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SUPRACONVERGENCE OF ELLIPTIC FINITE DIFFERENCE SCHEMES: GENERAL BOUNDARY CONDITIONS AND LOW REGULARITY

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ABSTRACT: In this paper we study the convergence properties of a finite difference discretization of a second order elliptic equation with mixed derivatives and variable coefficient in polygonal domains subject to general boundary conditions. We prove that the finite difference scheme on nonuniform grids exhibit the phenomenon of supraconvergence, more precisely, for $s \in [1,2]$ order $O(h^s)$-convergence of the finite difference solution and its gradient if the exact solution is in the Sobolev space $H^{s+1}(\Omega)$.

KEYWORDS: nonuniform grids, finite difference scheme, stability, supraconvergence, superconvergence of gradient.
AMS SUBJECT CLASSIFICATION (2000): 65M06, 65M20, 65M15.

1. Introduction

We consider the discretization of the differential equation

$$Au := -(au_x)_x-(bu_x)_y-(bu_y)_x-(cu_y)_y+(du)_x+(eu)_y+fu = g \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \tag{1}$$

subject to Dirichlet boundary condition

$$u = \psi \quad on \quad \partial\Omega \tag{2}$$

or third kind boundary conditions

$$Bu := (au_x + bu_y)\eta_x + (bu_x + cu_y)\eta_y \; + \; \alpha u = \psi \text{ on } \quad \partial\Omega, \tag{3}$$

by using finite difference operators defined on a general nonuniform rectangular grid $\overline{\Omega}_H$ satisfying certain compatibility conditions with the domain Ω.

Our aim is to study the behaviour of the scheme for a sequence of grids $\overline{\Omega}_H$, $H \in \Lambda$, with maximal mesh -size H_{max} converging to zero without any restriction on the non uniformity of $\overline{\Omega}_H$. In this case the scheme is first order consistent but our purpose is to show nevertheless that the finite difference

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approximations and their gradients are more accurate. This property of the FDM is usually called supraconvergence and was considered without be exhaustive in [5] - [12], [14], [19], [22]-[24].

Corresponding results have been obtained by the authors [7] for general second order elliptic equations in polygonal domains subject to Dirichlet boundary conditions and assuming that $u \in C^4(\overline{\Omega})$. Attending to this fact, the purpose of this paper to extend these results to boundary conditions of the third kind but assuming optimal smoothness assumptions for u. The one dimensional version of the results that we present in this paper were established in [8]. We note that the Laplacian in a square with Neumann boundary conditions has been previously considered by Marletta [24] using a different approach.

A main step in the proof of the supraconvergence result is to establish a relation to a linear finite element method combined with a special kind of quadrature. The rectangular grid $\overline{\Omega}_H$ has associated a triangulation $\mathcal{T}_H$ of the domain and the finite difference method that we study can be seen as an equivalent fully discrete Galerkin scheme. As a consequences we also show that in the context of FEM's the second convergence order of the gradient of the (fully discrete) FE approximation, a fact that wasn't nonstandard and which was firstly observed by the authors in [8]. In this way the results of this paper can be viewed as introducing a superconvergent finite element method. This property is usually known as a supercloseness of the gradient (see [30]) or superconvergence (see [1], [3], [21], [33]).

Attending that the superconvergence of the fully discrete FE approximation is obtained for general nonuniform rectangular grids, the triangulation can be nonuniform - the interior angles can go to zero. We mention that the standard linear piecewise linear FEM defined with a elliptic sesquilinear form (strongly coercive sesquilinear form) and quasi uniform triangulations enable us to compute a second order accurate approximations but with respect to L_2-norm.

The paper is organized as follows. In Section 2 we present the variational formulation of the boundary value problem problem (1), (3) and the fully discrete nonstandard piecewise linear FEM. In Section 3 the FD scheme is introduced. One main ingredient on the convergence analysis is the stability of the FDM. Such stability is established in Section 4 as a consequence of the stability of the fully discrete sesquilinear form associated with the fully discrete nonstandard piecewise linear FEM. In Section 5 is established an estimate for the truncation error. As a corollary of the estimate for the truncation error and of the stability inequality we establish the main result of this paper - Proposition 3- which stands that the H^1-norm of the error is an $O(H^s_{max})$ provided that $u \in H^{s+1}(\Omega)$, $s \in [1, 2]$.

Based on the results of this paper a joint work with R. D. Grigorieff is under preparation.

2. Fully discrete Galerkin approximation

In this section we describe our discretization. It is obtained as a non-standard linear finite element approximation in combination with certain quadrature rules which leads to a fully discrete method. The choice of our discretization has two consequences. On the one hand the approximation to the gradient shows superconvergence that it is second order accurate if the exact solution lies in $H^3(\Omega)$. On the other hand the method is equivalent to a familiar finite difference approximation for (1).

We start with the common variational formulation of (1). Let $\Omega \subset \mathbb{R}^2$ be a bounded regular polygonal domain, i.e. the boundary $\partial\Omega$ of Ω is the union of straight line segments which form no cuts. Let $g \in L^2(\Omega)$ and $\psi \in H^{1/2}(\partial\Omega)$ be given. By $(\cdot,\cdot)_0$ and $< \cdot,\cdot >$ we denote the standard inner product on $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. We also use $\|\cdot\|_1$ for the usual norm in the Sobolev space $H^1(\Omega)$. The variational formulation of our problem is:

$$\text{find } u \in H^1(\Omega) \text{ such that}$$
$$a(u,v) = (g,v)_0 + < \psi, v > \quad \forall\, v \in H^1(\Omega), \tag{4}$$

where $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$ are the usual Sobolev spaces and $a(\cdot\,,\cdot)$ is the sesquilinear form defined by

$$\begin{aligned} a(v,w) &= (av_x, w_x)_0 + (bv_x, w_y)_0 + (bv_y, w_x)_0 + (cv_y, w_y)_0 + (-dv, w_x)_0 \\ &+ (-ev, w_y)_0 + (fv, w)_0 + < dv\eta_x + ev\eta_y + \alpha v, w >, \end{aligned} \tag{5}$$

for $v, w \in H^1(\Omega)$, where (η_x, η_y) denotes the outer normal on $\partial\Omega$.

The coefficients of (1) in the given problem are assumed to be smooth enough, e.g. $a, b, c \in W^{3,\infty}(\Omega)$, $e, d, f \in W^{2,\infty}(\Omega)$ and $\alpha \in W^{2,\infty}(\partial\Omega)$ is sufficient. (Note that the use of the space $W^{2,\infty}(\partial\Omega)$ for a Lipschitz boundary requires some extra explanation: by $\alpha \in W^{2,\infty}(\partial\Omega)$ we mean that $\alpha \in W^{2,\infty}(\Gamma)$ for each straight section Γ.) Schemes for less regular coefficients are also known [15, 16, 18, 20, 28] which are based on earlier work by Samarskij [27] .

We also impose the general assumption that the homogeneous problem (4), i.e. with g and ψ taken to be equal to zero, has only the solution $u = 0$.

The discretization of (5) is obtained in the following way. We first introduce a nonequidistant rectangular grid in $\bar{\Omega}$. Let $h = (h_j)_{\mathbb{Z}}$ and $k = (k_\ell)_{\mathbb{Z}}$ be two sequences of mesh-sizes, i.e. of positive numbers. We define the grid

$$\mathbb{R}_1 = \{x_j \in \mathbb{R} : \, x_{j+1} = x_j + h_j, \, j \in \mathbb{Z}\}$$

with $x_0 \in \mathbb{R}$ given and a corresponding grid $\mathbb{R}_2$ with the meshsize vector k in place of h and y_0 in place of x_0. Let $\mathbb{R}_H$ be the two-dimensional rectangular grid

$$\mathbb{R}_H = \mathbb{R}_1 \times \mathbb{R}_2 \subset \mathbb{R}^2$$

and define

$$\Omega_H := \Omega \cap \mathbb{R}_H, \qquad \partial\Omega_H := \partial\Omega \cap \mathbb{R}_H, \qquad \bar{\Omega}_H = \bar{\Omega} \cap \mathbb{R}_H.$$

The grid $\bar{\Omega}_H$ is assumed to satisfy the following geometric condition with respect to the region Ω:

(Geom) The intersection of $\partial\Omega$ with the rectangle $(x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2})$ formed by midpoints of the grid $\mathbb{R}_H$ is, for all j, ℓ either is empty or it is a diagonal of the rectangle.

By W_H we denote the space of grid functions on $\bar{\Omega}_H$. Let $\square_{j\ell} := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega$ and $\omega_{j\ell} := |\square_{j\ell}|$. Then

$$(v_H, w_H)_H := \sum_{(x_j, y_\ell) \in \bar{\Omega}_H} \omega_{j\ell} \, v_{j,\ell} \, \bar{w}_{j,\ell} \quad \text{for} \quad v_H, w_H \in W_H \tag{6}$$

defines an inner product on W_H. Similarly,

$$< \varphi_H, \chi_H >_H := \sum_{(x_j, y_\ell) \in \partial\Omega_H} \sigma_{j\ell} \, \varphi_{j,\ell} \bar{\chi}_{j,\ell} \tag{7}$$

is an inner product for grid functions φ_H, χ_H on $\partial\Omega_H$, where $\sigma_{j\ell} := |\Gamma_{j\ell}|$, $\Gamma_{j\ell} := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \partial\Omega$. The discrete problem has the form:

$$\begin{aligned} &\text{find } u_H \in W_H \text{ such that} \\ &a_H(u_H, v_H) = (g_H, v_H)_H + < \psi_H, v_H >_H \quad \forall v_H \in W_H. \end{aligned} \tag{8}$$

Here a_H is a sesquilinear form which we are now going to define. Let $\mathcal{T}_H$ be a triangulation of Ω using the set $\bar{\Omega}_H$ as vertices. By $P_H v_H$ we denote the continuous piecewise linear interpolation of v_H with respect to $\mathcal{T}_H$. Then a_H is given as a sum

$$a_H = a + b + c + d + e + f + \gamma \tag{9}$$

of sesquilinear forms corresponding to the different terms in the continuous variational problem (5). They are all constructed in a similar way on the basis of linear triangular finite elements combined with an individual quadrature for each of the terms. Here the discretization of the mixed order derivative terms requires special attention (see below).

Let $\Delta \in \mathcal{T}_H$. We define $a_{\Delta,x}$ to be the value of a in the midpoint of the side of Δ parallel to the x-axis. Then let

$$a(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} a_{\Delta,x} \int_\Delta (P_H v_H)_x (P_H \bar{w}_H)_x \, dxdy. \tag{10}$$

Similarly, let $c_{\Delta,y}$ be the value of c in the midpoint of the side of Δ parallel to the y-axis and

$$c(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} c_{\Delta,y} \int_\Delta (P_H v_H)_y (P_H \bar{w}_H)_y \, dxdy. \tag{11}$$

The approximation of the first order terms is achieved by

$$d(v_H, w_H) := -\sum_{\Delta\in\mathcal{T}_H} [P_H(dv_H)]_{\Delta,x} \int_\Delta (P_H \bar{w}_H)_x \, dxdy, \tag{12}$$

$$e(v_H, w_H) := -\sum_{\Delta\in\mathcal{T}_H} [P_H(ev_H)]_{\Delta,y} \int_\Delta (P_H \bar{w}_H)_y \, dxdy. \tag{13}$$

Finally,

$$f(v_H, w_H) := \sum_{(x_j,y_\ell)\in\bar{\Omega}_H} \omega_{j\ell} f(x_j, y_\ell) v_{j,\ell} \bar{w}_{j,\ell}. \tag{14}$$

The function g on the right-hand side of (1) is discretized by the grid function

$$g_H(x_j, y_\ell) = \frac{1}{\omega_{j\ell}} \int_{\square_{j\ell}} g(x,y) dxdy, \quad (x_j, y_\ell) \in \bar{\Omega}_H. \tag{15}$$

The discretization of the function ψ in the boundary condition is given by

$$\psi_H(x_j, y_\ell) = \frac{1}{\sigma_{j\ell}} \int_{\Gamma_{j\ell}} \psi(x,y) \, d\sigma, \quad \text{for} \quad (x_j, y_\ell) \in \partial\Omega_H. \tag{16}$$

In Section 5 we will also consider the possibility of taking g_H to be the pointwise restriction of g to the grid Ω_H. The boundary term in (5) is simple approximated by

$$\gamma(v_H, w_H) :=< \alpha v_H + d v_H \eta_x + e v_H \eta_y, w_H >_H \quad \text{for} \quad v_H, w_H \in W_H. \tag{17}$$

For the discretization of the mixed derivative terms we need some preparations. We consider two special triangulations of Ω which we call $\mathcal{T}_H^{(1)}$ and $\mathcal{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \dot{\cup} \mathbb{R}_H^{(2)},$$

where the sum $j+\ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even or odd, respectively. To simplify the following definition we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. With each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the triangles $\Delta_{j,\ell}^{(i)}$, $i = 1,2,3,4$, which have a right angle at (x_j, y_ℓ) and two of the four closest neighbour grid points of (x_j, y_ℓ) as further vertices. We then define for $\nu \in \{1,2\}$ the triangulations

$$\begin{aligned}
\mathcal{T}_{H,1}^{(\nu)} &:= \{\Delta_{j,\ell}^{(i)} \subset \bar{\Omega} &&, (x_j, y_\ell) \in \mathbb{R}_H^{(\nu)} &&, i \in \{1,2,3,4\}\} \\
\mathcal{T}_{H,2}^{(\nu)} &:= \{\Delta_{j,\ell}^{(i)} \subset (\bar{\Omega} \backslash \bigcup_{\Delta \in \mathcal{T}_{H,1}^{(\nu)}} \overset{\circ}{\Delta}) &&, (x_j, y_\ell) \in \mathbb{R}_H^{(\nu+1)} &&, i \in \{1,2,3,4\}\} \\
\mathcal{T}_H^{(\nu)} &:= \mathcal{T}_{H,1}^{(\nu)} \cup \mathcal{T}_{H,2}^{(\nu)} && &&, \nu = 1,2,
\end{aligned} \tag{18}$$

($\overset{\circ}{\Delta}$ denotes the interior of Δ).

