. Marmo in [10]. In this section, we recall some basic facts concerning this notion and its relation with Dirac and Jacobi structures.

Definition 3.1 ([33])**.** *A generalized Courant algebroid over a differentiable manifold M is : i) a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $(\,,\,)$ on the bundle, a skew-symmetric bracket $[\,,\,]$ on $\Gamma(E)$ and a bundle map $\rho : E \to TM$ and ii) an E-1-form θ such that, for any $e_1, e_2 \in \Gamma(E)$, $\langle \theta, [e_1, e_2] \rangle = \rho(e_1)\langle \theta, e_2 \rangle - \rho(e_2)\langle \theta, e_1 \rangle$, verifying the following relations :*

(1) *for any $e_1, e_2, e_3 \in \Gamma(E)$,*
$$[[e_1, e_2], e_3] + c.p. = \mathcal{D}^{\theta} T(e_1, e_2, e_3);$$
(2) *for any $e_1, e_2 \in \Gamma(E)$,*
$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]; \tag{16}$$

(3) *for any* $e_1, e_2 \in \Gamma(E)$ *and* $f \in C^\infty(M, I\!R)$,

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathcal{D}f; \tag{17}$$

(4) *for any* $f, g \in C^\infty(M, I\!R)$,

$$(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0;$$

(5) *for any* $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)(e_1, e_2) + \langle\theta, e\rangle(e_1, e_2) = ([e, e_1] + \mathcal{D}^\theta(e, e_1), e_2) + (e_1, [e, e_2] + \mathcal{D}^\theta(e, e_2)).$$

T is the function on the base M defined, for any $e_1, e_2, e_3 \in \Gamma(E)$, by

$$T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p.$$

and $\mathcal{D}, \mathcal{D}^\theta : C^\infty(M, I\!R) \to \Gamma(E)$ are given, for any $f \in C^\infty(M, I\!R)$ and $e \in \Gamma(E)$, respectively, by

$$(\mathcal{D}f, e) = \frac{1}{2}\rho(e)f \quad \text{and} \quad (\mathcal{D}^\theta f, e) = \frac{1}{2}(\rho(e)f + \langle\theta, e\rangle f).$$

The above definition is based on the original definition of Courant algebroid presented in [21] by Z.-J. Liu and al. while its equivalent definition proposed in [10] is based on the alternative definition of Courant algebroid given by D. Roytenberg in [34]. Their equivalence is established in [33].

By defining, for any $e \in \Gamma(E)$, the first order differential operator $\rho^\theta(e)$ by

$$\rho^\theta(e) = \rho(e) + \langle\theta, e\rangle, \tag{18}$$

we have that Property (16) of Definition 3.1 is equivalent ([33]) to

$$\rho^\theta([e_1, e_2]) = [\rho^\theta(e_1), \rho^\theta(e_2)], \tag{19}$$

where the bracket on the right-hand side is the Lie bracket (2).

Definition 3.2. *A Dirac structure for a generalized Courant algebroid (E, θ) over M is a subbundle $L \subset E$ that is maximal isotropic under $(\,,\,)$ and integrable, i.e. $\Gamma(L)$ is closed under $[\,,\,]$.*

It is immediate from the above definition that a Dirac subbundle L of (E, θ) is a Lie algebroid under the restrictions of the bracket $[\,,\,]$ and of the anchor ρ to $\Gamma(L)$ and that the restriction of $\langle\theta, \cdot\rangle$ to $\Gamma(L)$ is an 1-cocycle for the Lie algebroid cohomology with trivial coefficients of $(L, [\,,\,]|_L, \rho|_L)$.

We consider now a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ over M and we denote by E its vector bundle direct sum, i.e. $E = A \oplus A^*$. On

E there exist two natural nondegenerate bilinear forms, one symmetric and another skew-symmetric : for any $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in E = A \oplus A^*$,

$$(e_1, e_2)_\pm = (X_1 + \alpha_1, X_2 + \alpha_2)_\pm = \frac{1}{2}(\langle \alpha_1, X_2 \rangle \pm \langle \alpha_2, X_1 \rangle). \tag{20}$$

On $\Gamma(E)$, which is identified with $\Gamma(A) \oplus \Gamma(A^*)$, we introduce the bracket $[\![\,,\,]\!]$ defined, for all $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in \Gamma(E)$, by

$$\begin{aligned} [\![e_1, e_2]\!] &= [\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = \\ &= ([X_1, X_2]^\phi + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1 - d^W_*(e_1, e_2)_-) + \\ &+ ([\alpha_1, \alpha_2]^W_* + \mathcal{L}^\phi_{X_1} \alpha_2 - \mathcal{L}^\phi_{X_2} \alpha_1 + d^\phi(e_1, e_2)_-). \end{aligned} \tag{21}$$

Finally, let $\rho : E \to TM$ be the bundle map given by $\rho = a + a_*$, i.e., for any $X + \alpha \in E$,

$$\rho(X + \alpha) = a(X) + a_*(\alpha). \tag{22}$$

The following result, which is proved in [33], shows that the notion of generalized Courant algebroid permits us to generalize the double construction for Lie bialgebras (the *Drinfeld double*, [5]) and Lie bialgebroids ([21]) to generalized Lie bialgebroids.

Theorem 3.3 ([33])**.** *If $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M, then $E = A \oplus A^*$ endowed with $([\![\,,\,]\!], (\,,\,)_+, \rho)$ and $\theta = \phi + W \in \Gamma(E^*)$ is a generalized Courant algebroid over M. The operators $\mathcal{D}$ and $\mathcal{D}^\theta$ are, respectively, $\mathcal{D} = (d + d_*)|_{C^\infty(M,\mathbb{R})}$ and $\mathcal{D}^\theta = (d^\phi + d^W_*)|_{C^\infty(M,\mathbb{R})}$.*

There are two important classes of Dirac structures for the generalized Courant algebroid $(E, \theta) = (A \oplus A^*, \phi + W)$ studied in [33].

The Dirac structure of the graph of an A-bivector field : Let Ω be an A-bivector field and $\Omega^\# : A^* \to A$ the associated vector bundle map. The graph of $\Omega^\#$ is the maximal isotropic vector subbundle

$$L = \{\Omega^\# \alpha + \alpha \,/\, \alpha \in A^*\}$$

of $(A \oplus A^*, (\,,\,)_+)$. L is a Dirac structure for $(A \oplus A^*, \phi + W)$ if and only if Ω satisfies the Maurer-Cartan type equation :

$$d^W_* \Omega + \frac{1}{2}[\Omega, \Omega]^\phi = 0.$$

Null Dirac structures : Let $D \subset A$ be a vector subbundle of A and $D^{\perp} \subset A^*$ its conormal bundle, i.e.

$$D^{\perp} = \{\alpha \in A^* \,/\, \langle \alpha, X\rangle = 0,\ \forall X \in A\}. \tag{23}$$

Then, $L = D \oplus D^{\perp}$ is a Dirac structure for $(A \oplus A^*, \phi + W)$ if and only if D and $D^{\perp}$ are Lie subalgebroids ([24]) of A and A^*, respectively. It is clear that in this context, as in the context of a Lie bialgebroid, $L = D \oplus D^{\perp}$ if and only if the skew-symmetric bilinear form $(\,,\,)_-$, defined on $E = A \oplus A^*$ by (20), vanishes on L. For this reason, L is said to be a *null Dirac structure*.

A third important category of Dirac structures for $(E, \theta) = (A \oplus A^*, \phi + W)$, also studied in [33], which generalizes both the above presented categories, is :

Dirac structures defined by a characteristic pair : We consider a pair (D, Ω) of a smooth subbundle $D \subset A$ and of an A-bivector field Ω. We construct, following [23], a subbunlde $L \subset A \oplus A^*$ by setting :

$$L = \{X + \Omega^{\#}\alpha + \alpha \,/\, X \in D \ and \ \alpha \in D^{\perp}\} = D \oplus graph(\Omega^{\#}|_{D^{\perp}}). \tag{24}$$

L is maximal isotropic with respect to $(\,,\,)_+$. The pair (D, Ω) is called the *characteristic pair of* L while the subbundle $D = L \cap (A \oplus \{0\})$, also denoted by $D = L \cap A$, is called the *characteristic subbundle of* L.

For simplicity, we will assume in the sequel that $D = L \cap A$ is of constant rank.

Moreover, since $D^{\perp}$ may be considered as the dual bundle $(A/D)^*$ of the quotient bundle A/D, the restricted vector bundle map $\Omega^{\#}|_{D^{\perp}}$ can be seen as the bundle map associated to an A/D-bivector field. Hence, two pairs (D_1, Ω_1) and (D_2, Ω_2) of a smooth subbundle and of an A-bivector field determine the same subbundle $L \subset A \oplus A^*$ (given by (24)) if and only if

$$D_1 = D_2 =: D \ \text{ and } \ \Omega_1^{\#}(\alpha) - \Omega_2^{\#}(\alpha) \in D, \ \ \forall \alpha \in D^{\perp}. \tag{25}$$

Let $pr : \Gamma(\bigwedge A) \to \Gamma(\bigwedge(A/D))$ be the map on the spaces of sections, induced by the natural projection $A \to A/D$. In order to express that the projection under pr of an A-multivector field $\Omega \in \Gamma(\bigwedge A)$ vanishes in $\Gamma(\bigwedge(A/D))$, we write $\Omega \equiv 0 (mod\, D)$. Thus, the second condition of (25) can be written as $\Omega_1 - \Omega_2 \equiv 0(mod\, D)$.

The conditions under which $L = D \oplus graph(\Omega^{\#}|_{D^{\perp}})$ is a Dirac subbundle of $(A \oplus A^*, \phi + W)$ are given by :

Theorem 3.4 ([33]). *Let $L = D \oplus graph(\Omega^{\#}|_{D^\perp})$ be a maximal isotropic subbundle of $A \oplus A^*$. Then, L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if*

i) *D is a Lie subalgebroid of A ;*
ii) $d_*^W \Omega + \frac{1}{2}[\Omega, \Omega]^\phi \equiv 0 (mod\, D)$ *;*
iii) *$D^\perp$ is integrable for the sum bracket $[\, ,\,]_* + [\, ,\,]_\Omega$, i.e., for all $\alpha, \beta \in \Gamma(D^\perp)$, $[\alpha, \beta]_* + [\alpha, \beta]_\Omega \in \Gamma(D^\perp)$, where $[\, ,\,]_\Omega$ is the bracket determined on $\Gamma(A^*)$ by (13).*

In the particular case where $((A, \phi), (A^*, W))$ is a triangular generalized Lie bialgebroid, (see, Definition 2.7), Theorem 3.4 takes the following form :

Corollary 3.5 ([33]). *Let $((A, \phi), (A^*, W), P)$ be a triangular generalized Lie bialgebroid and $L \subset A \oplus A^*$, $L = D \oplus graph(\Omega^{\#}|_{D^\perp})$, a maximal isotropic subbundle of $A \oplus A^*$ with a fixed characteristic pair (D, Ω). Then L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if*

i) *D is a Lie subalgebroid of A ;*
ii) $[P + \Omega, P + \Omega]^\phi \equiv 0 (mod\, D)$ *;*
iii) *for any $X \in \Gamma(D)$, $\mathcal{L}_X^\phi (P + \Omega) \equiv 0 (mod\, D)$.*

4. Relations between Dirac structures of $((A, \phi), (A^*, W))$ and associated structures of $(\tilde{A}, \tilde{A}^*)$

We consider a generalized Lie bialgebroid $((A, [\, ,\,], a, \phi), (A^*, [\, ,\,]_*, a_*, W))$ over a differentiable manifold M and we construct the associated generalized Courant algebroid $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$ over M, i.e. $[\![\, ,\,]\!]$ is determined by (21), $\rho = a + a_*$ and $\theta = \phi + W$. We introduce the notion of a *reducible Dirac structure* for $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$ and of an *admissible function* of a Dirac structure for $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$ in an analog manner as in the case of a Dirac structure for a Lie bialgebroid ([22]).

Definition 4.1. *We say that a Dirac subbundle L for $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$ is reducible if the image $a(D)$ of its characteristic subbundle $D = L \cap A$ by the anchor map a defines a simple foliation $\mathcal{F}$ of M. By the term "simple foliation", we mean that $\mathcal{F}$ is a regular foliation such that the space $M/\mathcal{F}$ is a nice manifold and the canonical projection $M \to M/\mathcal{F}$ is a submersion.*

Definition 4.2. *Let L be a Dirac subbundle for $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$. We say that a function $f \in C^\infty(M, I\!R)$ is L-admissible if there exists $Y_f \in \Gamma(A)$*

such that $Y_f + d^\phi f \in \Gamma(L)$. *We denote by* $C^\infty_L(M, \mathbb{R})$ *the set of all L-admissible functions of* $C^\infty(M, \mathbb{R})$.

