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JACOBI MANIFOLDS, DIRAC STRUCTURES AND NIJENHUIS OPERATORS

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ABSTRACT: In a recent paper [2], we studied the concept of Dirac-Nijenhuis structures. We defined them as deformations of the canonical Lie algebroid structure of a Dirac bundle D defined in the double of a Lie bialgebroid (A, A^*) which satisfy certain properties. In this paper, we introduce the concept of generalized Dirac-Nijenhuis structures as the natural analogue when we replace the double of the Lie bialgebroid by the double of a generalized Lie bialgebroid. We also show the usefulness of the concept by proving that a Jacobi-Nijenhuis manifold has an associated generalized Dirac-Nijenhuis structure of a certain type.

KEYWORDS: Jacobi manifold, Nijenhuis operator, generalized Courant algebroid, Jacobi-Nijenhuis manifold.
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1. Introduction

From a geometrical point of view, it is well known that Dirac structures are intimately related to Lie algebroids and Lie bialgebroids [3, 17, 16]. A Dirac structure on a manifold M was defined in [4, 3] as a subbundle D of the Whitney sum $TM \oplus T^*M$ satisfying certain properties, which correspond to the definition of a Lie algebroid structure. Later, the concept was generalized to similar subbundles defined on Whitney sums of the form $A \oplus A^*$ where (A, A^*) is a Lie bialgebroid [17]. These Whitney sums are examples of the so called Courant algebroids, and therefore, Dirac structures arise as suitable subbundles of those. Finally, in [24, 7] the concept was generalized again and defined as subbundles of generalized Courant algebroids, which are similar to the usual ones but include a suitable 1-cocycle in the definition.

On the other hand, the deformation of structures by using Nijenhuis operators is a concept often used in the Literature. Originally proposed within the framework of integrable systems (see the introduction and references of [21]), it allows a deformation of Lie algebra structures defined on different types of manifolds. It has been recently extended to the Lie algebroid case,

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and therefore a very interesting example seems to be the study of the deformation of Lie algebroid structure which corresponds to a Dirac manifold. In [5], the problem was discussed for the case of Poisson manifolds (corresponding to the case of Poisson-Nijenhuis manifolds [13, 14]). Within the Lie algebroid domain, the Jacobi-Nijenhuis case (i.e. the deformation associated to a Jacobi manifold) was also studied in [10, 23]. Recently, we studied [2] the problem of the geometric characterization of the deformation of a Dirac structure defined in the double of a Lie bialgebroid, problem which was also studied in [18] and within the framework of the deformation of Courant algebroids in [1]. In this paper, we study the same problem for the case of Dirac structures defined on generalized Courant algebroids, aiming to characterize the case of a Jacobi-Nijenhuis manifolds as a Dirac-Nijenhuis structure on this generalized setting.

The structure of the paper is as follows. In Section 2 we review some basic facts about Jacobi manifolds and its characterization in terms of the so called generalized Lie bialgebroids (or Jacobi bialgebroids). Section 3 is devoted to the summary of the basic properties of generalized Courant algebroids and their corresponding Dirac structures. In Section 4, we study the deformation of generalized structures at three levels: Lie algebroids, Lie bialgebroids and Courant algebroids; while Section 5 presents the deformation of their corresponding Dirac structures and proves that Jacobi-Nijenhuis manifolds are a particular example of these.

2. Generalized Lie (Jacobi) bialgebroids

Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over M, A^* its dual vector bundle and denote by $\bigwedge A = \oplus_{k\in\mathbb{Z}} \bigwedge^k A$ and $\bigwedge A^* = \oplus_{k\in\mathbb{Z}} \bigwedge^k A^*$ the graded exterior algebras of A and A^*, respectively.

Let $\phi \in \Gamma(A^*)$ be a 1-cocycle for the Lie algebroid cohomology complex with trivial coefficients (see [19] and [9]), i.e. for all $X, Y \in \Gamma(A)$,

$$\langle \phi, [X, Y]\rangle = \rho(X)\langle \phi, Y\rangle - \rho(Y)\langle \phi, X\rangle. \tag{1}$$

Using the 1-cocycle ϕ, we can define a new representation ρ^ϕ of the Lie algebra $(\Gamma(A), [\cdot,\cdot])$ on $C^\infty(M,\mathbb{R})$, by setting

$$\rho^\phi : \Gamma(A) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}), \quad \rho^\phi(X, f) = \rho(X)f + \langle \phi, X\rangle f.$$