Figure 1 shows an example of a triangulation.

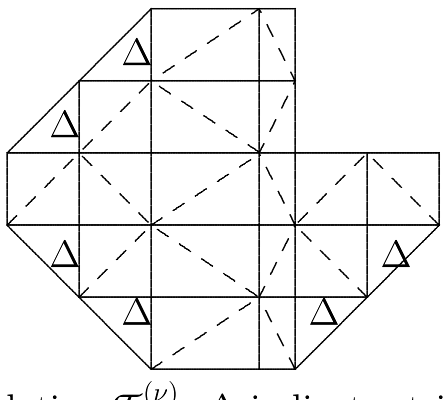

FIGURE 1. Triangulation $\mathcal{T}_H^{(\nu)}$. Δ indicates triangles of $\mathcal{T}_{H,2}^{(\nu)}$.

With respect to the triangulations $\mathcal{T}_H^{(\nu)}$, $\nu = 1, 2$, the continuous piecewise linear interpolation $P_H^{(\nu)} v_H$ of a grid function $v_H \in W_H$ is well-defined.

Let $b_\Delta := (x_\Delta, y_\Delta)$ where (x_Δ, y_Δ) is the vertex of Δ associated with the angle $\dfrac{\pi}{2}$ of Δ. Then

$$\begin{aligned} b^{(\nu)}(v_H, w_H) &:= \sum_{\Delta \in \mathcal{T}_H^{(\nu)}} b_\Delta \int_\Delta (P_H^{(\nu)} v_H)_x (P_H^{(\nu)} \bar{w}_H)_y \, dxdy \\ &+ \sum_{\Delta \in \mathcal{T}_H^{(\nu)}} b_\Delta \int_\Delta (P_H^{(\nu)} v_H)_y (P_H^{(\nu)} \bar{w}_H)_x dxdy \qquad (19) \\ &:= b_{xy}^{(\nu)}(v_H, w_H) + b_{yx}^{(\nu)}(v_H, w_H), \end{aligned}$$

and

$$b(v_H, w_H) := \frac{1}{2}(b^{(1)}(v_H, w_H) + b^{(2)}(v_H, w_H)) \qquad (20)$$

for $v_H, w_H \in W_H$.

3. Relations to finite differences

The discretized variational problem (8) is equivalent to a finite difference scheme which we will, at least in its main parts, derive in this section. Especially, in interior grid points we will obtain in (21) the standard finite difference discretization A_H of the given differential operator A on a nonuniform grid. It is this relation which shows, that our later convergence theorem is a supraconvergence result for the finite difference scheme (21).

The finite difference equations belonging to (8) are obtained by choosing grid functions v_H that vanish in all but one grid point in $\bar{\Omega}_H$. For their formulation we use the centered finite difference quotients

$$\delta_x^{(1/2)} v_{j,\ell} = \frac{v_{j+1/2,\ell} - v_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}}, \quad \delta_x^{(1/2)} v_{j+1/2,\ell} = \frac{v_{j+1,\ell} - v_{j,\ell}}{x_{j+1} - x_j},$$

$$\delta_x v_{j,\ell} = \frac{v_{j+1,\ell} - v_{j-1,\ell}}{x_{j+1} - x_{j-1}}$$

in x-direction and also correspondingly defined quantities in y-direction. First we take points in Ω_H. By collecting the terms arising from (8) it is straightforward to obtain the equations

$$\begin{aligned} A_H u_H \ := \ & -\delta_x^{(1/2)}(a\delta_x^{(1/2)}u_H) - \delta_y(b\delta_x u_H) - \delta_x(b\delta_y u_H) - \delta_y^{(1/2)}(c\delta_y^{(1/2)}u_H) \\ & +\delta_x(du_H) + \delta_y(eu_H) + fu_H = g_H \qquad \text{in} \ \ \Omega_H \, . \end{aligned} \tag{21}$$

If the operator A contains mixed derivatives then $A_H u_H$ acts on grid points outside $\bar{\Omega}_H$ near to oblique parts of the boundary. In this case the missing quantities informing $A_H u_H$ are determined by auxiliary equations. They can be given in the following way. Let $R := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ be a rectangle such that one of its diagonals forms part of $\partial\Omega$. In the approximation of $(bu_x)_y$ the the value of u_H belonging to the grid point outside of $\bar{\Omega}_H$ is determined by the equation

$$\delta_x^{(1/2)} \, u_{j+1/2,\ell} = \ \delta_x^{(1/2)} \, u_{j+1/2,\ell+1}. \tag{22}$$

For example, if b is a constant function and u is constant on $\partial\Omega$ then (22) simply means that

$$u_H(P) = -u_H(Q)$$

where P and Q are the vertices of the rectangle R lying inside and outside of Ω_H, respectively. Similarly, in the approximation of $(bu_y)_x$ the auxiliary equation is

$$\delta_y^{(1/2)} \, u_{j,\ell+1/2} = \delta_y^{(1/2)} \, u_{j+1,\ell+1/2}.$$

We now turn to the discretized form of the boundary conditions which are obtained by choosing the test function v_H to vanish in the whole of $\bar{\Omega}_H$ except in one point of $\partial\Omega_H$. Our aim is to give some examples the discrete boundary conditions in a familiar or at least intuitively understandable form. Frequently, boundary conditions containing derivatives are discretized with the aid of auxiliary grid points outside the solution domain $\bar{\Omega}$, and we proceed in the same way. We do not systematically consider all possible and most general cases because the only purpose is to provide some idea of how they look like. For example, in the following discussion there are no mixed derivatives included, i.e. we set

$$b = 0.$$

Let us first consider a boundary grid point (x_j, y_ℓ) on a bottom horizontal piece of $\partial\Omega$ which is not a vertex (see Figure 2).

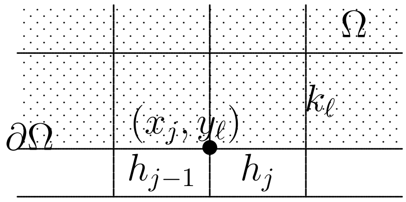

FIGURE 2

By choosing $v_{j,\ell} = 1$ and $v_{km} = 0$ elsewhere in (8) we obtain

$$-\delta_x^{(1/2)}(a\,\delta_x^{(1/2)}\,u_H)_{j,\ell}\,\omega_{j\ell} - (c\,\delta_y^{(1/2)}\,u_H)_{j,\ell+1/2}\,\frac{h_{j-1}+h_j}{2} + (\delta_x(du_H))_{j,\ell}\,\omega_{j\ell}$$
$$+(f\,u_H)_{j,\ell}\,\omega_{j\ell} + \left(\frac{1}{2}((eu_H)_{j,\ell+1} - (eu_H)_{j,\ell}) + (\alpha\,u_H)_{j,\ell}\right)\frac{h_{j-1}+h_j}{2}$$
$$= g_{j,\ell}\,\omega_{j\ell} + \psi_{j,\ell}\,\frac{h_{j-1}+h_j}{2}\,.$$

We introduce an auxiliary variable $u_{j,\ell-1}$ in the auxiliary point $(x_j, y_\ell - k_\ell)$ and the corresponding new unknown $u_{j,\ell-1}$. Then the last equation can be rewritten as

$$[(A_H u_H)_{j,\ell} - g_{j,\ell}]\,\omega_{j,\ell} \;+\; [-M_y(c\,\delta_y^{(1/2)}\,u_H)_{j,\ell} + (N_y(eu_H))_{j,\ell}$$
$$+\;(\alpha u_H)_{j,\ell} - \psi_{j,\ell}]\,\frac{h_{j-1}+h_j}{2} = 0\,,$$

where

$$(M_y\,v_H)_{j,\ell} := \frac{k_{\ell-1}}{k_{\ell-1}+k_\ell}\,v_{j,\ell+1/2} + \frac{k_\ell}{k_{\ell-1}+k_\ell}\,v_{j,\ell-1/2} \tag{23}$$

$$(N_y\,v_H)_{j,\ell} := \frac{1}{4}k_{\ell-1}k_\ell\left(\delta_y^{(1/2)}\delta_y^{(1/2)}v_H\right)_{j,\ell}. \tag{24}$$

The equation above then leads to the additional discretization

$$A_H u_H = g_H \quad \text{in} \;\; (x_j, y_\ell) \tag{25}$$

of the differential equation (1) on the boundary and the discretized boundary condition

$$-M_y\left(c\,\delta_y^{(1/2)}\,u_H\right)\eta_y + N_y(eu_H) + \alpha u_H = \psi_H \quad \text{in} \;\; (x_j, y_\ell)\,.$$

Here $\eta = (\eta_x, \eta_y) = (0, -1)$ is the outer normal in (x_j, y_ℓ).

As next case we consider a boundary grid point (x_j, y_ℓ) which lies on an oblique side of $\partial\Omega$ and is not a vertex as shown on Figure 3. We proceed in a a similar manner as before.

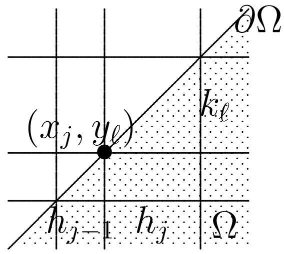

FIGURE 3

We don't give the details but write down only the result. One of the resulting equations is, as always, the discretized differential equation (25) in the boundary grid point (x_j, y_ℓ). After introducing the auxiliary variables $u_{j-1,\ell}$, $u_{j,\ell+1}$ in the auxiliary grid points $(x_j - h_{j-1}, y_\ell)$ and $(x_j, y_\ell + k_\ell)$. The discrete boundary condition reads as

$$[M_x(a\,\delta_x^{(1/2)}\,u_H) - N_x(du_H)]_{j,\ell}\,\eta_x + [M_y(c\,\delta_y^{(1/2)}\,u_H) - N_y(e\,u_H)]\,\eta_y + \alpha u_H = \psi_{j,\ell} \quad \text{in } (x_j, y_\ell)$$

The operators M_x and N_x are defined correspondingly to M_y and N_y (see (23) and (24)).

For the rest of this section we simplify the calculations further by assuming that there are no first order terms in A, i.e.

$$d = e = 0.$$

Let (x_j, y_ℓ) be a vertex obtained by the intersection of a horizontal and a vertical piece of $\partial\Omega$. We first consider the case that the associated interior angle is $\frac{\pi}{2}$ and $(x_{j+1}, y_{\ell-1})$ is an interior point (see Figure 4). In this case we define auxiliary variables $u_{j-1,\ell}$ and $u_{j,\ell+1}$ in the auxiliary points $(x_j - h_j, y_\ell)$ and $(x_j, y_\ell + k_{\ell-1})$.

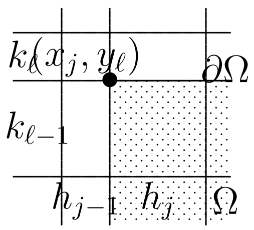

FIGURE 4

As always, one of the equations associated with the boundary grid point is (25). In addition, the discretized boundary condition is now expressed by the two separate equations

$$\begin{aligned} -M_x(a\,\delta_x^{(1/2)}\,u_H + \alpha u_H &= \psi_H, \\ M_y(c\,\delta_y^{(1/2)}\,u_H + \alpha u_H &= \psi_H \qquad \text{in } (x_j, y_\ell). \end{aligned}$$

For example, in the case $a = 1$ and $\alpha = 0$, $\psi_H = 0$ the first equation gives the well-known second order approximation $u_{j+1,\ell} = u_{j-1,\ell}$ for the homogeneous Neumann boundary condition.

Let now (x_j, y_ℓ) be a reentrance corner (i.e. the interior angle is $\frac{3\pi}{2}$) and $(x_{j+1}, y_{\ell-1})$ lies outside $\bar{\Omega}$. The linear equation associated with this vertex can be obtained from (8) in the form

$$\begin{aligned} (A_H u_H - g_H)(x_j, y_\ell)\,\omega_{j,\ell} + \frac{k_{\ell-1}}{2}\Big[-M_x(a\delta_x^{(1/2)}u_H)_{j,\ell} + (\alpha u_H)_{j,\ell} - (\psi_H)_{j,\ell}\Big] \\ + \frac{h_j}{2}\Big[-M_y(c\delta_y^{(1/2)}u_H)_{j,\ell})(\alpha u_H)_{j,\ell} - (\psi_H)_{j,\ell}\Big] = 0\ . \end{aligned} \tag{26}$$

It would be some artificial to interprete this equation in a similar way as in the cases before.

Next we consider the case of a vertex (x_j, y_ℓ) such that $(x_{j-1}, y_{\ell-1})$ and $(x_{j+1}, y_{\ell-1})$ are also on $\partial\Omega_H$ and $(x_j, y_{\ell-1})$ lies inside Ω (see Figure 5). It is convenient to introduce three auxiliary variables in points

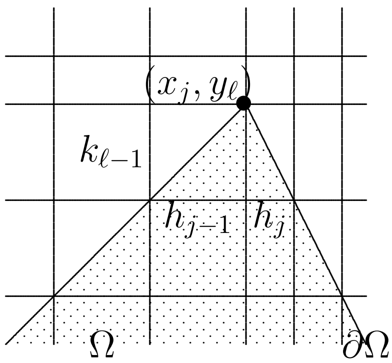

FIGURE 5

$(x_j, y_\ell+k_\ell)$, (x_j-h_{j-1}, y_ℓ) and (x_j+h_j, y_ℓ). The discrete boundary conditions can then be given in the form

$$(c\,\delta_y^{(1/2)}\,u_H)_{j,\ell+1/2} = (c\,\delta_y^{(1/2)}\,u_H)_{j,\ell-1/2} \tag{27}$$

$$\frac{1}{2}(a\,\delta_x^{(1/2)}\,u_H)_{j+1/2,\ell}\,\eta_x^+ + (c\,\delta_y^{(1/2)}\,u_H)_{j,\ell-1/2}\,\eta_y^+ + (\alpha u_H)_{j,\ell} = \psi_{j,\ell} \tag{28}$$

$$\frac{1}{2}(a\,\delta_x^{(1/2)}\,u_H)_{j-1/2,\ell}\,\eta_x^- + (c\,\delta_y^{(1/2)}\,u_H)_{j,\ell-1/2}\,\eta_y^- + (\alpha u_H)_{j,\ell} = \psi_{j,\ell}, \tag{29}$$

where (η_x^+, η_y^+) and (η_x^-, η_y^-) denote the outer normal vectors near (x_j, y_ℓ) with positive or negative x-component, respectively.