In the sequel, we consider the Lie bialgebroid $((\tilde{A}, [\,,\,]^{\tilde{}\phi}, \tilde{a}^\phi), (\tilde{A}^*, [\,,\,]_*^{\hat{}W}, \hat{a}_*^W))$ over $\tilde{M} = M \times \mathbb{R}$ defined by the given generalized Lie bialgebroid

$$((A, [\,,\,], a, \phi), (A^*, [\,,\,]_*, a_*, W))$$

over M as in Theorem 2.8. Then, $\tilde{A} \oplus \tilde{A}^*$ is a Courant algebroid over $\tilde{M}$ ([21]) when it is endowed with :

(1) the two natural nondegenerate bilinear forms on $\tilde{A} \oplus \tilde{A}^*$ determined, for all $\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2 \in \tilde{A} \oplus \tilde{A}^*$, as :

$$(\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2)_\pm = \frac{1}{2}(\langle \tilde{\alpha}_1, \tilde{X}_2 \rangle \pm \langle \tilde{\alpha}_2, \tilde{X}_1 \rangle);$$

(2) the bracket $[\![\,,\,]\!]^{\tilde{}}$ on $\Gamma(\tilde{A} \oplus \tilde{A}^*)$ determined, for all $\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2 \in \Gamma(\tilde{A} \oplus \tilde{A}^*)$, as:

$$\begin{aligned} &[\![\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2]\!]^{\tilde{}} = \\ &\quad ([\tilde{X}_1, \tilde{X}_2]^{\tilde{}\phi} + \hat{\mathcal{L}}^W_{\tilde{\alpha}_1}\tilde{X}_2 - \hat{\mathcal{L}}^W_{\tilde{\alpha}_2}\tilde{X}_1 - \hat{d}^W_*((\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2)_-)) + \\ &\quad +([\tilde{\alpha}_1, \tilde{\alpha}_2]_*^{\hat{}W} + \tilde{\mathcal{L}}^\phi_{\tilde{X}_1}\tilde{\alpha}_2 - \tilde{\mathcal{L}}^\phi_{\tilde{X}_2}\tilde{\alpha}_1 + \tilde{d}^\phi((\tilde{X}_1 + \tilde{\alpha}_1, \tilde{X}_2 + \tilde{\alpha}_2)_-)), \end{aligned} \tag{26}$$

where, for any $\tilde{f} \in C^\infty(\tilde{M}, \mathbb{R})$, $\tilde{d}^\phi \tilde{f} = \tilde{d}\tilde{f} + \dfrac{\partial \tilde{f}}{\partial t}\phi$ and $\hat{d}^W_* \tilde{f} = e^{-t}(\tilde{d}\tilde{f} + \dfrac{\partial \tilde{f}}{\partial t}\phi)$, (see [12]);

(3) the bundle map $\tilde{\rho} : \tilde{A} \oplus \tilde{A}^* \to T\tilde{M}$ defined by $\tilde{\rho} = \tilde{a}^\phi + \hat{a}^W_*$.

Let $\mathbf{E} : \Gamma(A \oplus A^*) \to \Gamma(\tilde{A} \oplus \tilde{A}^*)$ be the embedding of $\Gamma(A \oplus A^*)$ into $\Gamma(\tilde{A} \oplus \tilde{A}^*)$ defined, for all $X + \alpha \in \Gamma(A \oplus A^*)$, by

$$\mathbf{E}(X + \alpha) = X + e^t\alpha,$$

where X and α are regarded as time-independent sections of $\tilde{A}$ and $\tilde{A}^*$, respectively. We make the following convention. If $L \subset A \oplus A^*$ is a subbundle of $A \oplus A^*$, we write $\tilde{L} = \mathbf{E}(L)$ in order to denote the vector subbundle $\tilde{L}$ of $\tilde{A} \oplus \tilde{A}^*$ whose space of global cross sections is the image by $\mathbf{E}$ of the space of global cross sections of L, i.e. $\Gamma(\tilde{L}) = \mathbf{E}(\Gamma(L))$.

Proposition 4.3. *Let* $L \subset A \oplus A^*$ *be a vector subbundle of* $A \oplus A^*$ *and* $\tilde{L}$ *its embedding by* $\mathbf{E}$ *into* $\tilde{A} \oplus \tilde{A}^*$, *i.e.* $\tilde{L} = \mathbf{E}(L) = \{X + e^t\alpha \in \tilde{A} \oplus \tilde{A}^* \,/\, X +$

$\alpha \in L\}$. *Then, L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, [\![\,,\,]\!], (\,,\,)_+, \rho, \theta)$ if and only if $\tilde{L}$ is a Dirac structure for the Courant algebroid $(\tilde{A} \oplus \tilde{A}^*, [\![\,,\,]\!]^{\tilde{}}, (\,,\,)_+, \tilde{\rho})$.*

Proof : For all $X_1 + \alpha_1, X_2 + \alpha_2 \in L$, we have

$$\begin{aligned}
(X_1 + e^t\alpha_1, X_2 + e^t\alpha_2)_+ &= \frac{1}{2}(\langle e^t\alpha_1, X_2\rangle + \langle e^t\alpha_2, X_1\rangle) = \\
&= \frac{e^t}{2}(\langle \alpha_1, X_2\rangle + \langle \alpha_2, X_1\rangle) = \\
&= e^t(X_1 + \alpha_1, X_2 + \alpha_2)_+,
\end{aligned}$$

which means that $\tilde{L}$ is a maximally isotropic subbundle of $(\tilde{A} \oplus \tilde{A}^*, (\,,\,)_+)$ if and only if L is a maximally isotropic subbundle of $(A \oplus A^*, (\,,\,)_+)$. Moreover, by a straightforward calculation we get that, for any $X_1 + \alpha_1, X_2 + \alpha_2 \in \Gamma(L)$,

$$[\![\mathbf{E}(X_1 + \alpha_1), \mathbf{E}(X_2 + \alpha_2)]\!]^{\tilde{}} = \mathbf{E}([\![X_1 + \alpha_1, X_2 + \alpha_2]\!]),$$

which means that $\Gamma(\tilde{L})$ is closed under $[\![\,,\,]\!]^{\tilde{}}$ if and only if $\Gamma(L)$ is closed under $[\![\,,\,]\!]$. Thus, L is a Dirac structure for $(A \oplus A^*, [\![\,,\,]\!], (\,,\,)_+, \rho, \theta)$ if and only if $\tilde{L}$ is a Dirac structure for $(\tilde{A} \oplus \tilde{A}^*, [\![\,,\,]\!]^{\tilde{}}, (\,,\,)_+, \tilde{\rho})$. •

Proposition 4.4. *Let L be a Dirac structure for $(A \oplus A^*, [\![\,,\,]\!], (\,,\,)_+, \rho, \theta)$ and $\tilde{L} = \mathbf{E}(L)$ the associated Dirac structure of $(\tilde{A} \oplus \tilde{A}^*, [\![\,,\,]\!]^{\tilde{}}, (\,,\,)_+, \tilde{\rho})$. Then $\tilde{f} \in C^\infty(\tilde{M}, \mathbb{R})$ is an $\tilde{L}$-admissible function if and only if $\tilde{f} = e^t f$ and $f \in C^\infty(M, \mathbb{R})$ is an L-admissible function.*

Proof : Let $\tilde{f} \in C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R})$ be an $\tilde{L}$-admissible function, i.e. there exists a time-independent section Y of $\tilde{A}$, so Y may be considered as a section of A, such that $Y + \tilde{d}^\phi \tilde{f} \in \Gamma(\tilde{L})$. But, $Y + \tilde{d}^\phi \tilde{f} \in \Gamma(\tilde{L})$ implies that there exists $\xi \in \Gamma(A^*)$ such that $Y + \xi \in \Gamma(L)$ and $Y + \tilde{d}^\phi \tilde{f} = \mathbf{E}(Y + \xi)$, thus $\tilde{d}^\phi \tilde{f} = e^t\xi$. From *Theorem of normal forms for Lie algebroids* ([6], [7]), if the rank of $a(D)$, $D = L \cap A$, at a point $q \in M$ is k, we can construct on a neighborhood U of q in M a system of local coordinates $(x_1, \dots, x_k, \dots, x_n)$ ($n = \dim M$) and a basis of sections $(X_1, \dots, X_k, \dots X_r)$ of $\Gamma(A)$ (r denotes the dimension of the fibres of $A \to M$), with $(X_1, \dots, X_k)$ basis of $\Gamma(D)$, such that $a(X_i) = \dfrac{\partial}{\partial x_i}$, for every $i = 1, \dots, k$. Let $(\alpha_1, \dots, \alpha_k, \dots, \alpha_r)$ be the basis of $\Gamma(A^*)$, dual of $(X_1, \dots, X_k, \dots X_r)$. Since $\phi, \xi \in \Gamma(A^*)$, there exists $\phi_i, \xi_i \in C^\infty(U, \mathbb{R})$, $i = 1, \dots, r$, such that $\phi = \sum_{i=1}^r \phi_i\alpha_i$ and $\xi = \sum_{i=1}^r \xi_i\alpha_i$.

We have

$$\tilde{d}^\phi \tilde{f} = e^t\xi \Rightarrow \langle \tilde{d}\tilde{f} + \frac{\partial \tilde{f}}{\partial t}\phi, X_i\rangle = \langle e^t\xi, X_i\rangle \Leftrightarrow \langle \tilde{d}\tilde{f}, X_i\rangle + \frac{\partial \tilde{f}}{\partial t}\phi_i = e^t\xi_i, \quad (27)$$

for any $i = 1, \ldots, r$. But, for $i = 1, \ldots, k$,

$$\langle \tilde{d}\tilde{f}, X_i\rangle = \langle \delta\tilde{f}, \tilde{a}(X_i)\rangle = \langle \delta\tilde{f}, a(X_i)\rangle = \langle \delta\tilde{f}, \frac{\partial}{\partial x_i}\rangle = \frac{\partial \tilde{f}}{\partial x_i}.$$

Hence, the last equation of (27) can be written, for any $i = 1, \ldots, k$, as

$$\frac{\partial \tilde{f}}{\partial x_i} + \frac{\partial \tilde{f}}{\partial t}\phi_i = e^t\xi_i. \quad (28)$$

By resolving the characteristic system $\frac{\delta x_i}{1} = \frac{\delta t}{\phi_i} = \frac{\delta \tilde{f}}{e^t\xi_i}$ of (28), we obtain that $\tilde{f}$ must be, at least locally, of the form $\tilde{f} = e^t f$, where $f \in C^\infty(U, \mathbb{R})$. Taking into account Definition 4.2 and that $\tilde{L} = \mathbf{E}(L)$, we get

$$\begin{aligned} \tilde{f} = e^t f \in C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R}) \;&\Leftrightarrow\; \exists\, Y \in \Gamma(A) : Y + \tilde{d}^\phi(e^t f) \in \Gamma(\tilde{L}) \Leftrightarrow \\ &\Leftrightarrow\; \exists\, Y \in \Gamma(A) : Y + e^t d^\phi f \in \Gamma(\tilde{L}) \Leftrightarrow \\ &\Leftrightarrow\; \exists\, Y \in \Gamma(A) : Y + d^\phi f \in \Gamma(L) \Leftrightarrow \\ &\Leftrightarrow\; f \in C^\infty_L(M, \mathbb{R}). \bullet \end{aligned}$$

For the proof of Proposition 4.6, we need the result of the following lemma.

Lemma 4.5. *Let (B, B^*) be a Lie bialgebroid over a differentiable manifold N, with anchors b and b_* respectively, Δ a Dirac subbundle of $B \oplus B^*$ and $\mathcal{B}$ the (singular) foliation of N defined by the (singular) distribution $b(\Delta \cap B)$ on N. Then, $f \in C^\infty(N, \mathbb{R})$ is an Δ-admissible function if and only if f is constant along the leaves of $\mathcal{B}$.*

Proof : Let $f \in C^\infty(N, \mathbb{R})$ be an Δ-admissible function, i.e. there exists $Y_f \in \Gamma(B)$ such that $Y_f + d_B f \in \Gamma(\Delta)$, and $X \in \Gamma(b(\Delta \cap B))$ a section of the distribution $b(\Delta \cap B)$. $X \in \Gamma(b(\Delta \cap B))$ means that there exists $Y \in \Gamma(\Delta \cap B)$ such that $X = b(Y)$ and $Y \in \Gamma(\Delta \cap B)$ means that $Y + 0 \in \Gamma(\Delta)$. Since Δ is a Dirac subbundle of $B \oplus B^*$, it is maximally isotropic, thus $(Y_f + d_B f, Y + 0)_+ = 0$. But,

$$\begin{aligned} (Y_f + d_B f, Y + 0)_+ = 0 \;&\Leftrightarrow\; \frac{1}{2}(\langle d_B f, Y\rangle + \langle 0, Y_f\rangle) = 0 \\ &\Leftrightarrow\; \langle \delta f, b(Y)\rangle = 0 \Leftrightarrow \langle \delta f, X\rangle = 0. \end{aligned} \quad (29)$$

By the last equation of (29), which holds for any $X \in \Gamma(b(\Delta \cap B))$, we get that f is constant along the leaves of $\mathcal{B}$.

Conversely, let $f \in C^\infty(N, \mathbb{R})$ be a function on N constant along the leaves of $\mathcal{B}$, i.e, for any $X \in \Gamma(b(\Delta \cap B))$, $\langle \delta f, X \rangle = 0$. But, $X \in \Gamma(b(\Delta \cap B))$ means that there exists $Y \in \Gamma(\Delta \cap B)$ such that $X = b(Y)$ and $\langle \delta f, X \rangle = 0 \Leftrightarrow \langle \delta f, b(Y) \rangle = 0 \Leftrightarrow \langle d_B f, Y \rangle = 0$. If f is not an Δ-admissible function, we will have that, for any $Z \in \Gamma(B)$, $Z + d_B f$ is not a section of Δ. Hence, for any $Z \in \Gamma(B)$, $(Z + d_B f, Z + d_B f)_+ \neq 0 \Leftrightarrow \langle d_B f, Z \rangle \neq 0$; result contradictory to the fact that, for $Z = Y$, $\langle d_B f, Y \rangle = 0$. Thus, f is an Δ-admissible function. •

Proposition 4.6. *Let L be a Dirac subbundle for $(A \oplus A^*, [\![\, ,\,]\!], (\, ,\,)_+, \rho, \theta)$ and $\tilde{L} = \mathbf{E}(L)$ the associated Dirac subbundle of $(\tilde{A} \oplus \tilde{A}^*, [\![\, ,\,]\!]\tilde{}, (\, ,\,)_+, \tilde{\rho})$. Then, L is reducible if and only if $\tilde{L}$ is reducible.*

Proof : We denote by $D = L \cap A$ and $\tilde{D} = \tilde{L} \cap \tilde{A}$ the characteristic subbundles of L and $\tilde{L}$, respectively, and by $\mathcal{F}$ and $\tilde{\mathcal{F}}$ the foliations of M and $\tilde{M}$, respectively, defined by $a(D)$ and $\tilde{a}^\phi(\tilde{D})$, respectively. Obviously, $\tilde{D} \cong D$ and $\tilde{a}^\phi(\tilde{D}) = \{\tilde{a}^\phi(X) \,/\, X \in D\} = \{a(X) + \langle \phi, X \rangle \partial/\partial t \,/\, X \in D\}$. Hence, if (x_0, t_0) is a point of $\tilde{M} = M \times \mathbb{R}$, $\tilde{F}$ and F are the leaves of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ passing through $(x_0, t_0) \in \tilde{M}$ and $x_0 \in M$, respectively, and D_{x_0} is the fibre of D over x_0, we have :

i) if $ker(a|_{D_{x_0}})$ is not contained in $\langle \phi(x_0) \rangle^\circ$, then $\tilde{F} = F \times \mathbb{R}$, so $\dim \tilde{F} = \dim F + 1$ and the vector field $\partial/\partial t$ is tangent to $\tilde{F}$;

ii) if $ker(a|_{D_{x_0}})$ is contained in $\langle \phi(x_0) \rangle^\circ$ and $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection, then $\pi_1(\tilde{F}) = F$ and $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map, thus $\dim \tilde{F} = \dim F$ and the vector field $\partial/\partial t$ is not tangent to $\tilde{F}$;