We obtain a new cohomology complex, whose differential cohomology operator is given by

$$d^{\phi} : \Gamma(\bigwedge^{k} A^*) \to \Gamma(\bigwedge^{k+1} A^*), \quad d^{\phi}(\beta) = d\beta + \phi \wedge \beta.$$

For any $X \in \Gamma(A)$, the Lie derivative operator with respect to X is given by

$$\mathcal{L}^{\phi}_X : \Gamma(\bigwedge^{k} A^*) \to \Gamma(\bigwedge^{k} A^*), \quad \mathcal{L}^{\phi}_X(\beta) = \mathcal{L}_X\beta + \langle \phi, X\rangle\beta.$$

It is also possible to consider a ϕ-Schouten bracket on the graded algebra $\Gamma(\bigwedge A)$, denoted by $[\cdot,\cdot]^{\phi}$, which is defined as follows: for $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$,

$$[P, Q]^{\phi} = [P, Q] + (p-1)P \wedge (i_{\phi}Q) + (-1)^p(q-1)(i_{\phi}P) \wedge Q,$$

where $i_{\phi}Q$ can be interpreted as the usual contraction of a multivector field by a 1-form. For more details see [9] and [6].

Suppose that the vector bundle $(A, [\cdot,\cdot], \rho)$ and its dual vector bundle $(A^*, [\cdot,\cdot]_*, \rho_*)$ are both Lie algebroids over a manifold M. Let d (resp. d_*) denote the differential of A (resp. A^*). Let $\phi \in \Gamma(A^*)$ (resp. $W \in \Gamma(A)$) be a 1-cocycle in the Lie algebroid cohomology complex of $(A, [\cdot,\cdot], \rho)$ (resp. $(A^*, [\cdot,\cdot]_*, \rho_*)$).

Definition 2.1 ([6, 9])**.** *The pair $((A, \phi), (A^*, W))$ is a* generalized Lie bialgebroid *(or a* Jacobi bialgebroid*) if for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge A)$,*

$$d^W_*[P, Q]^{\phi} = [d^W_* P, Q]^{\phi} + (-1)^{p+1}[P, d^W_* Q]^{\phi}.$$

When $\phi = 0$ and $W = 0$, we recover the notion of *Lie bialgebroid*, [20] and [12].

Example 2.2. *If M is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [\cdot,\cdot], \pi)$ is a Lie algebroid over M, with π the projection over the first factor and $[\cdot,\cdot]$ given by*

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)), \quad (X, f), (Y, g) \in \mathfrak{X}(M) \times C^{\infty}(M, \mathbb{R}). \tag{2}$$

The pair $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (0, 0)))$ is a generalized Lie bialgebroid, where $T^*M \times \mathbb{R}$ is endowed with the null Lie algebroid structure, i.e. $[\cdot,\cdot]_* = 0$ and $\rho_* = 0$.

The next example, which appears in [9], is related with the notion of Jacobi manifold. We recall that a *Jacobi manifold* ([15]) is a smooth manifold M equipped with a bivector field Λ and a vector field E such that $[\Lambda,\Lambda] = -2E\wedge\Lambda$ and $[E,\Lambda]=0$.

Example 2.3. *Let (M,Λ,E) be a Jacobi manifold. Consider the associated Lie algebroid $(T^*M\times\mathbb{R},\ [\cdot,\cdot]_{(\Lambda,E)},\ \pi\circ(\Lambda,E)^\#)$ over M ([11]), where $(\Lambda,E)^\#$ is the vector bundle morphism given by*

$$(\Lambda,E)^\#(\alpha,f) = (\Lambda^\#(\alpha)+fE, -\langle\alpha,E\rangle), \tag{3}$$

for any section α of T^*M and $f\in C^\infty(M,\mathbb{R})$, and $[\cdot,\cdot]_{(\Lambda,E)}$ is the bracket given by

$$[(\alpha,f),(\beta,g)]_{(\Lambda,E)} := (\gamma,h), \tag{4}$$

with

$$\gamma := \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha\wedge\beta),$$
$$h := -\Lambda(\alpha,\beta) + \Lambda(\alpha,dg) - \Lambda(\beta,df) + \langle fdg - gdf, E\rangle.$$

The pair $((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0)))$ is a generalized Lie bialgebroid.

3. Generalized Courant algebroids and Dirac structures

The notion of Courant algebroid was introduced by Liu *et al.* ([17]) for describing the geometric structure of the double $A\oplus A^*$ of a Lie bialgebroid (A,A^*). In order to interpret the double of a generalized Lie bialgebroid, we introduced in [24] the notion of generalized Courant algebroid.