The calculations in this section underline the widely accepted superiority of the finite element over the finite difference formulation in the presence of general boundary conditions, even if the boundary is not curved. Of course in the special case of the Laplace operator in a rectangle as domain Ω it is standard knowledge how to set up the finite difference discretization of the boundary conditions for a second order convergent scheme. The approximation of the differential operator itself has the expected finite difference form which is expressed in the following.

Proposition 1. *Let the sesquilinear form $a_H(\cdot\,,\cdot)$ and the operator A_H be defined by (9) and (21), respectively. Then the equation*

$$a_H(v_H, w_H) = (A_H v_H, w_H)_H$$

holds for all v_H and $w_H \in W_H$ such that $w_H = 0$ on $\partial\Omega_H$. ■

4. Inverse stability

We now consider a sequence of grids $\mathbb{R}_H$ such that the maximal meshsize $H_{max} := max\{h_j,\, k_\ell,\, j, \ell \in \mathbb{Z}\}$ tends to zero. We use the symbol "Λ" for the sequence of mesh-size vectors and write "$(H \in \Lambda)$" for the convergence with respect to H running through this sequence.

One main ingredient for the convergence analysis is the following inverse stability result.

Theorem 1. *Let the grids $\bar{\Omega}_H$, $H \in \Lambda$, satisfy condition (Geom). Assume that the homogeneous variational problem (4), i.e. with $g = 0$, $\psi = 0$, has only the solution $u = 0$. For each $H \in \Lambda$ let $\mathcal{T}_H$ be a triangulation of Ω. Denote by P_H the corresponding piecewise linear interpolation operator.*

Then there exists a constant C such that for $H \in \Lambda$ with H_{max} small enough

$$\|P_H v_H\|_1 \leq C \sup_{0 \neq w_H \in W_H} \frac{|a_H(v_H, w_H)|}{\|P_H w_H\|_1}, \quad v_H \in W_H. \tag{30}$$

The proof of this theorem differs only in minor details from the one of Theorem 2 in [7] and is omitted.

5. Estimating the truncation error

Our error estimates are based on the inverse stability inequality in Theorem 1 applied to the global discretisation error $R_H u - u_H$ in place of v_H, where $R_h u \in W_H$ denotes the pointwise restriction of u to the grid $\overline{\Omega}_H$. Hence, since u_H solves (4) we have to bound the truncation error

$$a_H(R_H u, v_H) - (g_H, v_H)_H - < \psi_H, v_H >_H \tag{31}$$

in terms of $\|P_H v_H\|_1$. This will be done in the rest of this section.

Our starting point is the quantity $(g_H, v_H)_H$ in (31). According to the definition of g_H in (15) we have

$$(g_H, v_H)_H = \sum_{(x_j, y_\ell) \in \bar{\Omega}_H} \int_{\square_{j,\ell} \cap \Omega} (Au)(x,y) dx dy \, \bar{v}_{j,\ell}. \tag{32}$$

In order to simplify the presentation of the results we introduce in what follows some notations.

Let $\mathcal{T}_H^{obl} \subset \mathcal{T}_H$ denote the subset of triangles that have one side in common with the oblique part of $\partial\Omega$. $\mathcal{T}_H^{obl}$ is avoid for a domain Ω which is the union of rectangles.

If $\Delta \in \mathcal{T}_H^{obl}$ and (x_Δ, y_Δ) is the vertex associated with the angle $\frac{\pi}{2}$, then by I_{Δ_x} and I_{Δ_y} we denote the catheti of Δ and by $(x_\Delta, y_{\Delta/2})$ and $(x_{\Delta/2}, y_\Delta)$ we represent their midpoints. By $x_\square$ we denote x component of the midpoint of the neighbour rectangle of Δ which is in Ω being $y_\square$ defined analogously.

The introduced quantities are represented in Figure 6.

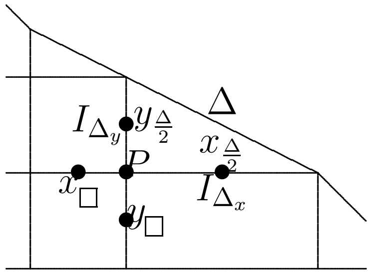

FIGURE 6

Let v_H be a grid function and $\Delta \in \mathcal{T}_H^{obl}$. By $\Delta_x v_\Delta$ we denote the backward (or the forward) difference of v_H involving the extreme points of the horizontal cathatus with respect to x component. Finally we represent by $\Delta_x v_{j,\ell}$ the difference $v_{j+1,\ell} - v_{j,\ell}$ being $\Delta_y v_\Delta$ and $\Delta_y v_{j,\ell}$ defined analogously.

We consider in what follows each contribution of Au in (32) separately. We star by the contribution of $-(au_x)_x$.

Theorem 2. *Let $a(R_H u, v_H)$ be defined by (10) and*

$$\tilde{a}(u, v_H) = \sum_{(x_j,y_\ell)\in\bar{\Omega}_H} \int_{\square_{j,\ell}\cap\Omega} (-au_x)_x dxdy\, \bar{v}_{j,\ell} \quad \textit{for } v_H \in W_H. \tag{33}$$

Then

$$a(R_H u, v_H) - \tilde{a}(u, v_H) = \sum_{(x_j,y_\ell)\in\partial\Omega_H} \int_{\Gamma_{j,\ell}} au_x \eta_x d\sigma\, \bar{v}_{j,\ell} + R_a\,,$$

where R_a satisfies:

(1) *If $u \in H^2(\Omega)$ then*

$$|R_a| \le C \left(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^2 \|u\|^2_{H^2(\Delta)} \right)^{1/2} \|P_H v_H\|_1\,,$$

(2) *If $u \in H^3(\Omega)$ then $R_a = R_{a_{stl}} + R_{a_{exe}} + R_{a_{rem}}$ with*

$$R_{a_{stl}} = \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} \eta_x \frac{k_\ell}{8} \int_{y_\ell}^{y_{\ell+1}} (au_x)_y\, d\sigma\, \Delta_y \bar{v}_{j,\ell}\,,$$

$$R_{a_{exe}} = \sum_{\Delta\in\mathcal{T}_H^{obl}} \frac{|I_{\Delta_y}|}{8} (- \int_{I_{\Delta_y}} (au_x)_y(x_\square)\, dy\, \Delta_y \bar{v}_\Delta + 2 \int_{I_{\frac{\Delta_y}{2}}} (au_x)_y(x_{\frac{\Delta}{2}})\, dy\, \Delta_x \bar{v}_\Delta)$$

and

$$|R_{a_{rem}}| \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (diam \Delta)^4 \|u\|^2_{H^3(\Delta)} \right)^{1/2} \|P_H v_H\|_1.$$

Proof: Let $I_{j,\ell+1/2} := \{(x, y_{\ell+1/2}) : x_{j-1/2} < x < x_{j+1/2}\} \cap \Omega$. By partial integration with respect to x and partial summation with respect to j we obtain

$$\begin{aligned} \tilde{a}(u, v_H) &= \sum_{(x_{j+1/2}, y_\ell) \in \bar{\Omega}} \int_{I_{j+1/2,\ell}} (au_x)(x_{j+1/2}, y) dy \, \Delta_x \bar{v}_{j,\ell} \\ &- \sum_{(x_j, y_\ell) \in \partial\Omega_H} \int_{\Gamma_{j,\ell}} au_x \eta_x d\sigma \, \bar{v}_{j,\ell}. \end{aligned} \tag{34}$$

From (10) we have

$$a(R_H u, v_H) = \sum_{(x_{j+1/2}, y_\ell) \in \bar{\Omega}} |I_{j+1/2,\ell}| \, (a\delta_x^{(1/2)} u)(x_{j+1/2}, y_\ell) \, \Delta_x \bar{v}_{j,\ell}. \tag{35}$$

In order to estimate $a(R_H u, v_H) - \tilde{a}(u, v_H)$ we consider the quantities

$$Q_1 := \sum_{(x_{j+1/2}, y_\ell) \in \bar{\Omega}} \int_{I_{j+1/2,\ell}} \left[(au_x)(x_{j+1/2}, y) - (a\delta_x^{(1/2)} u)(x_{j+1/2}, y) \right] dy \, \Delta_x \bar{v}_{j,\ell} \tag{36}$$

and

$$Q_2 := \sum_{(x_{j+1/2}, y_\ell) \in \bar{\Omega}} \int_{I_{j+1/2,\ell}} \left[(a\delta_x^{(1/2)} u)(x_{j+1/2}, y) - (a\delta_x^{(1/2)} u)(x_{j+1/2}, y_\ell) \right] dy \, \Delta_x \bar{v}_{j,\ell}. \tag{37}$$

Near oblique sections of $\partial\Omega$ the definition of $(\delta_x^{(1/2)} u)(x_{j+1/2}, y)$ for $y \in I_{j+1/2,\ell}$ requires that u is defined outside Ω. For this purpose we extend u for each involved triangle Δ into the mirror triangle outside the domain such that the norms in H^2 or H^3, respectively (depending on whether $u \in H^2(\Omega)$ or $u \in H^3(\Omega)$), with respect to the extended domain are bounded independently of j, ℓ and $H \in \Lambda$ by the corresponding norms with respect to Δ (see [26]).

Estimate for Q_1:
Introduce the variable ξ by $x = x_j + \xi h_j$ and let $w(\xi, y) = u(x_j + h_j\xi, y)$. Then

$$\begin{aligned}(au_x)(x_{j+1/2}, y) &- (a\delta_x^{(1/2)}u)(x_{j+1/2}, y)\\ &= a(x_{j+1/2}, y)\left[w_\xi(\frac{1}{2}, y) - w(1, y) + w(0, y)\right] h_j^{-1}.\end{aligned} \tag{38}$$

Let λ be the linear functional

$$\lambda(f) = f'(\frac{1}{2}) - f(1) + f(0),\ f \in W_1^2(0, 1).$$

The functional λ is bounded in $W_1^2(0, 1)$ and vanishes if f is a polynomial of degree less or equal two. Thus from the Bramble-Hilbert Lemma there exists a positive constant C such that

$$|\lambda(f)| \leq C\|f^{(s)}\|_{L_1(0,1)}, f \in W_1^s(0, 1)$$

for $s \in \{2, 3\}$.

Applying this result in (38) we obtain for $u \in H^s(\Omega)$

$$|(au_x - a\delta_x^{(1/2)}u)(x_{j+1/2}, y)| \leq C\|a\|_\infty h_j^{s-2}\|u_{x^s}(., y)\|_{L_1(x_j,x_{j+1})}. \tag{39}$$

We use the bound (39) in (36) to derive after an application of Schwarz's inequality for sums and estimating the L_1-norm over $(x_j, x_{j+1}) \times I_{j+1/2,\ell}$ by the corresponding L_2-norm

$$|Q_1| \leq C\|a\|_\infty(\sum_{(x_{j+1/2},y_\ell)\in\bar{\Omega}} h_j^{2s-2}\|u_{x^s}\|^2_{L_2((x_j,x_{j+1})\times I_{j+1/2,\ell})})^{1/2}\|P_H v_H\|_1. \tag{40}$$

The right-hand side can be further estimated to yield the more convenient bound

$$|Q_1| \leq C\|a\|_\infty(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^{2(s-1)}\|u\|^2_{H^s(\Delta)})^{1/2}\|P_H v_H\|_1,\ s \in \{2, 3\}. \tag{41}$$

Estimate for Q_2:
We assume for simplicity that Ω is the union of rectangles. The main problem in bounding Q_2 arises from the fact that on nonuniform grids we have in (37) no longer the midpoint rule. Thus we have to work a little bit more and exploit the alternating behaviour of the error in y-direction. Let

$$Q_{21} := \sum_{(x_{j+1/2},y_\ell)\in\Omega}\int_{I_{j+1/2,\ell}}(a\delta_x^{(1/2)}u)(x_{j+1/2}, y)dy\Delta_x\bar{v}_{j,\ell} - Q_{22} \tag{42}$$

and

$$Q_{22} := \frac{1}{4} \sum_{(x_{j+1/2},y_\ell)\in\Omega} \Big(k_{\ell-1}(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_{\ell-1/2}) \\ + (k_{\ell-1}+k_\ell)(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_\ell) + k_\ell(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_{\ell+1/2})\Big)\, \Delta_x \bar{v}_{j,\ell}\,. \tag{43}$$

In the first step we estimate Q_{21}. Since Q_{21} is a second order accurate quadrature formula we can derive in the same way as for Q_1 the bound

$$|Q_{21}| \le C \sum_{(x_{j+1/2},y_\ell)\in\Omega} |I_{j+1/2,\ell}|^t \|(a\delta_x^{(1/2)}u)_{y^t}(x_{j+1/2}, .)\|_{L_1(I_{j+1/2,\ell})} |\Delta_x \bar{v}_{j,\ell}|.$$

for $t \in \{1,2\}$. Since

$$(\delta_x^{(1/2)}u)(x_{j+1/2}, .) = \frac{1}{h_j}\int_{x_j}^{x_{j+1}} u_x(x,.)dx \tag{44}$$

we can further estimate Q_{21} by

$$|Q_{21}| \le C\|a\|_{t,\infty} \sum_{(x_{j+1/2},y_\ell)\in\Omega} (k_{\ell-1}+k_\ell)^t \|u\|_{W_1^{t+1}((x_j,x_{j+1})\times I_{j+1/2,\ell})} |\delta_x^{(1/2)}\bar{v}_{j,\ell}|.$$