(see also Theorem 2.9). Since every $\tilde{L}$-admissible function $\tilde{f}$ is of type $\tilde{f} = e^t f$, $f \in C^\infty_L(M, \mathbb{R})$, (see Proposition 4.4) and also it is constant along the leaves of $\tilde{\mathcal{F}}$ (see Lemma 4.5), it is not possible the leaves $\tilde{F}$ of $\tilde{\mathcal{F}}$ to be of type $\tilde{F} = F \times \mathbb{R}$ (because, in this case, $\partial/\partial t$ is tangent to $\tilde{F}$ and $\tilde{f} = e^t f$ is not constant along $\partial/\partial t$). Thus, for any leaf $\tilde{F}$ of $\tilde{\mathcal{F}}$ and for the corresponding leaf F of $\mathcal{F}$, we have that $\pi_1(\tilde{F}) = F$ and $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map, whence we get :

(1) every leaf $\tilde{F}$ of $\tilde{\mathcal{F}}$ is of the same dimension as the corresponding leaf F of $\mathcal{F}$, so $\mathcal{F}$ is a regular foliation of M if and only if $\tilde{\mathcal{F}}$ is a regular foliation of $\tilde{M}$;

(2) $\tilde{\mathcal{F}} \cong \mathcal{F}$, so $\tilde{M}/\tilde{\mathcal{F}} \cong (M \times \mathbb{R})/\mathcal{F} \cong (M/\mathcal{F}) \times \mathbb{R}$; thus, $M/\mathcal{F}$ is a nice manifold if and only if $\tilde{M}/\tilde{\mathcal{F}}$ is a nice manifold and the projection $M \to M/\mathcal{F}$ is a submersion if and only if the projection $M \times \mathbb{R} = \tilde{M} \to \tilde{M}/\tilde{\mathcal{F}} \cong (M/\mathcal{F}) \times \mathbb{R}$ is a submersion.

Consequently, L is a reducible Dirac structure for $A \oplus A^*$ if and only if $\tilde{L} = \mathbf{E}(L)$ is a reducible Dirac structure for $\tilde{A} \oplus \tilde{A}^*$. •

5. Jacobi structures and Dirac reducible subbundles

We consider a generalized Lie bialgebroid $((A, [\,,\,], a, \phi), (A^*, [\,,\,]_*, a_*, W))$ over a differentiable manifold M and we construct the associated generalized Courant algebroid $(A \oplus A^*, [\![\,,\,]\!], (\,,\,)_+, \rho, \theta)$ over M, i.e. $[\![\,,\,]\!]$ is determined by (21), $\rho = a + a_*$ and $\theta = \phi + W$. Let $L \subset A \oplus A^*$ be a reducible Dirac structure for $(A \oplus A^*, [\![\,,\,]\!], (\,,\,)_+, \rho, \theta)$, (see Definitions 3.2 and 4.1).

On $C^\infty_L(M, \mathbb{R})$ we define the bracket $\{\,,\,\}_L$ by setting, for all $f, g \in C^\infty_L(M, \mathbb{R})$,

$$\{f, g\}_L := \rho^\theta(e_f)g, \tag{30}$$

where $e_f = Y_f + d^\phi f$. An equivalent expression of (30) is :

$$\{f, g\}_L = \langle Y_f, d^\phi g\rangle + \{f, g\}_J, \tag{31}$$

where $\{\,,\,\}_J$ is the bracket (11) of the Jacobi structure on M defined by the generalized Lie bialgebroid structure $((A, \phi), (A^*, W))$ over M. Effectively,

$$\begin{aligned} \{f, g\}_L &= \rho^\theta(e_f)g \\ &= ((a^\phi + a_*^W)(Y_f + d^\phi f))g \\ &= a^\phi(Y_f)g + a_*^W(d^\phi f)g \\ &= \langle Y_f, d^\phi g\rangle + \langle d^\phi f, d_*^W g\rangle \\ &= \langle Y_f, d^\phi g\rangle + \{f, g\}_J. \end{aligned}$$

Theorem 5.1. *The space $C^\infty_L(M, \mathbb{R})$ endowed with the bracket $\{\,,\,\}_L$, given by (30), is a Jacobi algebra.*

Proof : We must prove that $C^\infty_L(M, \mathbb{R})$ is closed under $\{\,,\,\}_L$ and that $\{\,,\,\}_L$ is a bilinear, skew-symmetric first order differential operator on each of its arguments which satisfies the Jacobi identity.

Closeness of $\{\,,\,\}_L$ *in* $C^\infty_L(M,\mathbb{R})$: Let $f,g \in C^\infty_L(M,\mathbb{R})$ be two L-admissible functions. Then, there exist $Y_f, Y_g \in \Gamma(A)$ such that $e_f = Y_f + d^\phi f, e_g = Y_g + d^\phi g \in \Gamma(L)$. We consider the bracket $[\![e_f, e_g]\!]$; according to (21), its component in $\Gamma(A^*)$ is :

$$[d^\phi f, d^\phi g]^W_* + \mathcal{L}^\phi_{Y_f} d^\phi g - \mathcal{L}^\phi_{Y_g} d^\phi f + d^\phi (e_f, e_g)_-.$$

We have (see [12]),

$$[d^\phi f, d^\phi g]^W_* = -\mathcal{L}^\phi_{d^W_* f} d^\phi g = -d^\phi \langle d^W_* f, d^\phi g\rangle = -d^\phi \{g,f\}_J = d^\phi\{f,g\}_J$$

and, on the other hand,

$$\begin{aligned} &\mathcal{L}^\phi_{Y_f} d^\phi g - \mathcal{L}^\phi_{Y_g} d^\phi f + d^\phi (e_f, e_g)_- = \\ &\qquad -d^\phi (e_f, e_g)_- = -d^\phi (e_f, e_g)_- + \underbrace{d^\phi (e_f, e_g)_+}_{=0} = d^\phi \langle Y_f, d^\phi g\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} [d^\phi f, d^\phi g]^W_* + \mathcal{L}^\phi_{Y_f} d^\phi g - \mathcal{L}^\phi_{Y_g} d^\phi f + d^\phi (e_f, e_g)_- &= d^\phi\{f,g\}_J + d^\phi\langle Y_f, d^\phi g\rangle \\ &\overset{(31)}{=} d^\phi\{f,g\}_L, \end{aligned}$$

which means that $\{f,g\}_L$ is an L-admissible function, i.e. $\{f,g\}_L \in C^\infty_L(M,\mathbb{R})$, and that we can take $[\![e_f, e_g]\!] = e_{\{f,g\}_L}$.

Bilinearity and skew-symmetry of $\{\,,\,\}_L$: It is obvious that $\{\,,\,\}_L$ is bilinear. Also, for any $f \in C^\infty_L(M,\mathbb{R})$, we have $(e_f, e_f)_+ = 0 \Leftrightarrow \langle Y_f, d^\phi f\rangle = 0$, so $\{f,f\}_L \overset{(31)}{=} \langle Y_f, d^\phi f\rangle + \{f,f\}_J = 0+0 = 0$, which implies the skew-symmetry of $\{\,,\,\}_L$.

$\{\,,\,\}_L$ *is a first order differential operator on each of its arguments*. In fact, for any $f,g,h \in C^\infty_L(M,\mathbb{R})$,

$$\begin{aligned} \{f,gh\}_L &\overset{(31)}{=} \langle Y_f, d^\phi(gh)\rangle + \{f,gh\}_J \\ &= \langle Y_f, gdh + hdg + gh\phi\rangle + g\{f,h\}_J + h\{f,g\}_J - gh\{f,1\}_J \\ &= g(\langle Y_f, d^\phi h\rangle + \{f,h\}_J) + h(\langle Y_f, d^\phi g\rangle + \{f,g\}_J) \\ &\quad -gh(\langle Y_f, \phi\rangle + \{f,1\}_J) \\ &= g\{f,h\}_L + h\{f,g\}_L - gh\{f,1\}_L \end{aligned}$$

and by the skew-symmetry of $\{\,,\,\}_L$ we obtain the desired result.

Jacobi identity : By a straightforward, but long, calculation we get that, for any $f, g, h \in C^\infty_L(M, \mathbb{R})$, the Jacobi identity holds :

$$\{f, \{g, h\}_L\}_L + \{g, \{h, f\}_L\}_L + \{h, \{f, g\}_L\}_L = 0.$$

Hence, $(C^\infty_L(M, \mathbb{R}), \{\ ,\ \}_L)$ is a Jacobi algebra. •

The above result generalizes the one of A. Wade ([35]) for the $\mathcal{E}^1(M)$-Dirac structures.

In the sequel, we will prove that the Jacobi algebra structure of $C^\infty_L(M, \mathbb{R})$ is conformally equivalent to a Jacobi algebra on the space of the functions defined on a quotient manifold of M. The proof is based on the related results concerning the admissible functions of a reducible Dirac structure for the double of a Lie bialgebroid (see, [22]).

We consider the reducible Dirac subbundle $\tilde{L} = \mathbf{E}(L)$ of the Courant algebroid $(\tilde{A} \oplus \tilde{A}^*, [\![\ ,\ ]\!]\tilde{}, (\ ,\ )_+, \tilde{\rho})$ over $\tilde{M} = M \times \mathbb{R}$ and the simple foliation $\tilde{\mathcal{F}}$ of $\tilde{M}$ defined by $\tilde{a}^\phi(\tilde{D})$, $\tilde{D} = \tilde{L} \cap \tilde{A}$. By the results of [22] and of Lemma 4.5 and by the proof of Proposition 4.6, we have :

(1) A function $\tilde{f} \in C^\infty(\tilde{M}, \mathbb{R})$ is $\tilde{L}$-admissible if and only if $\tilde{f}$ is constant along $\tilde{\mathcal{F}}$, i.e.
$$C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R}) \cong C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R}).$$
(2) The bracket $\{\ ,\ \}_{\tilde{L}}$, given, for any $\tilde{f}, \tilde{g} \in C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R})$, by
$$\{\tilde{f}, \tilde{g}\}_{\tilde{L}} = \tilde{\rho}(\tilde{e}_{\tilde{f}})\tilde{g}, \tag{32}$$
where $\tilde{e}_{\tilde{f}} = Y_{\tilde{f}} + \tilde{d}^\phi \tilde{f}$, defines a Poisson algebra on $C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$.
(3) The vector field $\partial/\partial t$ is not tangent to $\tilde{\mathcal{F}}$.

By identifying locally, on a neighborhood $\tilde{U}$ of a point $(x_0, t_0) \in \tilde{M}$, the manifold $\tilde{M}$ with the product $(\tilde{M}/\tilde{\mathcal{F}}) \times \tilde{F}$, where $\tilde{F}$ is the leaf of $\tilde{\mathcal{F}}$ through (x_0, t_0), we have that $\partial/\partial t$ can be written, locally, as
$$\frac{\partial}{\partial t} = \tilde{T} + \tilde{X}, \tag{33}$$
where $\tilde{T}$ is a vector field tangent to $\tilde{M}/\tilde{\mathcal{F}}$, $\tilde{T}(x,t) \neq 0$ at every point $(x,t) \in \tilde{U}$, and $\tilde{X}$ is a vector field tangent to $\tilde{F}$.

Proposition 5.2. *The Poisson structure defined on $\tilde{M}/\tilde{\mathcal{F}}$ by (32) is an homogeneous Poisson structure, in the sense of* [4], *with respect to $\tilde{T}$.*

Proof : We remark that the functions of $C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$ are homogeneous of degree 1 with respect to $\tilde{T}$. Effectively, let $\tilde{f} \in C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R}) \cong C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R})$ be an $\tilde{L}$-admissible function, then there exists an L-admissible function f such that $\tilde{f} = e^t f$ (see Proposition 4.4) and $\tilde{f}$ is constant along $\tilde{\mathcal{F}}$. Thus,

$$\mathcal{L}_{\tilde{T}}\tilde{f} \stackrel{(33)}{=} \mathcal{L}_{\partial/\partial t - \tilde{X}}\tilde{f} = \mathcal{L}_{\partial/\partial t}(e^t f) - \mathcal{L}_{\tilde{X}}\tilde{f} = e^t f - 0 = \tilde{f}. \tag{34}$$

Moreover, for every pair $(\tilde{f}, \tilde{g})$ of functions on $\tilde{M}/\tilde{\mathcal{F}}$, $\tilde{f} = e^t f$ and $\tilde{g} = e^t g$ with $f, g \in C^\infty_L(M, \mathbb{R})$,

$$\begin{aligned}
\{\tilde{f}, \tilde{g}\}_{\tilde{L}} &= \{e^t f, e^t g\}_{\tilde{L}} \\
&\stackrel{(32)}{=} \tilde{\rho}(Y_f + \tilde{d}^\phi(e^t f))(e^t g) \\
&= (\tilde{a}^\phi(Y_f) + \hat{a}^W_*(e^t d^\phi f))(e^t g) \\
&\stackrel{(14),(15)}{=} (a(Y_f) + \langle \phi, Y_f\rangle \frac{\partial}{\partial t})(e^t g) + e^{-t}(e^t(a_*(d^\phi f) + \langle d^\phi f, W\rangle \frac{\partial}{\partial t}))(e^t g) \\
&= e^t(a(Y_f)g + \langle \phi, Y_f\rangle g) + e^t(a_*(d^\phi f) + \langle d^\phi f, W\rangle g) \\
&= e^t(a^\phi(Y_f) + a^W_*(d^\phi f))g \\
&= e^t \rho^\theta(e_f) g \\
&\stackrel{(30)}{=} e^t\{f, g\}_L.
\end{aligned} \tag{35}$$

From the last equation we have that the Poisson bracket of any pair of functions of $C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$ is also an homogeneous function of degree 1 with respect to $\tilde{T}$. Thus, if $\tilde{\Lambda}$ is the Poisson bivector field defined on $\tilde{M}/\tilde{\mathcal{F}}$ by the Poisson bracket (32), i.e., for all $\tilde{f}, \tilde{g} \in C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$, $\tilde{\Lambda}(\delta\tilde{f}, \delta\tilde{g}) = \{\tilde{f}, \tilde{g}\}_{\tilde{L}}$, by a straightforward calculation we can prove that $[\tilde{T}, \tilde{\Lambda}] = -\tilde{\Lambda}$, i.e. $\tilde{\Lambda}$ is an homogeneous Poisson structure with respect to $\tilde{T}$, in the sense of [4]. •

At this point, we need the next two well known propositions of Dazord-Lichnerowicz-Marle :

Proposition 5.3 ([4])**.** *Let (P, Λ_P, Z) be an homogeneous Poisson manifold and N a submanifold of P of codimension 1 transverse to the homothety vector field Z. Then, N has an induced Jacobi structure characterized by one of the next properties :*

(1) *For any pair (H_1, H_2) of homogeneous functions of degree 1 with respect to Z, defined on an open subset $\mathcal{O}$ of P, the Jacobi bracket of*

H_1 and H_2, restricted to $N \cap \mathcal{O}$, is the restriction to $N \cap \mathcal{O}$ of the Poisson bracket of H_1 and H_2.