Definition 3.1. *A* generalized Courant algebroid *is a pair (E,θ), where E is a vector bundle $E\to M$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$ and a bundle map $\rho^\theta : E\to TM\times\mathbb{R}$, which is a first-order differential operator, and where $\theta\in\Gamma(E^*)$ is such that, for any $e_1,e_2\in\Gamma(E)$, $\langle\theta,[e_1,e_2]\rangle = \rho(e_1)\langle\theta,e_2\rangle - \rho(e_2)\langle\theta,e_1\rangle$, $\rho(e_1)$ being the derivation associated with $\rho^\theta(e_1)$ (i.e., $\rho^\theta(e_1) = \rho(e_1)+\langle\theta,e_1\rangle$), satisfying, for all $e,e_1,e_2,e_3\in\Gamma(E)$ the following properties:*

i) $[[e_1,e_2],e_3] + c.p. = \mathcal{D}^\theta T(e_1,e_2,e_3)$,
where $T(e_1,e_2,e_3) = \frac{1}{3}([e_1,e_2],e_3) + c.p.$ and $\mathcal{D}^\theta : C^\infty(M,\mathbb{R})\to\Gamma(A)$ is the first-order differential operator given by $(\mathcal{D}^\theta f, e) = \frac{1}{2}\rho^\theta(e)f$.

ii) $\rho^\theta([e_1, e_2]) = [\rho^\theta(e_1), \rho^\theta(e_2)]$,
where the bracket on the right-hand side is the Lie bracket (2) on $\Gamma(TM \times \mathbb{R})$*;*

iii) $\rho^\theta(e)(e_1, e_2) = ([e, e_1] + \mathcal{D}^\theta(e, e_1), e_2) + (e_1, [e, e_2] + \mathcal{D}^\theta(e, e_2))$;

iv) *for any* $f, g \in C^\infty(M, \mathbb{R})$, $(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0$.

In [7], under the name of *Courant-Jacobi algebroid*, an equivalent definition is presented. This last version generalizes the Courant algebroid definition given by Roytenberg ([25]). The equivalence of both definitions is proved in [24].

When $\theta = 0$, the generalized Courant algebroid $(E, 0)$ is just the Courant algebroid E.

Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid over M. On the Whitney sum bundle $A \oplus A^*$ we can define two non-degenerate bilinear forms, one symmetric, denoted by $(\cdot, \cdot)_+$, and the other skew-symmetric, denoted by $(\cdot, \cdot)_-$, by setting, for any $X_1 + \alpha_1, X_2 + \alpha_2 \in A \oplus A^*$,

$$(X_1 + \alpha_1, X_2 + \alpha_2)_\pm = \frac{1}{2}(\langle \alpha_1, X_2 \rangle \pm \langle \alpha_2, X_1 \rangle). \tag{5}$$

On the space $\Gamma(A \oplus A^*)$ of the global cross sections of $A \oplus A^*$, which is identified with $\Gamma(A) \oplus \Gamma(A^*)$, we consider the following bracket:

$$\begin{aligned}
[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] &= \left([X_1, X_2]^\phi + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1 - d^W_*(e_1, e_2)_-\right) \\
&\quad + \left([\alpha_1, \alpha_2]^W_* + \mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1 + d^\phi(e_1, e_2)_-\right), \qquad (6)
\end{aligned}$$

where $e_1 = X_1 + \alpha_1$ and $e_2 = X_2 + \alpha_2$.

Using the anchor maps a and a_* of A and A^*, respectively, and the 1-cocycles ϕ and W, we define the vector bundle maps $\rho : A \oplus A^* \to TM$ and $\rho^{\phi+W} : A \oplus A^* \to TM \times \mathbb{R}$, which are given, for any section $X + \alpha$ of $A \oplus A^*$, by

$$\rho(X + \alpha) = a(X) + a_*(\alpha), \quad \rho^{\phi+W}(X + \alpha) = a(X) + a_*(\alpha) + \langle \phi, X \rangle + \langle \alpha, W \rangle, \tag{7}$$

respectively.