With $s := t+1$ this leads for $u \in H^s(\Omega)$ as in the case of Q_1 to the bound

$$|Q_{21}| \le C\|a\|_{s-1,\infty}\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^{2(s-1)}\|u\|^2_{H^s(\Delta)}\Big)^{1/2}\|P_H v_H\|_1, \;\; s\in\{2,3\}. \tag{45}$$

Let $Q_{22,p}$ the part of $Q_2 - Q_{21}$ for which holds the representation

$$Q_{22,p} = \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{4}\Big[(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_{\ell+1/2}) \\ -(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_\ell)\Big]\,\Delta_x\bar{v}_{j,\ell} \\ + \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{4}\Big[(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_{\ell+1/2}) \\ -(a\delta_x^{(1/2)}u)(x_{j+1/2}, y_{\ell+1})\Big]\,\Delta_x\bar{v}_{j,\ell+1}\,. \tag{46}$$

Its easy to show that $Q_{22,p} = Q^{(1)}_{22,p} + Q^{(2)}_{22,p}$ with

$$Q^{(1)}_{22,p} = -\sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{8}[(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1}) - 2(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1/2}) \\ +(a\delta_x^{(1/2)}u)(x_{j+1/2},y_\ell)]\,(\Delta_x\bar{v}_{j,\ell+1} + \Delta_x\bar{v}_{j,\ell}) \tag{47}$$

and

$$Q^{(2)}_{22,p} = \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{8}\Big[(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1/2}) \\ -(a\delta_x^{(1/2)}u)(x_{j+1/2},y_\ell)\Big]\,(-\Delta_x\Delta_y\bar{v}_{j,\ell}) \\ + \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{8}\Big[(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1/2}) \\ -(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1})\Big]\,\Delta_x\Delta_y\bar{v}_{j,\ell}\ . \tag{48}$$

Estimate for $Q^{(1)}_{22,p}$:
Let w be defined by

$$w(\xi) = (a\delta_x^{(1/2)}u)(x_{j+1/2}, y_\ell + \xi k_\ell)$$

then

$$(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1}) - 2(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1/2}) + (a\delta_x^{(1/2)}u)(x_{j+1/2},y_\ell) \\ = w(1) - 2w(\frac{1}{2}) + w(0)\,. \tag{49}$$

The linear functional

$$\lambda(f) = f(1) - 2f(\frac{1}{2}) + f(0),\ f \in W_1^1(0,1), \tag{50}$$

is bounded and vanishes if f is a polynomial of degree zero or one. Thus from the Bramble-Hilbert Lemma exists a positive constant C such that

$$|\lambda(f)| \leq C\|f^{(t)}\|_{L_1(0,1)}, f \in W_1^t(0,1), \tag{51}$$

for $t \in \{1,2\}$. Applying this result in (49) and considering (44) we obtain for $u \in H^{t+1}(\Omega)$

$$\Big|(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1}) - 2(a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1/2}) + (a\delta_x^{(1/2)}u)(x_{j+1/2},y_\ell)\Big| \\ \leq C\|a\|_{t,\infty} h_j^{-1}k_\ell^{t-1}\|u_{y^t x}\|_{L_1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}, \tag{52}$$

for $t \in \{1,2\}$. Following the lines considered in the estimation of Q_1 we establish

$$|Q_{22,p}^{(1)}| \leq C\|a\|_{s,\infty}(\sum_{\Delta\in\mathcal{T}_H}(diam\,\Delta)^{2(s-1)}\|u\|^2_{H^s(\Delta)})^{1/2}\|P_H v_H\|_1\,,\ s\in\{2,3\}. \tag{53}$$

Estimate for $Q_{22,p}^{(2)}$:
We have

$$Q_{22,p}^{(2)} = \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega}\frac{k_\ell}{8}\Big[(a\delta_x^{(1/2)}u)(x_{j+1/2},y_\ell) - (a\delta_x^{(1/2)}u)(x_{j+1/2},y_{\ell+1})\Big] \\ \Delta_x\Delta_y\bar{v}_{j,\ell}\,, \tag{54}$$

and we easily get

$$|Q_{22,p}^{(2)}| \leq C\|a\|_{1,\infty}\sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega}k_\ell\|u_{xy}\|_{L_1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \\ \Big(|(\delta_x^{(1/2)}\bar{v}_H)(x_{j+1/2},y_\ell)| + |(\delta_x^{(1/2)}\bar{v}_H)(x_{j+1/2},y_{\ell+1})|\Big).$$

The last inequality enable us to conclude that if $u \in H^2(\Omega)$ then

$$|Q_{22,p}^{(2)}| \leq C\|a\|_{1,\infty}(\sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega}k_\ell^2\|u_{yx}\|^2_{L_2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))})^{1/2}\|P_H v_H\|_1$$

and finally

$$|Q_{22,p}^{(2)}| \leq C\|a\|_{1,\infty}(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^2\|u\|^2_{H^2(\Delta)})^{1/2}\|P_H v_H\|_1.$$

In what follows another estimate to $Q^{(2)}_{22,p}$ is obtained assuming that $u \in H^3(\Omega)$. From (54) using partial summation with respect to j we obtain

$$
\begin{aligned}
Q^{(2)}_{22,p} = & \sum_{(x_{j+1/2},y_{\ell+1/2})\in\Omega} \frac{k_\ell}{8}\Big[\int_{y_\ell}^{y_{\ell+1}} (a\delta_x^{(1/2)}u)_y(x_{j+1/2},y)\,dy \\
& \qquad - \int_{y_\ell}^{y_{\ell+1}} (a\delta_x^{(1/2)}u)_y(x_{j-1/2},y)\,dy\Big]\,\Delta_y\bar{v}_{j,\ell} \\
& + \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} -\eta_x\frac{k_\ell}{8}\int_{y_\ell}^{y_{\ell+1}} (a\delta_x^{(1/2)}u)_y(x_{j-\eta_x/2},y)\,dy\,\Delta_y\bar{v}_{j,\ell}
\end{aligned}
\tag{55}
$$

Attending that for $(a\delta_x^{(1/2)})_y(x_{j\pm1/2},y) - (au_x)_y(x_{j\pm1/2},y)$ holds an estimate analogous to (39) with $s = 2$ and $u_{x^2}(.,y)$ replaced by $u_{x^2y}(.,y)$, and that

$$
(au_x)_y(x_{j\pm1/2},y) = (au_x)_y(x_j,y) + \int_{x_j}^{x_{j\pm1/2}} (au_x)_{yx}\,dx,
$$

we conclude

$$
Q^{(2)}_{22,p} = \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} -\eta_x\frac{k_\ell}{8}\int_{y_\ell}^{y_{\ell+1}} (au_x)_y(x_j,y)\,dy\,\Delta_y\bar{v}_{j,\ell} + R
$$

where

$$
\begin{aligned}
|R| & \le C\|a\|_{2,\infty}\Big(\sum_{(x_{j+1/2},y_\ell)\in\Omega} k_\ell^4\|u_{x^2y}\|^2_{L^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|P_Hv_H\|_1 \\
& \le C\|a\|_{2,\infty}\Big(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\|u\|^2_{H^3(\Delta)}\Big)^{1/2}\|P_Hv_H\|_1
\end{aligned}
$$

which conclude the proof.

■

Remark 1. *Let us assume in Theorem 2 that u is null on the boundary $\partial\Omega$. Let $\Delta^* \in \mathcal{T}_H^{obl}$ be a triangle with vertices (x_j,y_ℓ) (associated with the angle $\frac{\pi}{2}$), $(x_j,y_{\ell+1})$ and (x_{j+1},y_ℓ). Let R_{a_{exe},Δ^*} be the term of $R_{a_{exe}}$ correspondent* *to* Δ^**. We have*

$$R_{a_{exe},\Delta^*} = \frac{|I_{\Delta_y}|}{8}(-\int_{I_{\Delta_y}} (au_x)_y(x_\square)\, dy\, \Delta_y \bar{v}_\Delta + 2\int_{I_{\frac{\Delta_y}{2}}} (au_x)_y(x_{\frac{\Delta}{2}})\, dy\, \Delta_x \bar{v}_\Delta)$$

$$= \frac{k_\ell}{8}\left(w(1) - 2w(\frac{1}{2}) + w(0)\right)\bar{v}_{j,\ell} - \frac{k_\ell}{8}\int_{x_{j-1/2}}^{x_j}\int_{y_\ell}^{y_{\ell+1}} (au_x)_{xy}\bar{v}_{j,\ell}$$

$$+\frac{k_\ell}{4}\int_{x_j}^{x_{j+1/2}}\int_{y_\ell}^{y_{\ell+1/2}} (au_x)_{xy}\bar{v}_{j,\ell},$$

where $w(\xi) = (au_x)(x_j, y_\ell+\xi k_\ell)$, $\xi \in [0,1]$*. Using the Bramble-Hilbert Lemma the following estimate can be shown*

$$k_\ell|w(1) - 2w(\frac{1}{2}) + w(0)||\bar{v}_{j,\ell}| \leq Ck_\ell^t h_j \int_{y_\ell}^{y_{\ell+1}} |(au_x)_{y^t}(x_j, y)|\, dy|\delta_x^{(1/2)}\bar{v}_{j+1/2,\ell}|$$

$$\leq Ck_\ell \int_{y_\ell}^{y_{\ell+1}}\int_{x_j-\frac{h_j}{2}}^{x_{j+1/2}} (|(au_x)_{y^t}| + h_j|(au_x)_{y^t x}|)\, dx\, dy\, |\delta_x^{(1/2)}\bar{v}_{j+1/2,\ell}|$$

for $t \in \{1,2\}$.

Considering the last estimate in the representation of R_{a_{exe},Δ^*} *we obtain*

$$|R_{a_{exe},\Delta^*}| \leq \|a\|_{2,\infty}\sum_{\Delta\in V_1(\Delta^*)} (diam\Delta)^2\|u_{x^2y}\|_{L^1(\Delta)}|(P_H v_H)_{x,\Delta}|$$

$$+\sum_{\Delta\in V_2(\Delta^*)} (\|a\|_{t,\infty}(diam\Delta)^t\|u_{xy^t}\|_{L^1(\Delta)}$$

$$+\|a\|_{t+1,\infty}(diam\Delta)^{t+1}\|u_{x^2y^t}\|_{L^1(\Delta)})|(P_H v_H)_{x,\Delta}|,$$

where $t \in \{1,2\}$ *and* $V_1(\Delta^*)$ *and* $V_2(\Delta^*)$ *represent the union of* Δ^* *with the triangles which are respectively in* $(x_{j-1}, x_j) \times (y_\ell, y_{\ell+1})$ *and in smaller set of triangles containing* $(x_j - \frac{h_j}{2}, x_j) \times (y_\ell, y_{\ell+1})$.

Attending that Δ^* *is arbitrary in* $\mathcal{T}_H^{obl}$ *we deduce*

$$\begin{aligned}|R_{a_{exe}}| &\leq C\Bigg((\sum_{\Delta\in\mathcal{T}_H^{obl,1}} (diam\Delta)^4\|u\|^2_{H^3(\Delta)})^{1/2} \\ &+ (\sum_{\Delta\in\mathcal{T}_H^{obl,2}} (diam\Delta)^{2t}\|u\|^2_{H^{t+1}(\Delta)})^{1/2} \\ &+ (\sum_{\Delta\in\mathcal{T}_H^{obl,2}} (diam\Delta)^{2(t+1)}\|u\|^2_{H^{t+2}(\Delta)})^{1/2}\ \|P_Hv_H\|_1\,,\end{aligned} \tag{56}$$

where $t\in\{1,2\}$ *and* $\mathcal{T}_H^{obl,i}=\{\Delta\in V_i(\Delta^*),\Delta^*\in\mathcal{T}_H^{obl}\}, i=1,2,$ *being* $V_i(\Delta^*)$ *defined as above (with convenient adaptations).*

As a consequence, if $u\in H_0^3(\Omega)\cap W^{2,\infty}(\cup_{\Delta\in\mathcal{T}_H^{obl,2}}\Delta)$ *and* $\mathcal{T}_H^{obl}$ *is contained in a strip of width* $O(H_{max})$ *then*

$$|R_{a_{exe}}|\leq O(H_{max}^{3/2})\|P_Hv_H\|_1.$$

We establish is what follows the theorem correspondent to Theorem 2 for the contributions of $-(cu_y)_y$.

Theorem 3. *Let* $c(R_Hu,v_H)$ *be defined by (11) and* $\tilde{c}(u,v_H)$ *be*

$$\tilde{c}(u,v_H)=\sum_{(x_j,y_\ell)\in\bar{\Omega}_H}\int_{\Box_{j,\ell}\cap\Omega}(-cu_y)_y dxdy\,\bar{v}_{j,\ell}\quad for\ v_H\in W_H. \tag{57}$$

Then

$$c(R_Hu,v_H)-\tilde{c}(u,v_H)=\sum_{(x_j,y_\ell)\in\partial\Omega_H}\int_{\Gamma_{j,\ell}}cu_y\eta_y d\sigma\,\bar{v}_{j,\ell}+R_c\,,$$

where R_c *satisfies:*

(1) *If* $u\in H^2(\Omega)$ *then*

$$|R_c|\leq C\left(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^2\|u\|^2_{H^2(\Delta)}\right)^{1/2}\|P_Hv_H\|_1\,,$$

(2) *If* $u \in H^3(\Omega)$ *then* $R_c = R_{c_{stl}} + R_{c_{exe}} + R_{c_{rem}}$ *with*

$$R_{c_{stl}} = \sum_{(x_{j+1/2},y_\ell)\in\partial\Omega} \eta_y \frac{h_j}{8}\int_{x_j}^{x_{j+1}} (cu_y)_x \, d\sigma \, \Delta_x \bar{v}_{j,\ell} \,,$$

$$R_{c_{exe}} = \sum_{\Delta\in\mathcal{T}_H^{obl}} \frac{|I_{\Delta_x}|}{8}(-\int_{I_{\Delta_x}} (cu_y)_x(y_\Box)\, dx\, \Delta_x\bar{v}_\Delta + 2\int_{\frac{I_{\Delta_x}}{2}} (cu_y)_x(y_{\frac{\Delta}{2}})\, dx\, \Delta_y \bar{v}_\Delta)$$

and

$$|R_{c_{rem}}| \leq C(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^3(\Delta)})^{1/2}\|P_H v_H\|_1.$$

■

Remark 2. *Holds a remark analogous to Remark 1.*

Let us consider now the contribution of the mixed derivatives. We establish in the next result an estimation for the error associated with the discretization of $(bu_y)_x$. Let $b_{yx}(R_H u, v_H)$ be defined by

$$b_{yx}(R_H u, v_H) := \frac{1}{2}\left(b^{(1)}_{yx}(R_H u, v_H) + b^{(2)}_{yx}(R_H u, v_H)\right), \text{ for } v_H \in W_H,$$

where $b^{(\nu)}_{yx}(.,.)$ is defined by (19).