(2) *Let $\varpi : U \to N$ be the projection on N of a tubular neighborhood of N in P such that, for any $x \in N$, $\varpi^{-1}(x)$ is a connected arc of the integral curve of Z through x. Let λ be a nowhere zero function on U, equal to 1 on N and homogeneous of degree 1 with respect to Z. Then, the projection ϖ is a λ-conformal Jacobi map.*

Remark 5.4. Under some regularity conditions on Z, N can be locally considered as the manifold of the integral curves of Z. In this case, by identifying U with the product $N \times I$ of the submanifold N and an open interval I of $\mathbb{R}$ containing 0 and writing $Z = \partial/\partial z$, where z is the canonical coordinate on I, we have that $\lambda(x, z) = e^z$ and for every pair (h_1, h_2) of functions on N,

$$\varpi^*\{h_1, h_2\}_N = \frac{1}{e^z}\{e^z \varpi^* h_1, e^z \varpi^* h_2\}_P. \tag{36}$$

Proposition 5.5 ([4]). *Let (P, Λ_P, Z) be an homogeneous Poisson manifold and N and N' two submanifolds of P of codimension 1 transverse to the homothety vector field Z. We assume that there exists an integral curve of Z intersecting N at a point x_0 and N' at a point x_0'. We provide N and N' with the Jacobi structures induced by the homogeneous Poisson structure of P, in the sense of Theorem 5.3. Then, there exists a conformal Jacobi diffeomorphism of a neighborhood of x_0 in N onto a neighborhood of x_0' in N', mapping x_0 to x_0'.*

Therefore, since the integral curves of $\tilde{T}$ define a simple foliation $\tilde{\mathcal{T}}$ of $\tilde{M}/\tilde{\mathcal{F}}$, by applying Proposition 5.3 to the homogeneous Poisson manifold $(\tilde{M}/\tilde{\mathcal{F}}, \{\ ,\ \}_{\tilde{L}}, \tilde{T})$ and taking into account Remark 5.4, we obtain that the homogeneous Poisson structure of $\tilde{M}/\tilde{\mathcal{F}}$ induces a Jacobi structure on

$$(\tilde{M}/\tilde{\mathcal{F}})/\tilde{\mathcal{T}} \cong \tilde{M}/(\tilde{\mathcal{F}} \times \tilde{\mathcal{T}}) \cong (M \times \mathbb{R})/(\mathcal{F} \times \mathbb{R}) \cong M/\mathcal{F}.$$

Precisely, for any pair $(\bar{f}, \bar{g})$ of functions on $M/\mathcal{F}$,

$$\varpi^*\{\bar{f}, \bar{g}\}_{M/\mathcal{F}} = \frac{1}{e^{\tilde{\tau}}}\{e^{\tilde{\tau}} \varpi^* \bar{f}, e^{\tilde{\tau}} \varpi^* \bar{g}\}_{\tilde{L}}, \tag{37}$$

where $\varpi : \tilde{M}/\tilde{\mathcal{F}} \to (\tilde{M}/\tilde{\mathcal{F}})/\tilde{\mathcal{T}} \cong M/\mathcal{F}$ is the canonical projection and $\tilde{\tau}$ is a function on $\tilde{M}/\tilde{\mathcal{F}}$ such that $\tilde{T} = \partial/\partial\tilde{\tau}$.

Lemma 5.6. *1. The function $\tilde{\tau} \in C^\infty(\tilde{M}/\tilde{\mathcal{F}}, I\!R)$ is of the form :*

$$\tilde{\tau}(x,t) = \frac{t}{1 - \langle \phi, X \rangle} + \tau(x) \tag{38}$$

with τ and X defined below.
2. The hypersurface S of $\tilde{M}/\tilde{\mathcal{F}}$ defined by the equation $t = 0$ is transverse to $\tilde{T}$.

Proof : 1. By equation (33), we have that $\tilde{T} = \frac{\partial}{\partial t} - \tilde{X}$ with $\tilde{X}$ a section of $\tilde{a}^\phi(\tilde{D})$. The last fact implies that there exists $X \in \Gamma(D)$ such that $\tilde{X} = a(X) + \langle \phi, X \rangle \frac{\partial}{\partial t}$. So, $\tilde{T} = (1 - \langle \phi, X \rangle)\frac{\partial}{\partial t} - a(X)$ and we note that $\tilde{T}$ is locally non zero. Hence, by straightening out $\tilde{T}$ we can find a function $\tilde{\tau}$ such that $\tilde{T} = \frac{\partial}{\partial \tilde{\tau}}$, i.e. $\langle \delta\tilde{\tau}, \tilde{T} \rangle = 1$. But, $\langle \delta\tilde{\tau}, \tilde{T} \rangle = 1 \Leftrightarrow (1 - \langle \phi, X \rangle)\frac{\partial \tilde{\tau}}{\partial t} - \langle \delta\tilde{\tau}, a(X) \rangle = 1$. The solutions of this equation are the functions $\tilde{\tau}$ given by (38) with τ an appropriate "arbitrary" function on $\tilde{M}/\tilde{\mathcal{F}}$ independent of t.

2. By (38), we have that the hypersurface S of $\tilde{M}/\tilde{\mathcal{F}}$ defined by the equation $t = 0$ is the hypersurface determined by the equation $\tilde{\tau} - \tau = 0$. Since, $\langle \delta(\tilde{\tau} - \tau), \tilde{T} \rangle = \langle \delta\tilde{\tau}, \frac{\partial}{\partial \tilde{\tau}} \rangle - \langle \delta\tau, \frac{\partial}{\partial t} - \tilde{X} \rangle = 1 - 0 = 1 \neq 0$, we conclude that S is transverse to $\tilde{T}$. •

Now, by Proposition 5.3 and the above result, we get that the homogeneous Poisson structure $(\{\, ,\, \}_{\tilde{L}}, \tilde{T})$ of $\tilde{M}/\tilde{\mathcal{F}}$ induces a Jacobi structure on S. Obviously, the Jacobi bracket of this structure coincide with the Jacobi bracket $\{\, ,\, \}_L$ (see equation (35)). Finally, by applying Proposition 5.5 to the homogeneous Poisson manifold $(\tilde{M}/\tilde{\mathcal{F}}, \{\, ,\, \}_{\tilde{L}}, \tilde{T})$ and the induced Jacobi brackets $\{\, ,\, \}_{M/\mathcal{F}}$ and $\{\, ,\, \}_L$ on $C^\infty(M/\mathcal{F}, I\!R)$ and $C^\infty(S, I\!R) \cong C^\infty_L(M, I\!R)$, respectively, we get that these are conformally equivalent. In other words, we have that, there exists, locally, a diffeomorphism $\Psi : M/\mathcal{F} \to S$ and a nowhere zero function ψ on $M/\mathcal{F}$ such that, for any pair (f,g) of functions on S,

$$\Psi^*\{f,g\}_L = \frac{1}{\psi}\{\psi\Psi^* f, \psi\Psi^* g\}_{M/\mathcal{F}}. \tag{39}$$

In conclusion :

Theorem 5.7. *Let L be a reducible Dirac subbundle for*

$$(A \oplus A^*, [\![\, , \,]\!], (\, , \,)_+, \rho, \theta).$$

Then L induces a Jacobi bracket $\{\, , \,\}_{M/\mathcal{F}}$ on $M/\mathcal{F}$ (given by (37)) conformally equivalent to the Jacobi bracket $\{\, , \,\}_L$ defined by (30) or (31).

Hence, by applying the above theorem to the case of the generalized Lie bialgebroid defined by a Jacobi structure (Λ, E) on M (see Theorem 2.4) we deduce :

Corollary 5.8. *Let (M, Λ, E) be a Jacobi manifold,*

$$((TM \times I\!R, [\, , \,], \pi, (0,1)), (T^*M \times I\!R, [\, , \,]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\#, (-E, 0)))$$

the associated generalized Lie bialgebroid and L a reducible Dirac structure for the generalized Courant algebroid $((TM \times I\!R) \oplus (T^*M \times I\!R), [\![\, , \,]\!], (\, , \,)_+, \pi + \pi \circ (\Lambda, E)^\#, (0, 1) + (-E, 0))$. Then L induces a Jacobi bracket $\{\, , \,\}_{M/\mathcal{F}}$ on $M/\mathcal{F}$, where $\mathcal{F}$ is the foliation of M defined by the distribution $\pi(D)$, $D = L \cap (TM \times I\!R)$, which is conformally equivalent to the Jacobi bracket $\{\, , \,\}_L$ defined by (30) or (31).

Taking into account Corollary 2.6, Definition 4.2 and (31), we can easily establish :

Proposition 5.9. *Under the assumptions of Corollary 5.8,*

1. *if $L = graph(\Lambda', E')^\#$ is the graph of a $(TM \times I\!R)$-bivector field (Λ', E') on M, then $C^\infty_L(M, I\!R) = C^\infty(M, I\!R)$ and, for all $f, g \in C^\infty_L(M, I\!R)$,*
$$\{f, g\}_L = \{f, g\}_{(\Lambda',E')} + \{f, g\}_{(\Lambda,E)}\, ; \tag{40}$$
2. *if $L = D \oplus D^\perp$ is a null Dirac structure, then $C^\infty_L(M, I\!R) = \{f \in C^\infty(M, I\!R) \,/\, (\delta f, f) \in \Gamma(D^\perp)\}$ and, for all $f, g \in C^\infty_L(M, I\!R)$,*
$$\{f, g\}_L = \{f, g\}_{(\Lambda,E)}\, ; \tag{41}$$
3. *if $L = D \oplus graph(\Lambda', E')^\#|_{D^\perp}$ is defined by a characteristic pair $(D, (\Lambda', E'))$, then $C^\infty_L(M, I\!R) = \{f \in C^\infty(M, I\!R) \,/\, (\delta f, f) \in \Gamma(D^\perp)\}$ and, for all $f, g \in C^\infty_L(M, I\!R)$,*
$$\{f, g\}_L = \{f, g\}_{(\Lambda',E')} + \{f, g\}_{(\Lambda,E)}. \tag{42}$$

In what follows, we will prove that in the context of *"generalized Lie bialgebroids - Jacobi structures"*, as in the context of *"Lie bialgebroids - Poisson structures"* ([22]), the converse result of Corollary 5.8 also holds.

Theorem 5.10. *Let (M, Λ, E) be a Jacobi manifold, $\mathcal{F}$ a simple foliation of M defined by a Lie subalgebroid $D \subset TM \times \mathbb{R}$ that has no sections of type $(0, f)$ with $f \neq 0$, and $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ a Jacobi structure on the quotient manifold $M/\mathcal{F}$. Then $(M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ defines a reducible Dirac structure L in $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ such that $L \cap (TM \times \mathbb{R}) = D$ and the induced Jacobi structure by L on $M/\mathcal{F}$, in the sense of Corollary 5.8, is the initially given $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$.*

Proof : We make the proof in several steps.

First step : Let $D \subset TM \times \mathbb{R}$ be a Lie subalgebroid of $(TM \times \mathbb{R}, [\,,\,], \pi)$, which has no sections of type $(0, f)$ with $f \neq 0$, such that $\pi(D)$ defines a simple foliation $\mathcal{F}$ of M and let $D^\perp$ be its conormal bundle, i.e.

$$D^\perp = \{(\alpha, g) \in T^*M \times \mathbb{R} \,/\, \langle(\alpha, g), (X, f)\rangle = \langle \alpha, X\rangle + fg = 0, \ \forall (X, f) \in D\}.$$

We suppose that the quotient manifold $M/\mathcal{F}$ is endowed with a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ and we denote by $p : M \to M/\mathcal{F}$ the canonical projection.