Theorem 3.2 ([24]). *If* $((A, \phi), (A^*, W))$ *is a generalized Lie bialgebroid over* M*, then the pair* $(A \oplus A^*, \theta)$*, with* $\theta = \phi + W$*, is a generalized Courant algebroid with the bracket* $[\![\cdot, \cdot]\!]$ *on* $\Gamma(A \oplus A^*)$ *given by (6), the symmetric* *bilinear form* $(\cdot,\cdot)$ *given by* $(\cdot,\cdot)_+$ *of (5), the vector bundle map* ρ^θ *given by (7) and the operator* $\mathcal{D}^\theta$ *given by* $\mathcal{D}^\theta = (d^W + d_*^\phi)_{|C^\infty(M,\mathbb{R})}$.

The last theorem and the examples of generalized Lie bialgebroids given in the previous section, provide examples of generalized Courant algebroids. We are mainly interested in example 2.2 that enables us to conclude that $((TM\times\mathbb{R})\oplus(T^*M\times\mathbb{R}),(0,1)+(0,0))$ is a generalized Courant algebroid over M.

The bracket (6) on the space of sections of this generalized Courant algebroid, that was introduced by A. Wade in [26], is given, for all $(X_i,f_i)+(\alpha_i,g_i)\in\Gamma((TM\times\mathbb{R})\oplus(T^*M\times\mathbb{R}))$, $i=1,2$, by

$$\begin{aligned}
&[\![(X_1,f_1)+(\alpha_1,g_1),(X_2,f_2)+(\alpha_2,g_2)]\!] = \\
&= ([X_1,X_2],X_1(f_2)-X_2(f_1)) + \\
&\quad + \Big(f_1\alpha_2 - f_2\alpha_1 + \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \frac{1}{2}d(\langle\alpha_1,X_2\rangle - \langle\alpha_2,X_1\rangle) \\
&\quad + \frac{1}{2}(f_2dg_1 - g_1df_2 + g_2df_1 - f_1dg_2), \\
&\quad X_1(g_2) - X_2(g_1) + \frac{1}{2}(\langle\alpha_1,X_2\rangle - \langle\alpha_2,X_1\rangle) + \frac{1}{2}(f_1g_2 - f_2g_1)\Big). \qquad (8)
\end{aligned}$$

Let us now recall the notions of Dirac structure for a generalized Courant algebroid and for a Courant algebroid. Consider thus a generalized Courant algebroid (E,θ) (resp. Courant algebroid E). A subbundle $L\subset E$ of the generalized Courant algebroid (E,θ) (resp. Courant algebroid E) is said to be *integrable* if $\Gamma(L)$ is closed under the bracket $[\cdot,\cdot]$ on $\Gamma(E)$.

Definition 3.3. *A* Dirac structure *for the generalized Courant algebroid* (E,θ) *(resp. Courant algebroid* E*) is an integrable subbundle* L *of* E *which is maximally isotropic with respect to the symmetric bilinear form* $(\cdot,\cdot)$.

An immediate consequence of the previous definition is the following.

Proposition 3.4 ([24]). *If* L *is a Dirac structure for the generalized Courant algebroid* (E,θ) *and* $\theta\in\Gamma(L^*)$*, then* $(L,\rho|_L,[\cdot,\cdot]|_L)$ *is a Lie algebroid and* θ *is a* 1*-cocycle for the Lie algebroid cohomology complex with trivial coefficients.*

If $\theta=0$ in the previous proposition, i.e. if E is a Courant algebroid, then we recover a result from [17].

Let us now recall the notion of *characteristic pair*, introduced in [16]. Let A be a vector bundle, $D \subset A$ a subbundle of A and Ω a bivector field, $\Omega \in \Gamma(\bigwedge^2 A)$. Consider the subbundle L of $A \oplus A^*$, given by

$$L = \{X + \Omega^{\#}\alpha + \alpha,\ X \in D, \alpha \in D^{\perp}\} = D \oplus \mathrm{graph}(\Omega^{\#}|_{D^{\perp}}), \qquad (9)$$

where $D^{\perp}$ stands for the conormal bundle of D.

Clearly $L \subset A \oplus A^*$ is maximally isotropic with respect to the symmetric bilinear form (5). In what follows, we will assume that $D = L \cap A$ is of constant rank.