Theorem 4. *Let* $\tilde{b}_{yx}(u, v_H)$ *be*

$$\tilde{b}_{yx}(u, v_H) = \sum_{(x_j,y_\ell)\in\bar{\Omega}_H} \int_{\Box_{j,\ell}\cap\Omega} (-bu_y)_x dxdy \, \bar{v}_{j,\ell} \quad for\ v_H \in W_H. \tag{58}$$

Then

$$b_{yx}(R_H u, v_H) - \tilde{b}_{xy}(u, v_H) = \sum_{(x_j,y_\ell)\in\partial\Omega_H} \int_{\Gamma_{j,\ell}} bu_y\eta_x d\sigma \, \bar{v}_{j,\ell} + R_b \,,$$

where R_b *satisfies:*

(1) *If* $u \in H^2(\Omega)$ *then*

$$|R_b| \leq C(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^2 \|u\|^2_{H^2(\Delta)})^{1/2}\|P_H v_H\|_1 \,,$$

(2) *If* $u \in H^3(\Omega)$ *then* $R_b = R_{b_{stl}} + R_{b_{exe}} + R_{b_{rem}}$ *with*

$$R_{b_{stl}} = \sum_{(x_j, y_\ell+1/2)\in\partial\Omega} \eta_x\Big(\frac{1}{4}(b_{j,\ell+1} - b_{j,\ell})\Delta_y u(x_j, y_\ell) + \frac{1}{2}\int_{y_\ell}^{y_{\ell+1/2}} (bu_y)(x_j)\, dy - \frac{1}{2}\int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_y)(x_j)\, dy\Big)\Delta_y \bar{v}_{j,\ell}\,,$$

$$R_{b_{exe}} = \sum_{\Delta\in\mathcal{T}_H^{obl}} \Big(\frac{1}{2}b(x_{\frac{\Delta}{2}}, y_\Delta)\Delta_y u_\Delta - \frac{1}{2}\int_{I_{\frac{\Delta_y}{2}}} (bu_y)(x_{\frac{\Delta}{2}}, y)\, dy\Big)\Delta_x \bar{v}_\Delta + \left(\frac{1}{4}\Delta_y b(x_\Box, y_\Delta)\Delta_y u(x_\Box, y_\Delta) + \frac{1}{2}\int_{I_{\frac{\Delta_y}{2}}} (bu_y)(x_\Box, y)\, dy - \int_{I_{\Delta_y}} (bu_y)(x_\Box, y)\, dy\right)\Delta_y \bar{v}_\Delta$$

and

$$|R_{b_{rem}}| \le C\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^3(\Delta)}\Big)^{1/2}\|P_H v_H\|_1.$$

Proof: For simplicity we assume that Ω is the union of rectangles. By partial integration with respect to x and partial summation with respect to j we obtain

$$\begin{aligned}\tilde{b}_{yx}(u, v_H) &= \sum_{(x_{j+1/2}, y_\ell)\in\overline{\Omega}} \int_{I_{j+1/2,\ell}} (bu_y)(x_{j+1/2}, y)\, dy\, \Delta_x \bar{v}_{j,\ell} \\ &- \sum_{(x_j, y_\ell)\in\partial\Omega} \int_{\Gamma_{j\ell}} bu_y\eta_x\, d\sigma\, \bar{v}_{j,\ell}\,.\end{aligned} \tag{59}$$

Attending to the definition of $b_{yx}(.,.)$ and (19) we have

$$b_{yx}(R_H u, v_H) = \sum_{(x_{j+1/2}, y_\ell)\in\overline{\Omega}_H} \frac{k_\ell}{2}\frac{b_{j,\ell}\delta_y^{(1/2)}u_{j,\ell+1/2} + b_{j+1,\ell}\delta_y^{(1/2)}u_{j+1,\ell+1/2}}{2}\Delta_x \bar{v}_{j,\ell} + \frac{k_\ell}{2}\frac{b_{j,\ell+1}\delta_y^{(1/2)}u_{j,\ell+1/2} + b_{j+1,\ell+1}\delta_y^{(1/2)}u_{j+1,\ell+1/2}}{2}\Delta_x \bar{v}_{j,\ell+1}. \tag{60}$$

Let λ be the functional (50) and

$$f(\xi) = b(x_j + \xi h_j, y_\ell)\delta_y^{(1/2)} u(x_j + \xi h_j, y_{\ell+1/2}),\ \xi \in [0,1].$$

Using (51) it easy to show that

$$\begin{aligned} &\left|\frac{b_{j,\ell}\delta_y^{(1/2)} u_{j,\ell+1/2} + b_{j+1,\ell}\delta_y^{(1/2)} u_{j+1,\ell+1/2}}{2} - b_{j+1/2,\ell}\delta_y^{(1/2)} u_{j+1/2,\ell+1/2}\right| \\ &\qquad \le C\|b\|_{t,\infty} k_\ell^{-1} h_j^{t-1} \|u_{yx^t}\|_{L_1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \end{aligned} \tag{61}$$

and

$$\begin{aligned} &\left|\frac{b_{j,\ell+1}\delta_y^{(1/2)} u_{j,\ell+1/2} + b_{j+1,\ell+1}\delta_y^{(1/2)} u_{j+1,\ell+1/2}}{2} - b_{j+1/2,\ell+1}\delta_y^{(1/2)} u_{j+1/2,\ell+1/2}\right| \\ &\qquad \le C\|b\|_{t,\infty} k_\ell^{-1} h_j^{t-1} \|u_{yx^t}\|_{L_1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\,, \end{aligned} \tag{62}$$

for $t \in \{1,2\}$.

Considering (61) and (62) in (60) we obtain

$$\begin{aligned} b_{yx}(R_H u, v_H) = \sum_{(x_{j+1/2},y_\ell)\in\overline{\Omega}_H} & \frac{k_\ell}{2}\delta_x^{(1/2)} u_{j+1/2,\ell+1/2} \\ & \big(b_{j+1/2,\ell}\Delta_x \bar{v}_{j,\ell} + b_{j+1/2,\ell+1}\Delta_x \bar{v}_{j,\ell+1}\big) + R \end{aligned} \tag{63}$$

with

$$|R| \le C\|b\|_{t,\infty} \Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^{2t} \|u\|^2_{H^{t+1}(\Delta)}\Big)^{1/2} \|P_H v_H\|_1$$

for $t \in \{1,2\}$, or in a more convenient form

$$|R| \le C\|b\|_{s-1,\infty} \Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^{2(s-1)} \|u\|^2_{H^{s}(\Delta)}\Big)^{1/2} \|P_H v_H\|_1$$

for $s \in \{2,3\}$.

For $\tilde{b}_{yx}(u, v_H)$ holds the following representation

$$\begin{aligned} \tilde{b}_{yx}(u, v_H) = \sum_{(x_{j+1/2},y_\ell)\in\overline{\Omega}_H} & \int_{y_\ell}^{y_{\ell+1/2}} (bu_y)(x_{j+1/2}, y)\, dy \Delta_x \bar{v}_{j,\ell} \\ & + \int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_y)(x_{j+1/2}, y)\, dy \Delta_x \bar{v}_{j,\ell+1}\,, \end{aligned} \tag{64}$$

and so we deduce an estimate for $b_{yx}(R_H u, v_H) - \tilde{b}_{yx}(u, v_H)$ estimating separately the following terms

$$Q_{b,1} = \sum_{(x_{j+1/2}, y_\ell)\in\overline{\Omega}_H} \Big(k_\ell \frac{b_{j+1/2,\ell} + b_{j+1/2,\ell+1}}{2} \delta_y^{(1/2)} u_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1}} (bu_y)(x_{j+1/2}, y)\, dy \Big) \Delta_x \frac{\bar{v}_{j,\ell} + \bar{v}_{j,\ell+1}}{2},$$

$$Q_{b,2} = \frac{1}{2} \sum_{(x_{j+1/2}, y_\ell)\in\overline{\Omega}_H} \Big(\frac{k_\ell}{2} b_{j+1/2,\ell} \delta_y^{(1/2)} u_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1/2}} (bu_y)(x_{j+1/2}, y)\, dy \Big) (-\Delta_y \Delta_x \bar{v}_{j,\ell})$$

and

$$Q_{b,3} = \frac{1}{2} \sum_{(x_{j+1/2}, y_\ell)\in\overline{\Omega}_H} \Big(\frac{k_\ell}{2} b_{j+1/2,\ell+1} \delta_y^{(1/2)} u_{j+1/2,\ell+1/2} - \int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_y)(x_{j+1/2}, y)\, dy \Big) \Delta_y \Delta_x \bar{v}_{j,\ell}\,.$$

Estimate for $Q_{b,1}$:

First of all we note that

$$\begin{aligned} &k_\ell \frac{b_{j+1/2,\ell} + b_{j+1/2,\ell+1}}{2} \delta_y^{(1/2)} u_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1}} (bu_y)(x_{j+1/2}, y)\, dy \\ &= \int_{y_\ell}^{y_{\ell+1}} \big(b(x_{j+1/2}, y_{\ell+1/2}) - b(x_{j+1/2}, y) \big) u_y(x_{j+1/2}, y)\, dy + R \end{aligned} \tag{65}$$

where

$$|R| \le C\|b\|_{2,\infty} k_\ell^2 h_j^{-1} \int_{x_j}^{x_{j+1}} \int_{y_\ell}^{y_{\ell+1}} (|u_y| + h_j |u_{yx}|)\, dy\, dx\,.$$

Moreover

$$\begin{aligned} &\Big| \int_{y_\ell}^{y_{\ell+1}} (b(x_{j+1/2}, y_{\ell+1/2}) - b(x_{j+1/2}, y)) u_y(x_{j+1/2}, y)\, dy \Big| \\ &\qquad \le C\|b\|_{2,\infty} k_\ell^2 h_j^{-1} \int_{x_j}^{x_{j+1}} \int_{y_\ell}^{y_{\ell+1}} (|u_y| + |u_{y^2}| + h_j(|u_{yx}| + |u_{y^2x}|))\, dy\, dx\,. \end{aligned} \tag{66}$$

Considering (65) and (66) in the expression of $Q_{b,1}$ we obtain

$$|Q_{b,1}| \leq C\|b\|_{2,\infty}(\sum_{(x_{j+1/2},y_\ell)\in\overline{\Omega}_H} k_\ell^4 \int_{x_j}^{x_{j+1}}\int_{y_\ell}^{y_{\ell+1}} (|u_y|+|u_{y^2}| \\ h_j(|u_{yx}+|u_{y^2x}|))\,dy\,dx\,)^{1/2}\|P_Hv_H\|_1,$$

and so

$$|Q_{b,1}| \leq C\|b\|_{2,\infty}(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\|u\|^2_{H^3(\Delta)})^{1/2}\|P_Hv_H\|_1\,.$$

Estimate for $Q_{b,2}$:

Using in $Q_{b,2}$ partial summation with respect to j we obtain the following representation

$$Q_{b,2} = Q_{b,2}^{(1)} + Q_{b,2}^{(2)} \tag{67}$$

with

$$2Q_{b,2}^{(1)} = \sum_{(x_j,y_{\ell+1/2})\in\Omega} k_\ell \int_{x_{j-1/2}}^{x_{j+1/2}} \Bigg(b(x,y_\ell)\frac{1}{2}\left(u(x,y_{\ell+1})-2u(x,y_{\ell+1/2})+u(x,y_\ell)\right)_x \\ + \int_{y_\ell}^{y_{\ell+1/2}}\left((b(x,y_\ell)-b(x,y))u_y\right)_x\,dy\Bigg)\,dx\delta_y^{(1/2)}\bar{v}_{j,\ell+1/2}$$

and

$$2Q_{b,2}^{(2)} = -\sum_{(x_j,y_\ell+1/2)\in\partial\Omega}\eta_x\Bigg(\frac{k_\ell}{2}b_{j-\eta_x/2,\ell}\delta_y^{(1/2)}u_{j-\eta_x/2,\ell+1/2} \\ -\int_{y_\ell}^{y_{\ell+1/2}}(bu_y)(x_{j-\eta_x/2},y)\,dy\Bigg)\Delta_y\bar{v}_{j,\ell}\,.$$

By the Bramble-Hilbert Lemma holds the following inequality

$$|\left(b(x,y_\ell)\frac{1}{2}\left(u(x,y_{\ell+1})-2u(x,y_{\ell+1/2})+u(x,y_\ell)\right)\right)_x| \\ \leq C\|b\|_{1,\infty}k_\ell\int_{y_\ell}^{y_{\ell+1}}(|u_y|+|u_{xy^2}|)\,dy\,. \tag{68}$$

We also have

$$|\int_{y_\ell}^{y_{\ell+1/2}}\left((b(x,y_\ell)-b(x,y))u_y\right)_x\,dy| \leq \|b\|_{1,\infty}\int_{y_\ell}^{y_{\ell+1/2}}(|u_y|+|u_{yx}|)\,dy\,. \tag{69}$$

Considering (68) and (69) in $Q^{(1)}_{2,b}$ we easily obtain

$$|Q^{(1)}_{b,2}| \leq C\|b\|_{1,\infty}(\sum_{(x_j,y_{\ell+1/2})\in\Omega} k_\ell^4 \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{y_\ell}^{y_{\ell+1}} (|u_y|^2 + |u_{yx}|^2 + |u_{xy^2}|^2)\, dy\ dx\,)^{1/2} \|P_H v_H\|_1\,.$$

The last estimate enable us to conclude that

$$|Q^{(1)}_{b,2}| \leq C\|b\|_{1,\infty}(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^{2s} \|u\|^2_{H^{s+1}(\Delta)})^{1/2} \|P_H v_H\|_1\,, \qquad (70)$$

for $s = 2$.