Second step : We consider : i) the Poissonization $(\tilde{M}, \tilde{\Lambda})$, $\tilde{M} = M \times \mathbb{R}$ and $\tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$, of (M, Λ, E), t being the canonical coordinate on $\mathbb{R}$; ii) the Lie subalgebroid $\tilde{D}$ of $T\tilde{M} \cong TM \times T\mathbb{R}$ of time-independent vector fields on $\tilde{M}$ defined by :

$$\tilde{D} = \{X + f\frac{\partial}{\partial t} \in T\tilde{M} \,/\, (X, f) \in D\}.$$

Since, by hypothesis, D has no sections of type $(0, f)$ with $f \neq 0$, $\tilde{D}$ defines a simple foliation $\tilde{\mathcal{F}}$ of $\tilde{M}$ of the same dimension as $\mathcal{F}$. Let $\tilde{p} : \tilde{M} \to \tilde{M}/\tilde{\mathcal{F}}$ be the canonical projection and $\tilde{T} = \tilde{p}_*(\partial/\partial t)$ the projection of $\partial/\partial t$ on $\tilde{M}/\tilde{\mathcal{F}}$ ($\partial/\partial t$ is a projectable vector field). The integral curves of $\tilde{T}$ constitute a simple foliation $\tilde{\mathcal{T}}$ of $\tilde{M}/\tilde{\mathcal{F}}$. Since $(\tilde{M}/\tilde{\mathcal{F}})/\tilde{\mathcal{T}} \cong \tilde{M}/(\tilde{\mathcal{F}} \times \tilde{\mathcal{T}}) \cong (M \times \mathbb{R})/(\mathcal{F} \times \mathbb{R}) \cong M/\mathcal{F}$, by straightening out $\tilde{T}$, we can consider that $\tilde{M}/\tilde{\mathcal{F}} \cong (M/\mathcal{F}) \times \mathbb{R}$. We denote by $\tilde{\tau}$ the canonical coordinate on the factor $\mathbb{R}$, so $\tilde{T} = \partial/\partial\tilde{\tau}$, and we endow $\tilde{M}/\tilde{\mathcal{F}} \cong (M/\mathcal{F}) \times \mathbb{R}$ with the Poissonization $\Lambda_{\tilde{M}/\tilde{\mathcal{F}}} = e^{-\tilde{\tau}}(\Lambda_{M/\mathcal{F}} + \partial/\partial\tilde{\tau} \wedge E_{M/\mathcal{F}})$ of the Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ of $M/\mathcal{F}$. Therefore, $(\tilde{M}/\tilde{\mathcal{F}}, \Lambda_{\tilde{M}/\tilde{\mathcal{F}}}, \tilde{T})$ is an homogeneous Poisson manifold.

Now we consider the hypersurface S of $(\tilde{M}/\tilde{\mathcal{F}}, \Lambda_{\tilde{M}/\tilde{\mathcal{F}}}, \tilde{T})$ defined by the equation $t = 0$. Since $\tilde{T} = \partial/\partial t - (X + f\partial/\partial t) = (1 - f)\partial/\partial t - X$, where $X + f\partial/\partial t \in \Gamma(\tilde{D})$, we have that $\langle dt, \tilde{T}\rangle = 1 - f \neq 0$ (if $1 - f = 0$, $\tilde{T} = -X$, i.e. $\tilde{T}$ is tangent to $\mathcal{F}$, impossible !) which means that S is transverse to the homothety vector field $\tilde{T}$ of $(\tilde{M}/\tilde{\mathcal{F}}, \Lambda_{\tilde{M}/\tilde{\mathcal{F}}}, \tilde{T})$. Thus, the homogeneous Poisson structure $(\Lambda_{\tilde{M}/\tilde{\mathcal{F}}}, \tilde{T})$ of $\tilde{M}/\tilde{\mathcal{F}}$ induces a Jacobi structure on S (see, Proposition 5.3). We denote by $\{\, ,\, \}_S$ the associated Jacobi bracket of the induced structure on S. By Proposition 5.5 we deduce that the Jacobi manifolds $(M/\mathcal{F}, \{\, ,\, \}_{M/\mathcal{F}})$ and $(S, \{\, ,\, \}_S)$ are conformally equivalent. Namely, there exists, locally, a diffeomorphism $\Psi : M/\mathcal{F} \to S$ and a nowhere zero function ψ on $M/\mathcal{F}$ such that, for all $f, g \in C^\infty(S, \mathbb{R})$,

$$\Psi^*\{f, g\}_S = \frac{1}{\psi}\{\psi\Psi^* f, \psi\Psi^* g\}_{M/\mathcal{F}}. \tag{43}$$

Third step : We keep under control the fact that $\Psi \circ p : M \to S$ is not a Jacobi map by defining a "difference" bracket $\{\, ,\, \}_1 : C^\infty(S, \mathbb{R}) \times C^\infty(S, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ as follows :

$$\{f, g\}_1 = (\Psi \circ p)^*\{f, g\}_S - \{(\Psi \circ p)^* f, (\Psi \circ p)^* g\}_{(\Lambda, E)}, \quad \forall\, f, g \in C^\infty(S, \mathbb{R}). \tag{44}$$

Obviously, $\{\, ,\, \}_1$ is a bilinear, skew-symmetric, first order differential operator on each of its arguments. Thus, $\{\, ,\, \}_1$ induces a skew-symmetric bilinear form (Λ_1, E_1) on $T^*S \times \mathbb{R}$ so that, for all $f, g \in C^\infty(S, \mathbb{R})$,

$$\{f, g\}_1 = \Lambda_1(\delta f, \delta g) + \langle f\delta g - g\delta f, E_1\rangle.$$

In turn, (Λ_1, E_1) induces a vector bundle map $(\Lambda_1, E_1)^\# : T^*S \times \mathbb{R} \to TS \times \mathbb{R}$. But, $T^*S \times \mathbb{R} \cong D^\perp$ and $TS \times \mathbb{R} \cong (TM \times \mathbb{R})/D$. In fact, we have that $T^*(\tilde{M}/\tilde{\mathcal{F}}) \cong \tilde{D}^\perp$, where

$$\begin{aligned} \Gamma(\tilde{D}^\perp) \;=\; & \{\tilde{\alpha} + \tilde{g}dt \in \Gamma(T^*\tilde{M}) \,/\, \tilde{\alpha}\ is\ a\ t-dependent\ section\ of\ T^*M\ and \\ & \tilde{g} \in C^\infty(\tilde{M}, \mathbb{R}) : \langle \tilde{\alpha} + \tilde{g}dt, X + f\frac{\partial}{\partial t}\rangle = 0,\ \forall X + f\frac{\partial}{\partial t} \in \Gamma(\tilde{D})\}. \end{aligned}$$

Thus $T^*_S(\tilde{M}/\tilde{\mathcal{F}}) \cong (\tilde{D}^\perp)|_S = \{\tilde{\alpha}|_{t=0} + \tilde{g}|_{t=0}dt \,/\, \tilde{\alpha} + \tilde{g}dt \in \tilde{D}^\perp\} \cong \{\alpha + gdt \,/\, (\alpha, g) \in D^\perp\} \cong D^\perp$. Also, $T^*S = i^*(T^*_S(\tilde{M}/\tilde{\mathcal{F}}))$, where $i : S \to \tilde{M}/\tilde{\mathcal{F}}$ is the canonical injection, and $T^*_S(\tilde{M}/\tilde{\mathcal{F}}) \cong \{\alpha + gdt \,/\, (\alpha, g) \in D^\perp\} \cong \{(i^*\alpha, i^*g) \,/\, (\alpha, g) \in D^\perp\} \cong T^*S \times \mathbb{R}$. So, $T^*S \times \mathbb{R} \cong D^\perp$. Moreover, $T(\tilde{M}/\tilde{\mathcal{F}}) \cong T\tilde{M}/\tilde{D} \cong (TM \times \mathbb{R} \times \mathbb{R})/\tilde{D} \cong ((TM \times \mathbb{R})/D) \times \mathbb{R}$ and $T(\tilde{M}/\tilde{\mathcal{F}}) \cong T((M/\mathcal{F}) \times \mathbb{R}) \cong T(M/\mathcal{F}) \times \mathbb{R} \times \mathbb{R}$. So, $T(M/\mathcal{F}) \times \mathbb{R} \cong (TM \times \mathbb{R})/D$. Since $TS \cong T(M/\mathcal{F})$, we obtain that $TS \times \mathbb{R} \cong (TM \times \mathbb{R})/D$. Consequently, we can consider that $(\Lambda_1, E_1)^\# : D^\perp \to (TM \times \mathbb{R})/D$.

Fourth step : We denote by $pr : TM \times \mathbb{R} \to (TM \times \mathbb{R})/D$ the natural projection and we define a subbundle $L \subset (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ by

$$L = \{(X, f) + (\alpha, g) \in (TM \times \mathbb{R}) \oplus D^\perp \,/\, pr(X, f) = (\Lambda_1, E_1)^\#(\alpha, g)\}. \quad (45)$$

By construction, L is maximally isotropic and $C^\infty_L(M, \mathbb{R}) \cong C^\infty(S, \mathbb{R})$. Effectively, by a straightforward calculation we show that, for any $e_1 = (X_1, f_1) + (\alpha_1, g_1), e_2 = (X_2, f_2) + (\alpha_2, g_2) \in L$, $(e_1, e_2)_+ = 0$ and $f \in C^\infty_L(M, \mathbb{R}) \Leftrightarrow d^{(0,1)}f = (\delta f, f) \in \Gamma(D^\perp) \cong \Gamma(T^*S \times \mathbb{R}) \Leftrightarrow f \in C^\infty(S, \mathbb{R})$. Also, by Definition 4.2, $f \in C^\infty_L(M, \mathbb{R})$ if and only if there exists $(Y_f, \varphi_f) \in \Gamma(TM \times \mathbb{R})$ such that $e_f = (Y_f, \varphi_f) + (\delta f, f) \in \Gamma(L)$. Hence, we have that $\Gamma(L)$ is spanned by all the sections of the type he_f, where $h \in C^\infty(M, \mathbb{R})$ and $f \in C^\infty_L(M, \mathbb{R})$. To verify the integrability of L, it suffices to verify the closeness of the bracket $[\![\, , \,]\!]$ for the sections of L of the form $e_f = (Y_f, \varphi_f) + (\delta f, f)$ with $f \in C^\infty_L(M, \mathbb{R})$, since, according to (17) and because L is isotropic,

$$\begin{aligned} [\![e_f, he_g]\!] &= h[\![e_f, e_g]\!] + (\rho(e_f)h)e_g - (e_f, e_g)_+\mathcal{D}h = \\ &= h[\![e_f, e_g]\!] + (\rho(e_f)h)e_g, \end{aligned}$$

for all $e_f, e_g \in \Gamma(L)$, with $f, g \in C^\infty_L(M, \mathbb{R})$, and $h \in C^\infty(M, \mathbb{R})$.

Let $f, g \in C^\infty_L(M, \mathbb{R})$ be two L-admissible functions. Being $C^\infty_L(M, \mathbb{R}) \cong C^\infty(S, \mathbb{R})$, $\{f, g\}_S \in C^\infty_L(M, \mathbb{R})$, i.e. there is $(Y_{\{f,g\}_S}, \varphi_{\{f,g\}_S}) \in \Gamma(TM \times \mathbb{R})$ such that $e_{\{f,g\}_S} = (Y_{\{f,g\}_S}, \varphi_{\{f,g\}_S}) + (\delta\{f, g\}_S, \{f, g\}_S) \in \Gamma(L)$. We show that

$$\{f, g\}_S = \rho^\theta(e_f)g \overset{(30)}{=}: \{f, g\}_L. \quad (46)$$

Effectively,

$$\begin{aligned} \{f, g\}_L = \rho^\theta(e_f)g &= [(\pi^{(0,1)} + (\pi \circ (\Lambda, E)^\#)^{(-E,0)})((Y_f, \varphi_f) + (\delta f, f))]g = \\ &= (Y_f + \varphi_f + \Lambda^\#(\delta f) + fE - \langle \delta f, E\rangle)g = \\ &= (pr(Y_f, \varphi_f) + \textit{the component of } (Y_f, \varphi_f) \textit{ on } D)g \\ &\quad + \{f, g\}_{(\Lambda,E)} = \\ &= \langle(\delta g, g), (\Lambda_1, E_1)^\#(\delta f, f)\rangle + \{f, g\}_{(\Lambda,E)} = \\ &= \{f, g\}_1 + \{f, g\}_{(\Lambda,E)} = \\ &\overset{(44)}{=} \{f, g\}_S. \end{aligned}$$

On the other hand, since $\{\, , \,\}_S$ is a Jacobi bracket, thus it verifies the Jacobi identity, we have that, for any $f, g, h \in C^\infty_L(M, \mathbb{R}) \cong C^\infty(S, \mathbb{R})$,

$$\rho^\theta([\![e_f, e_g]\!] - e_{\{f,g\}_S})h \;=\; \rho^\theta([\![e_f, e_g]\!])h - \rho^\theta(e_{\{f,g\}_S})h =$$

$$\begin{aligned}
&\overset{(19)}{=} [\rho^\theta(e_f), \rho^\theta(e_g)]h - \rho^\theta(e_{\{f,g\}_S})h = \\
&= \rho^\theta(e_f)(\rho^\theta(e_g)h) - \rho^\theta(e_g)(\rho^\theta(e_f)h) - \rho^\theta(e_{\{f,g\}_S})h = \\
&\overset{(46)}{=} \{f, \{g,h\}_S\}_S - \{g, \{f,h\}_S\}_S - \{\{f,g\}_S, h\}_S = \\
&= 0. \qquad (47)
\end{aligned}$$

From the proof of Theorem 5.1, we have that the component of $[\![e_f, e_g]\!]$ in $\Gamma(T^*M \times \mathbb{R})$ is $d^{(0,1)}\{f,g\}_S$, therefore $[\![e_f, e_g]\!] - e_{\{f,g\}_S} \in \Gamma(TM \times \mathbb{R})$. So, (47) means that $\rho^\theta([\![e_f, e_g]\!] - e_{\{f,g\}_S}) \in \Gamma(D)$. But $\rho^\theta([\![e_f, e_g]\!] - e_{\{f,g\}_S}) = \pi^{(0,1)}([\![e_f, e_g]\!] - e_{\{f,g\}_S}) = [\![e_f, e_g]\!] - e_{\{f,g\}_S}$ and $\Gamma(D) \subset \Gamma(L)$. Consequently, $[\![e_f, e_g]\!] - e_{\{f,g\}_S} \in \Gamma(L)$ which implies $[\![e_f, e_g]\!] \in \Gamma(L)$, whence the integrability of L.