It is easy to show that the estimate (70) also holds for $s = 1$.

Estimate for $Q_{b,3}$:

For $Q_{b,3}$ holds the decomposition $Q_{b,3} = Q^{(1)}_{b,3} + Q^{(2)}_{b,3}$, analogous to the decomposition (67) established for $Q_{b,2}$, with $|Q^{(1)}_{b,3}|$ bounded by the the upper bound of (70), and $Q^{(2)}_{b,3}$ given by

$$Q^{(2)}_{b,3} = -\frac{1}{2} \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} \eta_x \left(\frac{k_\ell}{2} b_{j-\eta_x/2,\ell+1} \delta_y^{(1/2)} u_{j-\eta_x/2,\ell+1/2} - \int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_y)(x_{j-\eta_x/2}, y)\, dy \right) \Delta_y \bar{v}_{j,\ell}\,.$$

Estimate for $Q^{(2)}_{b,2} + Q^{(2)}_{b,3}$:

We have

$$Q^{(2)}_{b,2} + Q^{(2)}_{b,3} = \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} \eta_x B(x_{j-\eta_x/2}, y_\ell) \Delta_y \bar{v}_{j,\ell},$$

with

$$\begin{aligned} B(x_{j-\eta_x/2}, y_\ell) &= k_\ell \frac{b_{j-\eta_x/2,\ell} - b_{j-\eta_x/2,\ell+1}}{4} \delta_y^{(1/2)} u_{j-\eta_x/2,\ell+1/2} \\ &- \frac{1}{2}\int_{y_\ell}^{y_{\ell+1/2}} (bu_y)(x_{j-\eta_x/2}, y)\, dy + \frac{1}{2}\int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_y)(x_{j-\eta_x/2}, y)\, dy\,. \end{aligned}$$

We note that $B(x_{j-\eta_x/2}, y_\ell) = B(x_j, y_\ell) + R$, with $R = R_1 + R_2$ and

$$\begin{aligned} R_1 &= \int_{I_j} \left(\frac{b(y_\ell) + b(y_{\ell+1})}{4} (u(y_{\ell+1}) - 2u(y_{\ell+1/2}) + u(y_\ell)) \right)_x dx, \\ R_2 &= -\int_{I_j} \left(\frac{1}{2} \int_{y_\ell}^{y_{\ell+1/2}} (b(y) - b(y_\ell)) u_y \, dy \right. \\ &\qquad \left. + \frac{1}{2} \int_{y_{\ell+1/2}}^{y_{\ell+1}} (b(y) - b(y_{\ell+1})) u_y \, dy \right)_x dx \,, \end{aligned}$$

with $I_j = (x_j, x_{j+1})$ if $\eta_x = -1$ and $I_j = (x_{j-1}, x_j)$ if $\eta_x = 1$.
For R_1 applying the Bramble-Hilbert Lemma we obtain

$$|R_1| \le C \|b\|_{1,\infty} k_\ell^{t-1} \int_{I_j} \int_{y_\ell}^{y_{\ell+1}} (|u_{t^s}| + |u_{xy^t}|) \, dy \, dx \,, \quad t \in \{1, 2\}.$$

For R_2 holds

$$|R_2| \le C \|b\|_{2,\infty} k_\ell \int_{I_j} \int_{y_\ell}^{y_{\ell+1}} (|u_y| + |u_{yx}|) \, dy \, dx.$$

Then we conclude

$$Q_{b,2}^{(2)} + Q_{b,3}^{(2)} = R_{b_{stl}} + R,$$

where

$$\begin{aligned} |R| \; &\le \; C \|b\|_{2,\infty} \sum_{(x_j, y_{\ell+1/2}) \in \partial\Omega} \int_{I_j} \int_{y_\ell}^{y_{\ell+1}} \Big(k_\ell^t (|u_{y^t}| + |u_{xy^t}|) \\ &\qquad + k_\ell^2 (|u_y| + |u_{yx}|) \Big) \; dy \, dx |\delta_y^{(1/2)} \bar{v}_{j,\ell+1/2}| \,, \end{aligned}$$

for $t \in \{1, 2\}$, or in a more convenient form

$$\|R\| \le C \|b\|_{2,\infty} \Big(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^{2(s-1)} \|u\|^2_{H^s(\Delta)} \Big)^{1/2} \|P_H v_H\|_1$$

for $s \in \{2, 3\}$.

Finally, from the estimates for $|Q_{b,i}|$, $i = 1, 2, 3$, we conclude the proof. ∎

Remark 3. (1) *Let v_H in Theorem 4 be null on the boundary $\partial\Omega_H$. If Ω is the union of rectangles then*

$$|b_{yx}(R_H u, v_H) - \tilde{b}_{yx}(u, v_H)| \le C \Big(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^{2(s-1)} \|u\|^2_{H^s(\Delta)} \Big)^{1/2} \|P_H v_H\|_1 \,,$$

for $s \in \{2,3\}$.

Let Ω *be now a polygonal domain. In this case we have*

$$b_{yx}(R_H u, v_H) - \tilde{b}_{yx}(u, v_H) = R_{b_{exe}} + R_{b_{rem}}.$$

We consider now $R_{b_{exe}}$. *Let* $\Delta^* \in \mathcal{T}_H^{obl}$ *and we suppose that* Δ^* *has vertices* (x_j, y_ℓ), (x_{j+1}, y_ℓ) *and* $(x_j, y_{\ell+1})$ *being the first vertex associated with the* $\dfrac{\pi}{2}$ *angle. The term of* $R_{b_{exe}}$ *associated wit* Δ^*, $R_{b_{exe,\Delta^*}}$, *is given by*

$$\begin{aligned}
R_{b_{exe,\Delta^*}} &= \left(\frac{k_\ell}{2} b_{j+1/2,\ell}\delta_y^{(1/2)} u_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1/2}} (bu_j)(x_{j+1/2}, y)\, dy\right)\Delta_x \bar{v}_{j,\ell}\\
&+ \left(\frac{k_\ell}{4} b_{j-1/2,\ell+1}\delta_y^{(1/2)} u_{j-1/2,\ell+1/2} - \frac{1}{2}\int_{y_\ell}^{y_{\ell+1/2}} (bu_j)(x_{j-1/2}, y)\, dy\right.\\
&+ \left.\frac{k_\ell}{4} b_{j-1/2,\ell}\delta_y^{(1/2)} u_{j-1/2,\ell+1/2} - \frac{1}{2}\int_{y_{\ell+1/2}}^{y_{\ell+1}} (bu_j)(x_{j-1/2}, y)\right)\Delta_y \bar{v}_{j,\ell}\,.
\end{aligned} \tag{71}$$

Attending that $\bar{v}_{j+1,\ell} = \bar{v}_{j,\ell+1} = 0$, *for* $R_{b^*_{exe,\Delta}}$ *holds the following representation*

$$\begin{aligned}
R_{b_{exe,\Delta^*}} &= \frac{h_j}{2}\int_{x_{j-1/2}}^{x_{j+1/2}} \left(\int_{y_\ell}^{y_{\ell+1/2}} (bu_y)_x\, dy - \int_{y_\ell}^{y_{\ell+1}} (b(y_\ell)u_y)_x\, dy\right)\, dx\,\delta_x^{(1/2)}\bar{v}_{j+1/2,\ell}\\
&- h_j\left(\frac{b_{j+1/2,\ell}}{4}\big(u(x_{j+1/2}, y_{\ell+1}) - 2u(x_{j+1/2}, y_{\ell+1/2}) + u(x_{j+1/2}, y_\ell)\big)\right.\\
&\qquad\left. + \frac{1}{2}\int_{y_\ell}^{y_{\ell+1/2}} ((b - b(y_\ell))u_y)_{x_{j+1/2}}\, dy\right)\delta_x^{(1/2)}\bar{v}_{j+1/2,\ell}\\
&- h_j\left(\frac{b_{j-1/2,\ell+1}}{4}\big(u(x_{j-1/2}, y_{\ell+1}) - 2u(x_{j-1/2}, y_{\ell+1/2}) + u(x_{j-1/2}, y_\ell)\big)\right.\\
&\qquad\left. + \frac{1}{2}\int_{y_\ell}^{y_{\ell+1/2}} ((b - b(y_{\ell+1}))u_y)_{x_{j-1/2}}\, dy\right)\,\delta_x^{(1/2)}\bar{v}_{j+1/2,\ell}\,.
\end{aligned}$$

Using this representation and the Bramble-Hilbert Lemma it can be shown that

$$|R_{b_{exe,\Delta^*}}| \leq C\|b\|_{1,\infty} \sum_{\Delta\in V_1(\Delta^*)} (diam\Delta)\|u_{yx}\|_{L^1(\Delta)}\|(P_H\bar{v}_H)_{x,\Delta}\|_{L^1(\Delta)}$$

$$+ \|b\|_\infty \sum_{\Delta\in\tilde{V}_2(\Delta)} ((diam\Delta)^{t-1}\|u_{y^t}\|_{L^1(\Delta)} + (diam\Delta)^t\|u_{xy^t}\|_{L^1(\Delta)})\|(P_H\bar{v}_H)_{x,\Delta}\|_{L^1(\Delta)}$$

$$+ \|b\|_{1,\infty} \sum_{\Delta\in\tilde{V}_2(\Delta)} ((diam\Delta)\|u_y\|_{L^1(\Delta)} + (diam\Delta)^2\|u_{yx}\|_{L^1(\Delta)})\|(P_H\bar{v}_H)_{x,\Delta}\|_{L^1(\Delta)}$$

where $t\in\{1,2\}$, $V_1(\Delta^*)$ *is defined in Remark 1,* $\tilde{V}_2(\Delta^*)$ *is the union of* Δ^* *with the triangles of the smaller set of triangles containing* $(x_{j-1/2}-\frac{h_j}{2}, x_j)\times(y_\ell, y_{\ell+1})$. *Attending that* Δ^* *is arbitrary in* $\mathcal{T}_H^{obl}$ *we deduce*

$$\begin{aligned}|R_{b_{exe}}| &\leq C\Bigg((\sum_{\Delta\in\mathcal{T}_H^{obl,1}} (diam\Delta)^4\|u\|^2_{H^3(\Delta)})^{1/2} \\ &+ (\sum_{\Delta\in\tilde{\mathcal{T}}_H^{obl,2}} (diam\Delta)^2\|u\|^2_{H^2(\Delta)})^{1/2}\Bigg)\|P_Hv_H\|_1\end{aligned}$$

where $\mathcal{T}_H^{obl,1}$ *was defined in Remark 1 and* $\tilde{\mathcal{T}}_H^{obl,2}$ *is defined analogously to* $\mathcal{T}_H^{obl,2}$.

The last estimate enable us to conclude that if

$$u\in H_0^3(\Omega)\cap W^{2,\infty}(\cup_{\Delta\in\tilde{\mathcal{T}}_H^{obl,2}}\Delta) \text{ and } \sum_{\Delta\in\mathcal{T}_H^{obl}} |\Delta| = O(H_{max})$$

then

$$|R_{b_{exe}}| \leq O(H_{max}^{3/2})\|P_Hv_H\|_1.$$

(2) *For* $R_{b_{stl}}$ *holds the following*

$$|R_{b_{stl}}| \leq \sum_{(x_j,y_{\ell+1/2})\in\partial\Omega} |\eta_x|k_\ell\left(\|u(x_j)'\|_{L_1(y_\ell,y_{\ell+1})} + \|u(x_j)''\|_{L_1(y_\ell,y_{\ell+1})}\right)|\Delta_y\bar{v}_{j,\ell}|$$

where $u(x_j)(y) := u(x_j, y)$.

(3) *For $v_H \in W_H$, let $b_{xy}(R_H u, v_H)$ be defined by*

$$b_{xy}(R_H u, v_H) = \frac{b_{xy}^{(1)}(R_H u, v_H) + b_{xy}^{(2)}(R_H u, v_H)}{2}$$

where $b_{xy}^{(\nu)}(.,.)$ is given by (19). Let $\tilde{b}_{xy}(u, v_H)$ be defined changing in the definition of $\tilde{b}_{yx}(u, v_H)$ x with y.

For the difference

$$b_{xy}(R_H u, v_H) - \tilde{b}_{xy}(u, v_H),$$

holds a result analogous to Theorem 4.

Let us consider now the contribution of $(du)_x$ in (32).

Theorem 5. *Let $d(R_H u, P_H v_H)$ be defined by (12) and*

$$\tilde{d}(u, v_H) = - \sum_{(x_{j+1/2}, y_\ell) \in \bar{\Omega}} \int_{I_{j+1/2,\ell}} (du)(x_{j+1/2}, y)\, dy\, \Delta_x \bar{v}_{j,\ell} \quad for\, v_H \in W_H$$

Then

$$d(R_H u, v_H) = \tilde{d}(u, v_H) + R_d$$

where R_d satisfies:

(1) *If $u \in H^2(\Omega)$ then*

$$|R_d| \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (diam \Delta)^2 \|u\|^2_{H^2(\Delta)} \right)^{1/2} \|P_H v_H\|_1,$$

(2) *If $u \in H^3(\Omega)$ then $R_d = R_{d_{stl}} + R_{d_{exe}} + R_{d_{rem}}$ with*

$$R_{d_{stl}} = \sum_{(x_j, y_{\ell+1/2}) \in \partial\Omega} -\eta_x \frac{k_\ell}{8} \int_{y_\ell}^{y_{\ell+1}} (du)_y\, d\sigma\, \Delta_y \bar{v}_{j,\ell}\,,$$

$$R_{d_{exe}} = \sum_{\Delta \in \mathcal{T}_H^{obl}} \frac{|I_{\Delta_y}|}{8} (- \int_{I_{\Delta_y}} (du)_y(x_{\square})\, dy\, \Delta_y \bar{v}_\Delta + 2 \int_{I_{\frac{\Delta_y}{2}}} (du)_y(x_{\frac{\Delta}{2}})\, dy\, \Delta_x \bar{v})$$

and

$$|R_{d_{rem}}| \leq C(\sum_{\Delta \in \mathcal{T}_H} (diam \Delta)^4 \|u\|^2_{H^3(\Delta)})^{1/2} \|P_H v_H\|_1\,.$$

Proof: An estimate to $d(u, P_H v_H) - \tilde{d}(u, v_H)$ is obtained estimating separately

$$Q_3 := \sum_{(x_{j+1/2}, y_\ell)\in\bar{\Omega}} \int_{I_{j+1/2}} \big((du)(x_{j+1/2}, y) \; - Du(x_{j+1}, y)\big) \; dy \Delta_x \bar{v}_{j,\ell} \qquad (72)$$

and

$$Q_4 := \sum_{(x_{j+1/2}, y_\ell)\in\bar{\Omega}} \int_{I_{j+1/2}} \big(Du(x_{j+1/2}, y) - Du(x_{j+1/2}, y_\ell)\big) \; dy \Delta_x \bar{v}_{j,\ell} \qquad (73)$$

where $Du(x_{j+1/2}, y) := \frac{1}{2}\left((du)(x_j, y) + (du)(x_{j+1}, y)\right)$. The bounds to these two terms are obtained following the steps used in Theorem 2 on the estimation of Q_1 and Q_2 respectively.

■

Remark 4. *For Theorem 5 holds a remak analogous to Remark 1. In the context of a domain with an oblique side and with homogeneous boundary conditions, we have*

$$|R_{d_{exe}}| \leq C(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^3(\Delta)})^{1/2} \|P_H v_H\|_1,$$

provided that $u \in H^3(\Omega)$. In fact, if $\Delta \in \mathcal{T}_H^{obl}$ has the vertices (x_j, y_ℓ) (associated with the angle $\frac{\pi}{2}$), (x_{j+1}, y_ℓ) and $(x_j, y_{\ell+1})$ then

$$\begin{aligned} R_{d_{exe},\Delta} &= \frac{|I_{\Delta_y}|}{8}\Big(-\int_{I_{\Delta_y}} (du)_y(x_\Box)\, dy\, \Delta_y \bar{v}_\Delta + 2\int_{I_{\frac{\Delta_y}{2}}} (du)_y(x_{\frac{\Delta}{2}})\, dy\, \Delta_x \bar{v}\Big) \\ &= -\frac{k_\ell^2}{8} \int_{x_{j-1/2}}^{x_j} \int_{y_\ell}^{y_{\ell+1}} (du)_{xy}\, dy\, dx\, \delta_y^{(1/2)} \bar{v}_{j,\ell+1/2} + R\,, \end{aligned}$$

with

$$\begin{aligned} |R| &\leq Ck_\ell^2 h_j \int_{x_j}^{x_{j+1}} |(du)_{x^2}(x, y_\ell)|\, dx |\delta_y^{(1/2)} \bar{v}_{j,\ell+1/2}| \\ &\leq Ck_\ell h_j \int_{y_\ell - \frac{k_\ell}{2}}^{y_{\ell+1/2}} \int_{x_j}^{x_{j+1}} \left(|(du)_{x^2}| + k_\ell |(du)_{x^2 y}|\right)\, dx\, dy |\delta_y^{(1/2)} \bar{v}_{j,\ell+1/2}|\,. \end{aligned}$$

For the contribution of $(eu)_y)$ holds the following result:

Theorem 6. *Let $e(R_H u, P_H v_H)$ be defined by (13)*

$$\tilde{e}(u, v_H) = -\sum_{(x_{j+1/2}, y_\ell)\in\bar{\Omega}} \int_{I_{j+1/2,\ell}} (eu)(x, y_{\ell+1/2})\, dx\, \Delta_y \bar{v}_{j,\ell} \ for\, v_H \in V_H\,.$$

Then

$$e(R_H u, P_H v_H) = \tilde{e}(u, v_H) + R_e$$

where R_e satisfies:

(1) *If $u \in H^2(\Omega)$ then*

$$|R_e| \le C\left(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^2 \|u\|^2_{H^2(\Delta)}\right)^{1/2} \|P_H v_H\|_1,$$

(2) *If $u \in H^3(\Omega)$ then $R_e = R_{e_{stl}} + R_{e_{exe}} + R_{e_{rem}}$ with*

$$R_{e_{stl}} = \sum_{(x_{j+1/2}, y_\ell)\in\partial\Omega} -\eta_y \frac{h_j}{8}\int_{x_j}^{x_{j+1}} (eu)_x\, d\sigma\, \Delta_x \bar{v}_{j,\ell}\,,$$

$$R_{e_{exe}} = \sum_{\Delta\in\mathcal{T}_H^{obl}} \frac{|I_{\Delta_x}|}{8}\Big(-\int_{I_{\Delta_x}} (eu)_x(y_\Box)\, dx\, \Delta_x \bar{v}_\Delta + 2\int_{I_{\frac{\Delta_x}{2}}} (eu)_x(y_{\frac{\Delta}{2}})\, dx \Delta_y \bar{v}_\Delta\Big)\,,$$

and

$$|R_{e_{rem}}| \le C\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^3(\Delta)}\Big)^{1/2} \|P_H v_H\|_1\,.$$

■

Remark 5. *Holds a remark analogous to Remark 4.*

Let us look now to the difference

$$f(R_H u, v_H)_H - (fu, v_H)_H \quad \text{for } v_H \in W_H,$$

where $f(.,.)$ is defined by (14). As a preparation for the study of the last term we prove the following:

Lemma 1. *The following identity holds for all $a_j, b_j \in \mathbb{C}, j = 1, \ldots, 4$:*

$$\begin{aligned} 4\sum_{i=1}^4 a_i b_i &= \sum_{i=1}^4 a_i \sum_{i=1}^4 b_i + (a_1 + a_2 - a_3 - a_4)(b_1 + b_2 - b_3 - b_4) \\ &+ (a_1 - a_2 + a_3 - a_4)(b_1 - b_2 + b_3 - b_4) \\ &+ (a_1 - a_2 - a_3 + a_4)(b_1 - b_2 - b_3 + b_4). \end{aligned}$$

Proof: The assertion follows applying the identity $2(ab+cd) = (a+c)(b+d) + (a-c)(b-d)$ to $2(a_1b_1 + a_2b_2)$ and $2(a_3b_3 + a_4b_4)$ and to the resulting terms.

■

Theorem 7. *Assume that* $u \in H^2(\Omega)$. *Let* $(fu)_H \in W_H$ *be defined by (15) then*

$$|f(R_Hu, v_H)_H - ((fu)_H, v_H)_H| \le C\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^2(\Delta)} + S_H\Big)^{1/2} \|P_H v_H\|_1, \tag{74}$$

where S_H *can be estimated either by*

$$S_H \le \sum_{\Delta\in\mathcal{T}_H^{obl}} (diam\Delta)^2 \|u\|^2_{H^2(\Delta)} \tag{75}$$

or, assuming $u \in H^3(\Omega)$, *by*

$$S_H \le \sum_{\Delta\in\mathcal{T}_H^{obl}} (diam\Delta)^2 |\Delta| \|u\|^2_{C^1(\Delta)}. \tag{76}$$

Proof: We decompose

$$((fu)_H, v_H)_H - f(R_Hu, v_H)_H \tag{77}$$

into the contributions belonging to full rectangles contained in Ω and the remaining triangles. We begin considering such rectangle $S = (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ and subdivide it in four congruent subrectangles $S_1, \dots, S_4$ of equal size, numbering them from bottom left to top right. By P_i we denote the common vertex of S and S_i and we use the abbreviations $v_i = v_H(P_i)$, $(fu)_i = (fu)_H(P_i)$.
The contribution $E(S)$ of (77) belonging to S is then

$$E(S) = \sum_{i=1}^{4} \rho_i \bar{v}_i\,, \tag{78}$$

where

$$\rho_i = \int_{S_i} fu\,dxdy - \frac{|S|}{4}(fu)_i\,, i = 1, \dots, 4. \tag{79}$$

We apply Lemma 1 to $4E(S)$ and study the behaviour of the four resulting summands. The first one is

$$E_1(S) = \sum_{i=1}^{4} \rho_i \sum_{i=1}^{4} \bar{v}_i = \left(\int_S fu\, dxdy - \frac{|S|}{4} \sum_{i=1}^{4} (fu)_i \right) \sum_{i=1}^{4} \bar{v}_i.$$

The Bramble-Hilbert Lemma furnishes

$$\left| \int_S fu\, dxdy - \frac{|S|}{4} \sum_{i=1}^{4} (fu)_i \right| \leq C \|f\|_{2,\infty} (diamS)^2 |u|_{1,2}^{(S)}$$

and we obtain

$$|E_1(S)| \leq C \|f\|_{2,\infty} (diamS)^2 |u|_{1,2}^{(S)} \sum_{i=1}^{4} |v_i| \,, \tag{80}$$

where $|u|_{1,2}^{(S)} = \max_{s_1+s_2=2} \|u_{x^{s_1}y^{s_2}}\|_{L_1(S)}$, $s_1, s_2 \in \{0,1,2\}$.
Next we estimate the second summand that have the form

$$\begin{aligned} E_2(S) &:= (\rho_1 + \rho_2 - \rho_3 - \rho_4)(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \\ &= (\rho_1 + \rho_2 - \rho_3 - \rho_4)(\bar{v}_{j,\ell} - \bar{v}_{j,\ell+1} + \bar{v}_{j+1,\ell} - \bar{v}_{j+1,\ell+1}) \,. \end{aligned}$$

Thus we obtain

$$|E_2(S)| \leq |\rho_1 + \rho_2 - \rho_3 - \rho_4| k_\ell \left(|(P_H v_H)_y(x_j, y_{\ell+1/2})| + |(P_H v_H)_y(x_{j+1}, y_{\ell+1/2})| \right) .$$

To estimate further consider

$$\rho_1 - \rho_3 = \int_{S_1} fu\, dxdy - \int_{S_3} fu\, dxdy + \frac{|S|}{4}((fu)_1 - (fu)_3).$$

From the Bramble-Hilbert Lemma follows

$$\left| \int_{S_1} fu\, dxdy - \int_{S_3} fu\, dxdy \right| \leq C \|f\|_{1,\infty} diamS |u|_{1,1}^{(S)}.$$

On the other hand

$$h_j((fu)_1 - (fu)_3) = h_j \int_{y_\ell}^{y_{\ell+1}} (fu)_y(x_j, y)\, dy = \int_S (fu)_y\, dxdy + R$$

where from another application of the Bramble-Hilbert Lemma (or more elementary from the error bound of the rectangle rule) we have

$$|R| \leq C \|f\|_{2,\infty} h_j \int_S |u_{xy}| dxdy \leq C \|f\|_{2,\infty} h_j |u|_{1,2}^{(S)}.$$

In the same way $\rho_2 - \rho_4$ can be bounded. Altogether we have shown

$$|E_2(S)| \le C\|f\|_{2,\infty}(diam S)^2\left(|u|_{1,1}^{(S)} + |u|_{1,2}^{(S)}\right) \left(|(P_H v_H)_y(x_j, y_{\ell+1/2})| + |(P_H v_H)_y(x_{j+1}, y_{\ell+1/2})|\right). \quad (81)$$

The two other summands $E_3(S)$ and $E_4(S)$ coming from the application of Lemma 1 can be bounded in the same way as $E_2(S)$.

We turn now to the contribution of (77) belonging to triangles in $\mathcal{T}_H^{obl}$ if there any at all. To be specific, let S be such a triangle with vertices $P_1 := P_2 := (x_j, y_\ell)$, $P_3 := (x_{j+1}, y_\ell)$ and $P_4 := (x_j, y_{\ell+1})$. Let S_i, $i = 1, \ldots, 4$, denote the four congruent triangles that partition S with the aid of the midpoints of the sides of S, where S_i has P_i as one vertex. Then the representation (78) of $E(S)$ with the quantities ρ_i from (79) remaining the same. Lemma 1 is then applied and the summands $E_i(S)$, $i = 2, 3, 4$, can be estimated as before. But there is a difference with $E_1(S)$ which we write as

$$E_1(S) = E_{11}(S) + E_{12}(S)$$

with

$$E_{11} := \left(\int_S fu\,dxdy - \frac{|S|}{6}((fu)_1 + (fu)_2) - \frac{|S|}{3}((fu)_3 + (fu)_4)\right)\sum_{i=1}^{4} \bar{v}_i\,,$$

$$E_{12} := \frac{|S|}{12}\left(-(fu)_1 - (fu)_2 + (fu)_3 + (fu)_4\right)\sum_{i=1}^{4} \bar{v}_i\,.$$

The quantity $E_{11}(S)$ has the same bound as $E_1(S)$ in (80). In $E_{12}(S)$ the differences $(fu)_3 - (fu)_1$ and $(fu)_4 - (fu)_2$ can be bounded as in the first part of the proof leading to the lower order estimate

$$|E_{12}| \le C\|f\|_{2,\infty} diam S(|u|_{1,1}^{(S)} + |u|_{1,2}^{(S)})\sum_{i=1}^{4} |v_i|\,.$$

The assertion with (75) is now derived from the presentation

$$f(R_H u, v_H) - ((fu)_H, v_H)_H = \sum_{S\subset\Omega} E(S)$$

by an application of Schwarz's inequality for sums. All terms in the sum are estimated in the form

$$\|u\|_{L^1(S)}|w_H(\tilde{x},\tilde{y})| \;\le\; \|u\|^2_{L^2(S)} + |S||w_H(\tilde{x},\tilde{y})|^2$$

$$\leq \|u\|^2_{L^2(S)} + \|w_H\|^2_{L^2(S)}$$

where the last inequality stems from the fact that w_H is either equal to a function of type $P_H v_H$ or a derivative of it, so that a norm equivalence can be applied.