For the constructed L we have $L \cap (TM \times \mathbb{R}) = \{(X,f) + (0,0) \in (TM \times \mathbb{R}) \oplus \{(0,0)\} \,/\, pr(X,f) = (\Lambda_1, E_1)^\#(0,0)\} = \{(X,f) \in TM \times \mathbb{R} \,/\, pr(X,f) = (0,0)\} = D$. Taking into account (46) and (43), we conclude that the induced Jacobi structure on $M/\mathcal{F}$, in the sense of Corollary 5.8, is the initially given $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$. •

Remark 5.11. The condition *D has no sections of type $(0,f)$ with $f \neq 0$* is indispensable. In the opposite case, $D \cong \pi(D) \times \mathbb{R}$ and $\tilde{D} = \{X + f\partial/\partial t \,/\, (X,f) \in D\}$ defines a simple foliation $\tilde{\mathcal{F}}$ of $\tilde{M}$ whose leaves $\tilde{F}$ are of the type $\tilde{F} = F \times \mathbb{R}$, where F is the leaf of $\mathcal{F}$ corresponding to $\tilde{F}$, and the vector field $\partial/\partial t$ is tangent to $\tilde{\mathcal{F}}$. We suppose that we can construct, by using D and $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$, a reducible Dirac structure L for $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), [\![\,,\,]\!], (\,,\,)_+, \pi + \pi \circ (\Lambda, E)^\#, (0,1) + (-E, 0))$ such that $L \cap (TM \times \mathbb{R}) = D$. Then, $\tilde{L} = \mathbf{E}(L)$ is a reducible Dirac subbundle of $T\tilde{M} \oplus T^*\tilde{M}$ (see, Proposition 4.6) such that $\tilde{L} \cap T\tilde{M} = \tilde{D}$. By Lemma 4.5 and Proposition 4.4 we have that $C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R}) \cong C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$ and $\tilde{f} \in C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R})$ if and only if $\tilde{f} = e^t f$, $f \in C^\infty_L(M, \mathbb{R})$. But, $\partial/\partial t$ is tangent to $\tilde{\mathcal{F}}$ and, for any $\tilde{f} \in C^\infty_{\tilde{L}}(\tilde{M}, \mathbb{R}) \cong C^\infty(\tilde{M}/\tilde{\mathcal{F}}, \mathbb{R})$, $\partial/\partial t(\tilde{f}) = \partial/\partial t(e^t f) = e^t f \neq 0$; contradiction. Thus, when D has sections of type $(0,f)$ with $f \neq 0$, it is not possible to construct a reducible Dirac subbundle $L \subseteq (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ such that $L \cap (TM \times \mathbb{R}) = D$.

In conclusion, we have proved :

Theorem 5.12. *Let (M, Λ, E) be a Jacobi manifold. There is a one-one correspondence between reducible Dirac subbundles for the generalized Courant* *algebroid* $((TM\times I\!R)\oplus(T^*M\times I\!R),[\![\,,\,]\!],(\,,\,)_+,\pi+\pi\circ(\Lambda,E)^{\#},(0,1)+(-E,0))$ *and quotient Jacobi manifolds* $M/\mathcal{F}$ *of* M*, where* $\mathcal{F}$ *is a simple foliation of* M *defined by a Lie subalgebroid* $D\subset TM\times I\!R$ *that has no sections of type* $(0,f)$ *with* $f\neq 0$*.*

Remark 5.13. If, in the proof of Theorem 5.10,

$$p:(M,\Lambda,E)\to(M/\mathcal{F},\Lambda_{M/\mathcal{F}},E_{M/\mathcal{F}})$$

is a $1/p^*\psi$-conformal Jacobi map, then $\Psi\circ p:M\to S$ is a Jacobi map and $(\Lambda_1,E_1)=(0,0)$. Hence, in this case, $L=D\oplus D^{\perp}$ is a null Dirac structure. Thus, we can deduce :

Corollary 5.14. *A Lie subalgebroid* $D\subset TM\times I\!R$*, which has no sections of type* $(0,f)$ *with* $f\neq 0$*, defines a simple foliation* $\mathcal{F}$ *of* (M,Λ,E) *such that* $p:(M,\Lambda,E)\to(M/\mathcal{F},\Lambda_{M/\mathcal{F}},E_{M/\mathcal{F}})$ *is a* $1/p^*\psi$*-conformal Jacobi map if and only if* $L=D\oplus D^{\perp}$*.*

Remark 5.15. In the case where $D=\{(0,0)\}$, a Jacobi structure on $M/\mathcal{F}\cong M$ is a new Jacobi structure (Λ',E') on M and the constructed L is the graph of $(\Lambda'-\Lambda,E'-E)$. Since, by construction, L is a Dirac subbundle of $(TM\times\mathbb{R})\oplus(T^*M\times\mathbb{R})$, $(\Lambda'-\Lambda,E'-E)$ is a Jacobi structure on M ([33]), fact which implies that (Λ,E) and (Λ',E') are compatible Jacobi structures in the sense of [32].

A geometric interpretation of Corollary 3.5 : In the context of this paragraph, Corollary 3.5 can be formulated as : *Let* (M,Λ,E) *be a Jacobi manifold,* $((TM\times I\!R,(0,1)),(T^*M\times I\!R,(-E,0)),(\Lambda,E))$ *the associated triangular generalized Lie bialgebroid over* M *and* (Λ',E') *a* $(TM\times I\!R)$*-bivector field such that* $L=D\oplus graph((\Lambda',E')^{\#}|_{D^{\perp}})$ *is a maximal isotropic subbundle of* $(TM\times I\!R)\oplus(T^*M\times I\!R)$ *with fixed characteristic pair* $(D,(\Lambda',E'))$*. Then* L *is a Dirac structure for* $((TM\times I\!R)\oplus(T^*M\times I\!R),(0,1)+(-E,0))$ *if and only if*

(i) *D is a Lie subalgebroid of $TM\times I\!R$;*
(ii) $[(\Lambda+\Lambda',E+E'),(\Lambda+\Lambda',E+E')]^{(0,1)}\equiv 0(modD)$ *;*
(iii) *for any* $(X,f)\in\Gamma(D)$, $\mathcal{L}^{(0,1)}_{(X,f)}(\Lambda+\Lambda',E+E')\equiv 0(modD)$*.*

If $L=D\oplus graph((\Lambda',E')^{\#}|_{D^{\perp}})$ is a reducible Dirac structure, after the proofs of Theorems 5.7 and 5.10, we get that condition (iii) is equivalent to that $(\Lambda+\Lambda',E+E')$ can be reduced to a $(TM\times\mathbb{R})/D\cong(TS\times\mathbb{R})$-bivector field on S and the condition (ii) is equivalent to the fact that the reduced bivector field is a Jacobi structure on S. Furthermore, by Proposition 5.9 (case 3) we get that the induced Jacobi structure on S is exactly the one defined by the bracket of L-admissible functions. Consequently, it is conformally equivalent to the Jacobi structure induced on $M/\mathcal{F}$ by L in the sense of Corollary 5.8.

6. Dirac structures and Jacobi reduction

In this paragraph, we will establish a Jacobi reduction theorem in terms of Dirac structures. For its proof, we need to adapt the results concerning the pull-back Dirac structures of a Lie bialgebroid ([23]) to the pull-back Dirac structures for a generalized Lie bialgebroid.

Proposition 6.1. *Let (A_1, ϕ_1) be a Lie algebroid over a differentiable manifold M_1 with an 1-cocycle, $((A_2, \phi_2), (A_2^*, W_2), P_2)$ a triangular generalized Lie bialgebroid over a differentiable manifold M_2 and $\Phi : A_1 \to A_2$ a Lie algebroid morphism of constant rank, which covers a surjective map between the bases, such that $\Phi^*(\phi_2) = \phi_1$. Then the following two statements are equivalent.*

1. *There exists a Dirac structure for the triangular generalized Lie bialgebroid $((A_1, \phi_1), (A_1^*, 0), 0)$ whose characteristic pair is $(\ker\Phi, P_1)$ and $\Phi(P_1) = P_2$.*
2. $\mathrm{Im}P_2^{\#} \subseteq \mathrm{Im}\Phi$.

We note that, since $\Phi : A_1 \to A_2$ is a Lie algebroid morphism such that $\Phi^*(\phi_2) = \phi_1$ then, for any $P \in \Gamma(\bigwedge^p A_1)$ and $Q \in \Gamma(\bigwedge^q A_1)$, $\Phi([P,Q]^{\phi_1}) = [\Phi(P), \Phi(Q)]^{\phi_2}$.

Proof : According to Corollary 3.5, it suffices to show that the following two statements are equivalent.

1. There exists $P_1 \in \Gamma(\bigwedge^2 A_1)$ such that $\Phi(P_1) = P_2$ and
   - (a) $\ker\Phi$ is a Lie subalgebroid of A_1 ;
   - (b) $[0 + P_1, 0 + P_1]^{\phi_1} \equiv 0 (mod \ker\Phi) \Leftrightarrow [P_1, P_1]^{\phi_1} \equiv 0 (mod \ker\Phi)$;
   - (c) for any $X \in \Gamma(\ker\Phi)$, $\mathcal{L}_X^{\phi_1}(0 + P_1) \equiv 0 (mod \ker\Phi) \Leftrightarrow \mathcal{L}_X^{\phi_1}(P_1) \equiv 0 (mod \ker\Phi)$.
2. $\mathrm{Im}P_2^{\#} \subseteq \mathrm{Im}\Phi$.

Obviously, $\ker\Phi$ is a Lie subalgebroid of A_1 since, for all $X, Y \in \Gamma(\ker\Phi)$, $\Phi([X,Y]) = [\Phi(X), \Phi(Y)] = [0,0] = 0$, which means that $[X,Y] \in \Gamma(\ker\Phi)$. On the other hand, we have that the subbundle $\ker\Phi^{\perp} = \{\alpha \in A_1^* / \langle\alpha, X\rangle =$

$0, \forall X \in \ker\Phi\}$ of A_1^* can be identified with the dual bundle $(A_1/\ker\Phi)^*$ of $A_1/\ker\Phi$. Also, $\ker\Phi^\perp = \mathrm{Im}\Phi^*$, where $\Phi^* : A_2^* \to A_1^*$ is the dual map of Φ. Effectively, it is clear that, $\mathrm{Im}\Phi^* \subseteq \ker\Phi^\perp$ and, since Φ is of constant rank, $\dim \mathrm{Im}\Phi^* = \dim\ker\Phi^\perp$, thus $\mathrm{Im}\Phi^* = \ker\Phi^\perp \cong (A_1/\ker\Phi)^*$. Hence, $\Phi^* : A_2^* \to (A_1/\ker\Phi)^*$ is a surjective map, i.e., for any $\bar\alpha_1, \bar\beta_1 \in \Gamma((A_1/\ker\Phi)^*)$, there exist $\alpha_2, \beta_2 \in \Gamma(A_2^*)$ such that $\bar\alpha_1 = \Phi^*(\alpha_2)$ and $\bar\beta_1 = \Phi^*(\beta_2)$. If there is some $\bar P_1 \in \Gamma(\bigwedge^2(A_1/\ker\Phi))$ which is Φ-related to P_2, i.e. $\Phi(\bar P_1) = P_2$, then it should be defined, for all $\bar\alpha_1, \bar\beta_1 \in \Gamma((A_1/\ker\Phi)^*)$, by

$$\bar P_1(\bar\alpha_1, \bar\beta_1) = P_2(\alpha_2, \beta_2).$$

It is clear that $\bar P_1$ is well-defined if and only if $\ker\Phi^* \subseteq \ker P_2^\#$, or equivalently, if and only if $\mathrm{Im}P_2^\# \subseteq \mathrm{Im}\Phi$. Let P_1 be an arbitrary representative of $\bar P_1$ in $\Gamma(\bigwedge^2 A_1)$. Since $\Phi : A_1 \to A_2$ is a Lie algebroid morphism such that $\Phi^*(\phi_2) = \phi_1$ and $((A_2, \phi_2), (A_2^*, W_2), P_2)$ is a triangular generalized Lie bialgebroid, we have that

$$\Phi([P_1, P_1]^{\phi_1}) = [\Phi(P_1), \Phi(P_1)]^{\phi_2} = [P_2, P_2]^{\phi_2} = 0 \;\Leftrightarrow\; [P_1, P_1]^{\phi_1} \equiv 0(mod\ker\Phi).$$

Moreover, for any $X \in \Gamma(\ker\Phi)$,

$$\Phi(\mathcal{L}_X^{\phi_1} P_1) = \Phi([X, P_1]^{\phi_1}) = [\Phi(X), \Phi(P_1)]^{\phi_2} = [0, \Phi(P_1)]^{\phi_2} = 0 \;\Leftrightarrow$$

$$\Leftrightarrow\; \mathcal{L}_X^{\phi_1} P_1 \equiv 0(mod\ker\Phi).$$

Consequently, there exists $P_1 \in \Gamma(\bigwedge^2 A_1)$ such that $\Phi(P_1) = P_2$ and $(\ker\Phi, P_1)$ defines a Dirac structure for the triangular generalized Lie bialgebroid

$$((A_1, \phi_1), (A_1^*, 0), 0)$$

if and only if $\mathrm{Im}P_2^\# \subseteq \mathrm{Im}\Phi$. •

Reduction of Jacobi manifolds: Let (M, Λ, E) be a Jacobi manifold, $N \subseteq M$ a submanifold of M and $i : N \hookrightarrow M$ the canonical inclusion, $D \subset TM \times \mathbb{R}$ a Lie subalgebroid of $(TM \times \mathbb{R}, [\,,\,], \pi)$ that has no sections of type $(0, f)$ with $f \neq 0$ and $D_0 = D \cap (TN \times \mathbb{R})$. We suppose that D and D_0 define, respectively, a simple foliation $\mathcal{F}$ of M and a simple foliation $\mathcal{F}_0$ of N and we denote by $p : M \to M/\mathcal{F}$ and $p_0 : N \to N/\mathcal{F}_0$ the canonical projections. Thus, we have the following commutative diagram :

$$\begin{array}{ccc} N & \stackrel{i}{\hookrightarrow} & M \\ p_0 \downarrow & & \downarrow p \\ N/\mathcal{F}_0 & \stackrel{\varphi}{\to} & M/\mathcal{F} \end{array} \tag{48}$$

Since any leaf of $\mathcal{F}_0$ is a connected component of the intersection between N and some leaf of $\mathcal{F}$, we can always suppose, under some clean intersection condition, that $\varphi : N/\mathcal{F}_0 \to M/\mathcal{F}$ is an immersion, locally injective.