For the proof of the assertion with (76) we note that

$$|(fu)_3 - (fu)_1| \leq \|f\|_{1,\infty} h |u_x|_{C(\bar{S})} \text{ and } |(fu)_4 - (fu)_2| \leq \|f\|_{1,\infty} k |u_y|_{C(\bar{S})}.$$

Hence

$$|E_{12}(S)| \leq C\|f\|_{1,\infty}(diam S)|S|^{1/2}\|u\|_{C^1(\bar{S})}|S|^{1/2}\sum_{i=1}^{4}|v_i|.$$

and the result follows along the same lines as in the case (75).

■

Remark 6. *If in Theorem 7 we take v_H null on the boundary $\partial\Omega_H$ then we obtain (74) with S_H bounded by*

$$\Big(\sum_{\Delta\in\mathcal{T}_H^{obl}} (diam\Delta)^4\|u\|^2_{H^2(\Delta)}\Big)^{1/2}\|P_H v_H\|_1.$$

As a preparation for the estimate of the boundary contributions to the global error we provide an estimate that is the key for the gain of an additional power of $h^{1/2}$ of supraconvergence.

Let $Q_\ell, \ell = 1, \ldots, N+1$, be a consecutive numbering of the points in $\partial\Omega_H$ with $Q_{N+1} = Q_1$. Denote by Σ_ℓ the line segment joining the points Q_ℓ and $Q_{\ell+1}$ and by σ_ℓ its length.

Lemma 2. *Holds the following*

$$\Big(\sum_{\ell=1}^{N}|v_H(Q_{\ell+1}) - v_H(Q_\ell)|^2\Big)^{1/2} \leq C\|P_H v_H\|_1, \quad v_H \in W_H. \tag{82}$$

Proof: The assertion follows from the following chain of inequalities

$$\begin{aligned}\sum_{\ell=1}^{N}|v_H(Q_{\ell+1}) - v_H(Q_\ell)|^2 &= \sum_{l=1}^{N}\int_{Q_\ell}^{Q_{\ell+1}}\int_{Q_\ell}^{Q_{\ell+1}}\Big|\frac{P_H v_H(\sigma) - P_H v_H(\tau)}{\sigma-\tau}\Big|^2 d\sigma\, d\tau \\ &\leq \int_{\partial\Omega}\int_{\partial\Omega}\Big|\frac{P_H v_H(\sigma) - P_H v_H(\tau)}{\sigma-\tau}\Big|^2 d\sigma\, d\tau\end{aligned}$$

$$= \|P_H v_H\|^2_{1/2,\partial\Omega} \le C\|P_H v_H\|_1^2.$$

The last estimate is a well-known trace inequality for functions in $H^1(\Omega)$. ■

The following result has an important role on the estimation of

$$< (du)\eta_x + (eu)\eta_y - ((du)\eta_x + (eu)\eta_y)_H, v_H >_H$$

which arises in (31) and also on the estimation of

$$< R_H\psi - \psi_H, v_H >_H$$

which arises in (31) when in (8) ψ_H is replaced by $R_H\psi$.

The points in $\partial\Omega_H$ define a partition of $\partial\Omega$. We denote by $\mathcal{T}_H^b$ the collection of all line segments from this partition.

Proposition 2. *Let $\psi \in H^2(\partial\Omega)$ and let ψ_H be define by (16). Then, for all $v_H \in W_H$,*

$$| < R_H\psi - \psi_H, v_H >_H | \le C\, (\sum_{\Sigma\in\mathcal{T}_H^b} |\Sigma|^4\|\psi''\|^2_{L^2(\Sigma)} + |\Sigma|^3\|\psi'\|^2_{L^2(\Sigma)})^{\frac{1}{2}}\, \|P_H v_H\|_1. \tag{83}$$

Proof: Let $Q_{\ell+1/2}$ be the midpoint of the line segment Σ_ℓ joining the points Q_ℓ and $Q_{\ell+1}$. Then we obtain by a summation by parts and a reordering of terms

$$< R_H\psi - \psi_H, v_H >_H = \sum_{\ell=1}^N \left(\frac{\sigma_{\ell-1}+\sigma_\ell}{2}\phi(Q_\ell) - \int_{Q_{\ell-1/2}}^{Q_{\ell+1/2}} \psi(\sigma)\, d\sigma \right) \bar{v}_\ell$$

$$= \sum_{\ell=1}^N \left(\sigma_\ell \frac{\psi(Q_\ell)+\psi(Q_{\ell+1})}{2} - \int_{\Sigma_\ell} \psi(\sigma)\, d\sigma \right) \frac{\bar{v}_\ell + \bar{v}_{\ell+1}}{2}$$

$$+ \sum_{\ell=1}^N \left(\frac{\sigma_\ell}{2}(\psi_\ell - \psi_{\ell+1}) - \int_{Q_\ell}^{Q_{\ell+1/2}} \psi(\sigma)\, d\sigma + \int_{Q_{\ell+1/2}}^{Q_{\ell+1}} \psi(\sigma)\, d\sigma \right) \frac{\bar{v}_{\ell+1} - \bar{v}_\ell}{2}.$$

An estimate of the error of the trapezoidal rule and the rectangle rule, respectively, yields

$$| < R_H\psi - \psi_H, v_H >_H | \tag{84}$$

$$\le C \sum_{\ell=1}^N \left(\sigma_\ell^2 \|\psi''\|_{L^1(\Sigma_\ell)} |v_\ell + v_{\ell+1}| + \sigma_\ell \|\psi'\|_{L^1(\Sigma_\ell)} |v_\ell - v_{\ell+1}| \right) .$$

Since

$$\sum_{\ell=1}^{N} \sigma_\ell |v_\ell + v_{\ell+1}|^2 \leq C\|P_H v_H\|^2_{L^2(\partial\Omega)} \leq C\|P_H v_H\|_1^2$$

the asserted estimate follows from (84) by an application of Schwarz' inequality for sums and integrals taking Lemma 2 into account.

■

Finally in order to get an estimate to (31), attending to Theorems 2-7 we should estimate

$$R_{stl} = R_{a_{stl}} + R_{b_{stl}} + R_{c_{stl}} + R_{d_{stl}} + R_{e_{stl}},$$

$$R_{exe} = R_{a_{exe}} + R_{b_{exe}} + R_{c_{exe}} + R_{d_{exe}} + R_{e_{exe}},$$

$$< (R_H(du) - (du)_H)\eta_x + (R_H(eu) - (eu)_H)\eta_y - (\alpha u)_H + R_H(\alpha u), v_H >_H \quad (85)$$

and

$$< (Bu)_H - \psi_H, v_H >_H$$

where Bu is defined by (3) and $(v)_H$ represents the grid function defined by (16).

Assuming that $u \in H^3(\Omega)$ is such that the partial derivatives of first and second order are in $L^2(\partial\Omega)$, applying Lemma 2 we obtain for R_{stl} the estimate

$$\begin{aligned} |R_{stl}| &\leq C(\sum_{\Sigma\in\mathcal{T}_H^b} |\Sigma|^3 \|u\|^2_{H^2(\Sigma_\ell)})^{1/2} (\sum_{\ell=1}^{N} |v_H(Q_{\ell+1}) - v_H(Q_\ell)|^2)^{1/2} \\ &\leq C(\sum_{\Sigma\in\mathcal{T}_H^b} |\Sigma|^3 \|u\|^2_{H^2(\Sigma)})^{1/2} \|P_H v_H\|_1. \end{aligned}$$

For R_{exe} assuming that $u \in H_0^3(\Omega) \cap W^{2,\infty}(\cup_{\Delta\in\mathcal{T}_H^{obl,2}} \Delta)$ we have

$$\begin{aligned} |R_{exe}| &\leq C(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \|u\|^2_{H^3(\Delta)})^{1/2} \|P_H v_H\|_1 \\ &+ C(\sum_{\Delta\in\tilde{\mathcal{T}}_H^{obl,2}} (diam\Delta)^2 |\Delta| \|u\|^2_{W^{2,\infty}(\Delta)})^{1/2} \|P_H v_H\|_1 . \end{aligned} \quad (86)$$

From (86) we conclude if $\sum_{\Delta\in\mathcal{T}_H^{obl}} |\Delta| = O(H_{max})$ then

$$|R_{exe}| \leq O(H_{max}^{3/2}) \|P_H v_H\|_1.$$

Let us consider the boundary terms (85). Applying Proposition 2 we get

$$| < (R_H(du) - (du)_H)\eta_x + (R_H(eu) - (eu)_H)\eta_y + R_H(\alpha u) - (\alpha u)_H, v_H >_H |$$

$$\leq C(\sum_{\Sigma \in \mathcal{T}_H^b} |\Sigma|^3(\|u_x\|^2_{L^2(\Sigma)} + \|u_y\|^2_{L^2(\Sigma)}) + |\Sigma|^4 \|u_{xy}\|^2_{L^2(\Sigma)})^{1/2} \|P_H v_H\|_1. \tag{87}$$

If ψ_H is defined by (16) then attending to (3) we have $< (Bu)_H - \psi_H, v_H >_H = 0$. Otherwise, if $\psi_H = R_H\psi$ an estimate to $< (Bu)_H - \psi_H, v_H >_H$ is obtained using Proposition 2.

In the following proposition we summarize the previous considerations and, attending that the estimates were obtained for $s \in \{2, 3\}$, by interpolation, the estimates also hold for $s \in [2, 3]$.

Proposition 3. *Let the grids $\overline{\Omega}_H$, $H \in \Lambda$, satisfy condition (Geom). Assume that the homogeneous variational problem 4, i.e with $g = 0$ and $\psi = 0$ has only the null solution. Then the discretized problem (8) has a unique solution u_H for $H \in \Lambda$ with H_{max} sufficiently small. Moreover for $s \in [2, 3]$,*

(1) *if the solution u of (1) and (2) lies in $H_0^s(\Omega)$ then*

(a) *if Ω is a union of rectangles then*

$$\|P_H(R_H u - u_H)\|_1 \leq C(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^{2(s-1)} \|u\|^2_{H^{s-1}(\Delta)})^{1/2} \leq C H^s_{max} \|u\|_{H^s(\Omega)}\,, \tag{88}$$

(b) *if Ω has at least an oblique side and $u \in H_0^s(\Omega) \cap W^{2,\infty}(\cup_{\Delta \in \tilde{\mathcal{T}}_H^{obl,2}} \Delta)$ then*

$$\|P_H(R_H u - u_H)\|_1 \leq \mathcal{E}_{\mathcal{T}_H,p}(u, u_h) + \mathcal{E}_{\mathcal{T}_H^{obl},p}(u, u_h) \tag{89}$$

with

$$\mathcal{E}_{\mathcal{T}_H,p}(u, u_h) = C(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^{2(s-1)} \|u\|^2_{H^s(\Delta)})^{1/2} \leq C H^{s-1}_{max} \|u\|_{H^s(\Omega)}$$

and

$$\begin{aligned} \mathcal{E}_{\mathcal{T}_H^{obl},p}(u, u_h) &= C(\sum_{\Delta \in \tilde{\mathcal{T}}_H^{obl,2}} (diam\Delta)^2 |\Delta| \|u\|^2_{W^{2,\infty}(\Delta)})^{1/2} \\ &\leq C H_{max}(\sum_{\Delta \in \mathcal{T}_H^{obl}} |\Delta|)^{1/2} \|u\|_{W^{2,\infty}(\tilde{\mathcal{T}}_H^{obl})}, \end{aligned}$$

(2) *if the solution u of (1) and (3) lies in $H^s(\Omega)$ where Ω represents a union of rectangles and*

(a) *if ψ_H is defined by (16) then*

$$\|P_H(R_H u - u_H)\|_1 \leq \mathcal{E}_{\mathcal{T}_H,r}(u,u_h) + \mathcal{E}_{\mathcal{T}_H^b,r}(u,u_h) \tag{90}$$

with

$$\mathcal{E}_{\mathcal{T}_H,r}(u,u_h) = C(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^{2(s-1)}\|u\|^2_{H^s(\Delta)})^{1/2} \leq CH^{s-1}_{max}\|u\|_{H^s(\Omega)}$$

and

$$\begin{aligned}\mathcal{E}_{\mathcal{T}_H^b,r}(u,u_h) &= C(\sum_{\Sigma\in\mathcal{T}_H^b} |\Sigma|^3(\|u_x\|^2_{L^2(\Sigma)} + \|u_y\|^2_{L^2(\Sigma)}) + |\Sigma|^4\|u_{xy}\|^2_{L^2(\Sigma)})^{1/2} \\ &\leq C(H^{3/2}_{max}(\|u_x\|_{L^2(\partial\Omega)} + \|u_y\|_{L^2(\partial\Omega)}) + H^2_{max}\|u_{xy}\|_{L^2(\partial\Omega)}),\end{aligned}$$

(b) *if $\psi_H = R_H\psi$ then*

$$\|P_H(R_H u - u_H)\|_1 \leq \mathcal{E}_{\mathcal{T}_H,r}(u,u_h) + \mathcal{E}_{\mathcal{T}_H^b,r}(u,u_h) + \mathcal{E}_{\mathcal{T}_H^b,r}(\psi) \tag{91}$$

with

$$\begin{aligned}\mathcal{E}_{\mathcal{T}_H^b,r}(\psi) &= C(\sum_{\Sigma\in\mathcal{T}_H^b} |\Sigma|^3\|\psi'\|^2_{L^2(\Sigma)} + |\Sigma|^4\|\psi''\|^2_{L^2(\Sigma)})^{1/2} \\ &\leq C(H^{3/2}_{max}\|\psi'\|_{L^2(\partial\Omega)} + H^2_{max}\|\psi''\|_{L^2(\partial\Omega)}).\end{aligned}$$

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