We consider $L = D \oplus D^{\perp}$ and we suppose that L is a null Dirac structure for the triangular generalized Lie bialgebroid $((TM \times \mathbb{R}, (0,1)), (T^*M \times \mathbb{R}, (-E, 0)), (\Lambda, E))$. By the hypothesis on D, we have that L is also reducible. Then, by Corollary 5.8 and the proofs of Theorems 5.7 and 5.10, we get that L induces a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ on $M/\mathcal{F}$ conformally equivalent to the Jacobi structure (Λ_S, E_S) on S defined by the bracket (30) of L-admissible functions. As we have seen, S has the property $TS \times \mathbb{R} \cong (TM \times \mathbb{R})/D$ and $T^*S \times \mathbb{R} \cong D^{\perp}$. If $\Psi : (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}) \to (S, \Lambda_S, E_S)$ is the ψ-conformal Jacobi diffeomorphism which, locally, maps $M/\mathcal{F}$ to S, i.e. $\Psi_*(\Lambda^{\psi}_{M/\mathcal{F}}) = \Lambda_S$ and $\Psi_*(E^{\psi}_{M/\mathcal{F}}) = E_S$, by Corollary 5.14 and Remark 5.13, we obtain that $p : (M, \Lambda, E) \to (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ is $1/p^*\psi$-conformal Jacobi map and that $\Psi \circ p : (M, \Lambda, E) \to (S, \Lambda_S, E_S)$ is a Jacobi map. We have :

$$\begin{array}{ccccc} N & \stackrel{i}{\hookrightarrow} & M & & \\ p_0 \downarrow & & \downarrow p & \searrow^{\Psi\circ p} & \\ N/\mathcal{F}_0 & \stackrel{\varphi}{\to} & M/\mathcal{F} & \stackrel{\Psi}{\to} & S \end{array}$$

We consider the triangular generalized Lie bialgebroids

$$((TS \times \mathbb{R}, (0,1)), (T^*S \times \mathbb{R}, (-E_S, 0)), (\Lambda_S, E_S))$$

over S and

$$((TN \times \mathbb{R}, (0,1)), (T^*N \times \mathbb{R}, (0,0)), (0,0))$$

over N. We note that any function $\bar{f} \in C^{\infty}(N, \mathbb{R})$ can be seen as the image by $(\Psi \circ p \circ i)^*$ of a function $f \in C^{\infty}(S, \mathbb{R})$, i.e $\bar{f} = (\Psi \circ p \circ i)^* f$. Since $\mathcal{F}$ is a regular foliation, p has constant rank, thus the map $\Psi\circ p\circ i : N \to S$ has also constant rank. Hence, the application $\Phi : TN\times\mathbb{R} \to TS\times\mathbb{R} \cong (TM\times\mathbb{R})/D$ defined, for any $(X,\bar{f}) \in \Gamma(TN\times\mathbb{R})$, by

$$\Phi(X,\bar{f}) = ((\Psi\circ p\circ i)_* X, f) \tag{49}$$

can be considered as a Lie algebroid morphism of constant rank such that $\Phi^*(0,1) = (0,1)$ and $\ker\Phi = D\cap(TN\times\mathbb{R}) = D_0$. Therefore, by Proposition 6.1, there exists a pull-back Dirac structure L_0 for the triangular generalized Lie bialgebroid $((TN\times\mathbb{R},(0,1)),(T^*N\times\mathbb{R},(0,0)),(0,0))$ with characteristic pair $(D_0,(\Lambda_N,E_N))$ verifying $\Phi(\Lambda_N,E_N) = (\Lambda_S,E_S)$ if and only if $\mathrm{Im}(\Lambda_S,E_S)^{\#} \subseteq \mathrm{Im}\Phi$ holds on $TS\times\mathbb{R}$, i.e.

$$\Gamma((\Lambda_S,E_S)^{\#}(D^{\perp})) \subseteq \{((\Psi\circ p\circ i)_* X, f)\,/\,X\in\Gamma(TN) \text{ and } f\in C^\infty(S,\mathbb{R})\}. \tag{50}$$

But, $\Psi\circ p : (M,\Lambda,E) \to (S,\Lambda_S,E_S)$ being a Jacobi map, $(\Lambda_S,E_S) = (\Psi\circ p)_*(\Lambda,E)$. Thus, on the submanifold $N\subseteq M$, by identifying $i_*(TN)$ with TN, condition (50) is equivalent to

$$(\Lambda,E)^{\#}(D^{\perp}) \subseteq TN\times\mathbb{R} + D. \tag{51}$$

Also, since $L_0 = D_0 \oplus graph(\Lambda_N,E_N)^{\#}|_{D_0^{\perp}}$ is a reducible Dirac structure of $((TN\times\mathbb{R})\oplus(T^*N\times\mathbb{R}),(0,1)+(0,0))$, it induces a Jacobi structure $(\Lambda_{N/\mathcal{F}_0},E_{N/\mathcal{F}_0})$ on $N/\mathcal{F}_0$ (see, Corollary 5.8). Now, we consider : i) the manifold $\tilde{N} = N\times\mathbb{R}$, $\tilde{N}\subseteq\tilde{M} = M\times\mathbb{R}$, endowed with the null Poisson structure, which can be viewed as the Poissonization of the null Jacobi structure on N ; ii) the reducible Dirac structure $\tilde{L}_0 = \mathbf{E}(L_0)$ for the triangular Lie bialgebroid $(T\tilde{N},T^*\tilde{N},0)$ (of course, $\tilde{L}_0\cap T\tilde{N} = \tilde{D}_0 \cong D_0$) ; iii) the simple foliation $\tilde{\mathcal{F}}_0$ defined by $\tilde{D}_0$ whose leaves are of the same dimension as the leaves of $\mathcal{F}_0$; iv) the manifold $\tilde{N}/\tilde{\mathcal{F}}_0$ which is immersed in $\tilde{M}/\tilde{\mathcal{F}}$; v) the submanifold S_0 of $\tilde{N}/\tilde{\mathcal{F}}_0$ defined by the equation $t=0$ which is an immersed submanifold of S and has the property that $TS_0\times\mathbb{R} \cong (TN\times\mathbb{R})/D_0$ and $T^*S_0\times\mathbb{R} \cong D_0^{\perp}$ (see, proofs of Theorems 5.7 and 5.10). By Corollary 3.5 and its geometric interpretation and by the fact that $L_0 = D_0\oplus graph((\Lambda_N,E_N)^{\#}|_{D_0^{\perp}})$ is a reducible Dirac structure, we get that (Λ_N,E_N) induces a Jacobi structure (Λ_{S_0},E_{S_0}) on S_0 which is the Jacobi structure defined by the Jacobi bracket of L_0-admissible functions on N and which is conformally equivalent to $(\Lambda_{N/\mathcal{F}_0},E_{N/\mathcal{F}_0})$. Precisely, there exist, locally, a nowhere zero function $\psi_0 \in C^\infty(N/\mathcal{F}_0,\mathbb{R})$ and a diffeomorphism

$\Psi_0 : N/\mathcal{F}_0 \to S_0$ such that $\Psi_{0*}(\Lambda^{\psi_0}_{N/\mathcal{F}_0}) = \Lambda_{S_0}$ and $\Psi_{0*}(E^{\psi_0}_{N/\mathcal{F}_0}) = E_{S_0}$. Also, $(\Psi_0 \circ p_0)_*(\Lambda_N, E_N) = (\Lambda_{S_0}, E_{S_0})$ and $p_{0*}(\Lambda_N, E_N) = (\Lambda^{\psi_0}_{N/\mathcal{F}_0}, E^{\psi_0}_{N/\mathcal{F}_0})$. Hence,

$$
\begin{array}{ccccccccccccc}
 & & & & N & \overset{i}{\hookrightarrow} & M & & & & \\
 & & {}^{\Psi_0\circ p_0}\swarrow & & p_0\downarrow & & \downarrow p & & \searrow^{\Psi\circ p} & & \\
S & \overset{i_0}{\hookleftarrow} & S_0 & \overset{\Psi_0}{\leftarrow} & N/\mathcal{F}_0 & \overset{\varphi}{\rightarrow} & M/\mathcal{F} & \overset{\Psi}{\rightarrow} & S & \overset{i_0}{\hookleftarrow} & S_0
\end{array} \tag{52}
$$

where $i_0 : S_0 \hookrightarrow S$ is the immersion of S_0 in S and $i_0 \circ \Psi_0 = \Psi \circ \varphi$. By the above results and by the commutativity of the diagram (52), we obtain :

$$
\begin{aligned}
(\Lambda_S, E_S) &= \Phi(\Lambda_N, E_N) = \\
&= ((\Psi \circ p \circ i)_*\Lambda_N, (\Psi \circ p \circ i)_* E_N) = \\
&= ((\Psi \circ \varphi)_* \circ p_{0*}\Lambda_N, (\Psi \circ \varphi)_* \circ p_{0*}E_N) = \\
&= ((\Psi \circ \varphi)_*\Lambda^{\psi_0}_{N/\mathcal{F}_0}, (\Psi \circ \varphi)_* E^{\psi_0}_{N/\mathcal{F}_0}) = \\
&= (\Psi \circ \varphi)_*(\Lambda^{\psi_0}_{N/\mathcal{F}_0}, E^{\psi_0}_{N/\mathcal{F}_0}) = (i_0 \circ \Psi_0)_*(\Lambda^{\psi_0}_{N/\mathcal{F}_0}, E^{\psi_0}_{N/\mathcal{F}_0}) \qquad (53)
\end{aligned}
$$

which means that $\Psi \circ \varphi = i_0 \circ \Psi_0$ is a ψ_0-conformal Jacobi map. Taking into account that $\Psi : M/\mathcal{F} \to S$ is a ψ-conformal Jacobi map and that $\Psi_0 : N/\mathcal{F}_0 \to S_0$ is a ψ_0-conformal Jacobi map, (53) implies that $\varphi : N/\mathcal{F}_0 \to M/\mathcal{F}$ is a $(\psi_0/\varphi^*\psi)$-conformal Jacobi map and that $i_0 : S_0 \to S$ is a Jacobi map.

The above study led us to the following theorem :

Theorem 6.2 (Reduction Theorem of Jacobi manifolds). *Let (M, Λ, E) be a Jacobi manifold, $N \subseteq M$ a submanifold of M, $D \subset TM \times I\!R$ a Lie subalgebroid of $(TM \times I\!R, [\,,\,], \pi)$ that has no sections of type $(0, f)$ with $f \neq 0$ and $D_0 = D \cap (TN \times I\!R)$. We suppose that D and D_0 define, respectively, a simple foliation $\mathcal{F}$ of M and a simple foliation $\mathcal{F}_0$ of N and that $L = D \oplus D^\perp$ is a reducible Dirac structure for the triangular generalized Lie bialgebroid $((TM \times I\!R, (0,1)), (T^*M \times I\!R, (-E, 0)), (\Lambda, E))$. Then, the following two statements are equivalent.*

(1) *There exists a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on $N/\mathcal{F}_0$ and a function $\bar{\psi}_0 \in C^\infty(N/\mathcal{F}_0, I\!R)$ (in the above notation, $\bar{\psi}_0 = \psi_0/\varphi^*\psi$) such that*

$$p_*(\Lambda, E) = \varphi_*(\Lambda^{\bar{\psi}_0}_{N/\mathcal{F}_0}, E^{\bar{\psi}_0}_{N/\mathcal{F}_0}).$$

(2) $(\Lambda, E)^{\#}(D^{\perp}) \subseteq TN \times I\!R + D$ *holds on* N.

Remarks 6.3.

1. We remark that, in the context of the Reduction Theorem 6.2, the initial Jacobi manifold (M, Λ, E) and the reduced Jacobi manifold

$$(N/\mathcal{F}_0, \Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$$

are connected by means of the Jacobi manifold $(M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ with two conformal Jacobi maps.

2. Reduction Theorem 6.2 holds for any reducible Dirac structure $L \subset (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ having a characteristic pair $(D, (\Lambda', E'))$, i.e. $L = D \oplus graph((\Lambda', E')^{\#}|_{D^{\perp}})$. Effectively, by Corollary 5.8 we get that L induces a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ on $M/\mathcal{F}$. Also, by the geometric interpretation of Corollary 3.5 we conclude that $(\Lambda + \Lambda', E + E')$ is reduced to a Jacobi structure (Λ_S, E_S) on S, which is exactly the Jacobi structure defined by the Jacobi bracket of L-admissible functions on M and which is ψ-conformally equivalent to $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$. If (Λ_S, E_S) verifies (50) or, equivalently, $(\Lambda + \Lambda', E + E')$ verifies (51), then, by Proposition 6.1, there exists a pull-back Dirac structure L_0 for $((TN \times \mathbb{R}, (0,1)), (T^*N \times \mathbb{R}, (0,0)), (0,0))$ with characteristic pair $(D_0, (\Lambda_N, E_N))$ such that $\Phi(\Lambda_N, E_N) = (\Lambda_S, E_S)$. The reducible Dirac subbundle $L_0 \subset (TN \times \mathbb{R}) \oplus (T^*N \times \mathbb{R})$ induces a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on $N/\mathcal{F}_0$ and

$$p_{0*}(\Lambda_N, E_N) = (\Lambda^{\psi_0}_{N/\mathcal{F}_0}, E^{\psi_0}_{N/\mathcal{F}_0}),$$

where ψ_0 is an appropriate nowhere zero function on $N/\mathcal{F}_0$. Applying the calculus of (53) to the relation $(\Lambda_S, E_S) = \Phi(\Lambda_N, E_N)$, we conclude that $\varphi : N/\mathcal{F}_0 \to M/\mathcal{F}$ is always a conformal Jacobi map. But, the projection $p : M \to M/\mathcal{F}$ is a conformal Jacobi map if and only if L is a null Dirac structure, fact which is equivalent to $(\Lambda', E') \equiv 0(mod D)$.

3. We note that the Reduction Theorem of Jacobi manifolds 6.2 differs at same points from that proved in [30] by the second author and independently by K. Mikami in [29], which generalizes the Reduction Theorem of Poisson manifolds of Marsden and Ratiu [28]. Precisely, in Theorem 6.2 we suppose that we have two simple foliations, a foliation $\mathcal{F}$ of the initial phase space M determined by $\pi(D)$ and a foliation $\mathcal{F}_0$ of the considered submanifold N of M determined by $\pi(D_0) = \pi(D) \cap TN$, while in the theorem of [30] we only suppose that we have a subbundle Δ of $T_N M$ such that $\Delta \cap TN$ defines a simple foliation of N. In the particular case where $D = \Delta \times \{0\}$, Δ being an integrable subbundle of TM, the reducibility condition

$$(\Lambda, E)^{\#}(D^{\perp}) \subseteq TN \times \mathbb{R} + D$$

of Theorem 6.2 reduced to the one of the Reduction Theorem of [30] given by

$$\Lambda^{\#}(\Delta^{\perp}) \subseteq TN + \Delta \;\text{ and }\; E|_N \in \Gamma(TN + \Delta)$$

which is a stricter condition.

7. Applications and Examples

1. Jacobi submanifolds : From Theorem 6.2 we obtain sufficient conditions under which a Jacobi structure (Λ, E) on a differentiable manifold M induces a Jacobi structure on a submanifold N of M. Effectively, under the assumptions of the above mentioned theorem, if $D_0 = D \cap (TN \times \mathbb{R}) = \{(0,0)\}$ and $(\Lambda, E)^{\#}(D^{\perp}) \subseteq TN \times \mathbb{R} + D$ holds on N, then there exists a $(TN \times \mathbb{R})$-bivector field (Λ_N, E_N) on N such that $L_0 = D_0 \oplus graph(\Lambda_N, E_N)^{\#}|_{D_0^{\perp}} = graph(\Lambda_N, E_N)^{\#}$ is a reducible Dirac structure for the triangular generalized Lie bialgebroid $((TN \times \mathbb{R}, (0,1)), (T^*N \times \mathbb{R}, (0,0)), 0)$ and $\Phi(\Lambda_N, E_N) = (\Lambda_S, E_S)$. But, the fact *"$L_0 = graph(\Lambda_N, E_N)^{\#}$ is Dirac for $((TN \times \mathbb{R}, (0,1)), (T^*N \times \mathbb{R}, (0,0)), 0)$"* is equivalent to the fact *"(Λ_N, E_N) is a Jacobi structure on N"* (see Proposition 5.2 in [33]) and

$$\begin{aligned} (\Lambda_S, E_S) = \Phi(\Lambda_N, E_N) \;&\Leftrightarrow\; (\Psi \circ p)_*(\Lambda, E) = (\Psi \circ p \circ i)_*(\Lambda_N, E_N) \\ &\Leftrightarrow\; p_*(\Lambda, E) = (p \circ i)_*(\Lambda_N, E_N) \\ &\Leftrightarrow\; p_*((\Lambda, E) - i_*(\Lambda_N, E_N)) = (0,0). \end{aligned}$$

By the last equality we conclude either that $(\Lambda, E) - i_*(\Lambda_N, E_N) = (0,0) \Leftrightarrow (\Lambda, E) = i_*(\Lambda_N, E_N)$, i.e. $i : (N, \Lambda_N, E_N) \to (M, \Lambda, E)$ is a Jacobi map, or that $\Lambda = i_*\Lambda_N + \sum_{j=1}^{k} X_j \wedge Y_j$ and $E = i_*E_N + X$, where $X_j, X \in \Gamma(\pi(D))$, $Y_j \in \Gamma(TM)$, $j = 1, \ldots, k$, are convenable vector fields such that $[\Lambda, \Lambda] = -2E \wedge \Lambda$ and $[E, \Lambda] = 0$.

Particular cases

a) When $D = \{(0,0)\}$, then $D^{\perp} = T^*M \times \mathbb{R}$, and they verify the assumptions of Theorem 6.2. Condition $D_0 = D \cap (TN \times \mathbb{R}) = \{(0,0)\}$ is automatically satisfied and (51) is equivalent to

$$\Lambda^{\#}(T^*M) \subseteq TN \;\text{ on }\; N \;\text{ and }\; E|_N \in \Gamma(TN),$$

which are exactly the conditions given in [4] and [26] for the submanifolds N of (M,Λ,E) of the first kind.

b) When $D=(\Lambda,E)^{\#}((TN\times \mathbb{R})^{\perp})$, we have $D_0=D\cap(TN\times\mathbb{R})=\{(0,0)\}$ if and only if

$$TN\cap\Lambda^{\#}(TN^{\perp})=\{0\} \;\text{ on }\; N \;\text{ and }\; E|_N\in\Gamma(TN). \tag{54}$$

Effectively, $(TN\times\mathbb{R})^{\perp}=TN^{\perp}\times\{0\}$ and $D=(\Lambda,E)^{\#}(TN^{\perp}\times\{0\})=\{(\Lambda^{\#}(\alpha),-\langle\alpha,E\rangle)\,/\,(\alpha,0)\in TN^{\perp}\times\{0\}\}$. Hence, $D_0=D\cap(TN\times\mathbb{R})=\{(0,0)\}$ if and only if

(i) for all $\alpha\in\Gamma(TN^{\perp})$, $\langle\alpha,E\rangle=0$, i.e. $E|_N\in\Gamma(TN)$, and

(ii) $\Lambda^{\#}(\alpha)$, $\alpha\in\Gamma(TN^{\perp})$, is a section of TN only in the case where $\Lambda^{\#}(\alpha)=0$.

Obviously, (ii) is equivalent to $TN\cap\Lambda^{\#}(TN^{\perp})=\{0\}$. Thus, if (54) holds,

$$D=\Lambda^{\#}(TN^{\perp})\times\{0\} \;\text{ and }\; D^{\perp}=(\Lambda^{\#}(TN^{\perp}))^{\perp}\times\mathbb{R}.$$

Taking into account (54), by a simple calculation we show that $D=\Lambda^{\#}(TN^{\perp})\times\{0\}$ is a Lie subalgebroid of $(TM\times\mathbb{R},[\,,\,],\pi)$ if and only if Λ belongs to the ideal generated by the space of smooth sections of TN. Also, since $\Lambda^{\#}((\Lambda^{\#}(TN^{\perp}))^{\perp})\subseteq TN$ and $E|_N\in\Gamma(TN)$, it is easy to prove that $D^{\perp}=(\Lambda^{\#}(TN^{\perp}))^{\perp}\times\mathbb{R}$ is a Lie subalgebroid of $(T^*M\times\mathbb{R},[\,,\,]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#})$.

Consequently, under the assumptions that (54) holds and that Λ belongs to the ideal generated by the space of smooth sections of TN, we have that the requirements of Theorem 6.2 as the reducibility condition (51) are verified, therefore (Λ,E) induces a Jacobi structure on N. We note that conditions (54) are exactly the ones given in [14].

2. Reduction of Jacobi manifolds with symmetry : Let (M,Λ,E) be a Jacobi manifold, G a connected Lie group acting on M by a Jacobi action, $\mathcal{G}$ the Lie algebra of G, $\mathcal{G}^*$ the dual space of $\mathcal{G}$ and $J:M\to\mathcal{G}^*$ an Ad^*-equivariant moment map for the considering action. Let D be the vector subbundle of $TM\times\mathbb{R}$ formed by the pairs $(X_M,0)$, where X_M is the fundamental vector field on M associated to an element $X\in\mathcal{G}$, and $D^{\perp}$ its conormal bundle which is $D^{\perp}=\{X_M\in TM\,/\,X\in\mathcal{G}\}^{\perp}\times\mathbb{R}$. It is easy to check that D and $D^{\perp}$ are Lie subalgebroids of $(TM\times\mathbb{R},[\,,\,],\pi)$ and $(T^*M\times\mathbb{R},[\,,\,]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#})$, respectively. (For $D^{\perp}$, we take into account that the action of G on M is a Jacobi action, thus, for any fundamental vector field X_M on M, $\mathcal{L}_{X_M}\Lambda = 0$ and $\mathcal{L}_{X_M}E = 0$.) Consequently, $L = D \oplus D^\perp$ is a Dirac subbundle of $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), (0,1) + (-E, 0))$. We suppose that 0 is a weakly regular value of the moment map J. Hence, $N = J^{-1}(0)$ is a submanifold of M and $D_0 = D \cap (TN \times \mathbb{R}) = \{(X_M, 0) \,/\, X \in \mathcal{G}_0\}$, where $\mathcal{G}_0$ is the Lie algebra of the isotropy subgroup G_0 of 0. Also, we suppose that $\pi(D)$ and $\pi(D_0)$ define, respectively, a simple foliation $\mathcal{F}$ of M and a simple foliation $\mathcal{F}_0$ of N. Since, $(\Lambda, E)^\#(D^\perp) \subseteq TN \times \mathbb{R} + D$ holds on N, from the Reduction Theorem 6.2 we get that (Λ, E) induces a Jacobi structure on $N/\mathcal{F}_0$. For more details, see [31], [29] and [11].

3. An example : Let M be a five-dimensional C^∞-differentiable manifold equipped with a transitive Jacobi structure (Λ, E) and $(x_0, x_1, x_2, x_3, x_4)$ a system of local coordinates of M in which

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} - (x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3}) \wedge \frac{\partial}{\partial x_0} \quad \text{and} \quad E = \frac{\partial}{\partial x_0}. \tag{55}$$

We consider the open submanifold $M' = M \setminus \{(x_0, x_1, x_2, x_3, x_4) \in M \,/\, x_0 = x_2 = x_4 = 0\}$ of M equipped with (Λ', E'), the Jacobi structure induced by (Λ, E), that is also given by (55). Let D be the subbundle of $TM' \times \mathbb{R}$ generated by

$$D = \langle (Z, -1), (\frac{\partial}{\partial x_1}, 0), (\frac{\partial}{\partial x_3}, 0) \rangle, \quad \text{where } Z = x_0 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_4}.$$

Then, $D^\perp = \langle (\delta x_0, x_0), (\delta x_2, x_2), (\delta x_4, x_4) \rangle$ and, by a simple calculation, we confirm that D and $D^\perp$ are Lie subalgebroids of $(TM' \times \mathbb{R}, [\,,\,], \pi)$ and $(T^*M' \times \mathbb{R}, [\,,\,]_{(\Lambda', E')}, \pi \circ (\Lambda', E')^\#)$, respectively. Hence, $L = D \oplus D^\perp$ is a reducible Dirac structure of $((TM' \times \mathbb{R}) \oplus (T^*M' \times \mathbb{R}), (0,1) + (-E', 0))$, i.e. $\pi(D) = \langle Z, \partial/\partial x_1, \partial/\partial x_3 \rangle$ defines a simple foliation $\mathcal{F}$ of M'. Let N be the submanifold of M' defined by the equation $x_0 = c$, where c is a nonzero constant. We have $D_0 = D \cap (TN \times \mathbb{R}) = \langle (\partial/\partial x_1, 0), (\partial/\partial x_3, 0) \rangle$, which defines a simple foliation $\mathcal{F}_0$ of N. Since,

$$\begin{aligned} \text{-}\ (\Lambda', E')^\#(\delta x_0, x_0) &= (x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3} + x_0 \frac{\partial}{\partial x_0}, -1) = \\ &= (x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_4}, 0) + (Z, -1) \in \Gamma(TN \times \mathbb{R} + D), \end{aligned}$$

- $(\Lambda', E')^{\#}(\delta x_2, x_2) = (-\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_0}, 0) =$

$$= (-\frac{\partial}{\partial x_1} - \frac{x_2^2}{x_0}\frac{\partial}{\partial x_2} - \frac{x_2 x_4}{x_0}\frac{\partial}{\partial x_4}, \frac{x_2}{x_0}) + \frac{x_2}{x_0}(Z, -1) \in \Gamma(TN \times \mathbb{R} + D)$$

and

- $(\Lambda', E')^{\#}(\delta x_4, x_4) = (-\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_0}, 0) =$

$$= (-\frac{\partial}{\partial x_3} - \frac{x_2 x_4}{x_0}\frac{\partial}{\partial x_2} - \frac{x_4^2}{x_0}\frac{\partial}{\partial x_4}, \frac{x_4}{x_0}) + \frac{x_4}{x_0}(Z, -1) \in \Gamma(TN \times \mathbb{R} + D),$$

we obtain that $(\Lambda', E')^{\#}(D^{\perp}) \subseteq TN \times \mathbb{R} + D$ holds on N (where $x_0 = c \neq 0$). Therefore, by Reduction Theorem 6.2, we conclude that there exists a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on $N/\mathcal{F}_0$ such that

$$p_*(\Lambda', E') = \varphi_*(\Lambda^{\bar{\psi}_0}_{N/\mathcal{F}_0}, E^{\bar{\psi}_0}_{N/\mathcal{F}_0}),$$

where p, φ and $\bar{\psi}_0$ as in Theorem 6.2. Because $p_*\Lambda' = 0$, $\Lambda_{N/\mathcal{F}_0} = 0$, thus the induced Jacobi structure on $N/\mathcal{F}_0$ is the trivial one defined by a differentiable non null vector field. To verify the last statement, it is sufficient to remark that, in this example,

(1) $\dim S_0 = \dim N/\mathcal{F}_0 = \dim M/\mathcal{F} = \dim S = 2$ and,
(2) for the L-admissible functions $x_2 - x_0, x_4 - x_0 \in C^\infty_L(M', \mathbb{R}) \cong C^\infty(S, \mathbb{R})$,

$$\{x_2 - x_0, x_4 - x_0\}_S \overset{(46)}{=} \{x_2 - x_0, x_4 - x_0\}_L \overset{(41)}{=} \{x_2 - x_0, x_4 - x_0\}_{(\Lambda', E')} = x_4 - x_2 \neq 0,$$

thus the induced Jacobi structure on $N/\mathcal{F}_0$ is not null. (For the notation, see the proof of Theorem 6.2.)

Remark 7.1. This is an example where we may see that the reducibility condition $(\Lambda, E)^{\#}(D^{\perp}) \subseteq TN \times \mathbb{R} + D$ of Theorem 6.2 is less strict than that of Reduction Theorem proved in [30] and [29], which requires $E|_N \in \Gamma(TN)$. In this example, $E|_N = \partial/\partial x_0$ and it is transverse to N.

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