Definition 3.5 ([16])**.** *The pair (D, Ω) is called the* characteristic pair *of the subbundle L of $A \oplus A^*$ given by (9), while $D = L \cap A$ is called the* characteristic subbundle *of L.*

Theorem 3.6 ([24])**.** *Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid and $L \subset A \oplus A^*$ a maximal isotropic subbundle of $A \oplus A^*$ defined by a characteristic pair (D, Ω), i.e*

$$L = \{X + \Omega^{\#}\alpha + \alpha,\ X \in D, \alpha \in D^{\perp}\} = D \oplus \mathrm{graph}(\Omega^{\#}|_{D^{\perp}}).$$

Then L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if:

i) D is a Lie subalgebroid of A;

ii) $d_*^W \Omega + \frac{1}{2}[\Omega, \Omega]^{\phi} = 0\,(mod D)$;

iii) for any $\alpha, \beta \in \Gamma(D^{\perp})$,

$$[\alpha, \beta]_* + [\alpha, \beta]_{\Omega} \in \Gamma(D^{\perp}),$$

where $[\alpha, \beta]_{\Omega} = \mathcal{L}_{\Omega^{\#}(\alpha)}\beta - \mathcal{L}_{\Omega^{\#}(\beta)}\alpha - d(\Omega(\alpha, \beta))$.

Remark 3.7. *Under the assumptions of the above theorem, we can also call L a Dirac structure for the generalized Lie bialgebroid $((A, \phi), (A^*, W))$.*

If we consider now the generalized Courant algebroid structure $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), (0, 1) + (0, 0))$ described above, theorem 3.6 implies that (M, Λ, E) is a Jacobi manifold if and only if $\mathrm{graph}(\Lambda, E)^{\#}$ is a Dirac structure for the generalized Courant algebroid $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), (0, 1) + (0, 0))$. In other words,

$$(\mathrm{graph}(\Lambda, E)^{\#}, [\![\cdot,\cdot]\!]_{|\mathrm{graph}(\Lambda,E)^{\#}}, \pi_{|\mathrm{graph}(\Lambda,E)^{\#}}),$$

with $[\![\cdot,\cdot]\!]$ given by (8), is a Lie algebroid over M.

A simple computation shows that,

Lemma 3.8. *For any sections (α, f) and (β, g) of $T^*M \times \mathbb{R}$, the Lie bracket on the Lie algebroid* $\mathrm{graph}(\Lambda, E)^{\#}$ *takes the following form:*

$$\begin{aligned}[\![(\Lambda, E)^{\#}(\alpha, f) + (\alpha, f), (\Lambda, E)^{\#}(\beta, g) + (\beta, g)]\!]_{|\mathrm{graph}(\Lambda,E)^{\#}} &= \\ = [(\Lambda, E)^{\#}(\alpha, f), (\Lambda, E)^{\#}(\beta, g)] + [(\alpha, f), (\beta, g)]_{(\Lambda,E)}. \qquad (10)\end{aligned}$$

Hence, on $\mathrm{graph}(\Lambda, E)^{\#}$, the skew-symmetric product (8) factorizes in two operations, one defined on $TM \times \mathbb{R}$ and the other on $T^*M \times \mathbb{R}$, which are dual to each other. This result will be very useful for us in the next section when studying the deformation of the Lie algebroid structure by Nijenhuis operators.

4. Nijenhuis operators on Lie algebroids and generalized Lie bialgebroids and their doubles

The concept of Nijenhuis operator on Lie algebroids is well known ([14, 8]) as a simple generalization of the usual concept for vector fields.

Definition 4.1. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid. A linear transformation*

$$N : A \to A$$

is said to be a Nijenhuis transformation on A if and only if the torsion tensor $\mathcal{T}_N$ defined as

$$\mathcal{T}_N(X, Y) = [N(X), N(Y)] - N([X, N(Y)]) - N([N(X), Y]) + N^2([X, Y]) \qquad (11)$$

vanishes for any $X, Y \in \Gamma A$.

This is equivalent to have a new Lie structure on the sections of A, N being a homomorphism for them. But in the case of a Lie algebroid, this also implies that another Lie algebroid structure is available for A.

Lemma 4.2 ([8])**.** *Consider a Lie algebroid $(A, [\cdot,\cdot], \rho)$ and a Nijenhuis operator $N : A \to A$. Define the following bracket on the sections of A:*

$$[X, Y]_N = -N([X, Y]) + [X, N(Y)] + [N(X), Y]. \qquad (12)$$

Consider the mapping

$$\hat{N} = \rho \circ N : A \to TM. \tag{13}$$

Then, $(A, [\cdot,\cdot]_N, \hat{N})$ *is a Lie algebroid. This new Lie algebroid has a new exterior derivative, defined as*

$$d^N = [i_N, d] = i_N \circ d - d \circ i_N, \tag{14}$$

where i_N *is the superderivation of degree zero on the forms* $\Gamma(\bigwedge A^*)$ *defined by*

$$i_N\theta(X_1, \cdots, X_p) = \sum_{i=1}^{p} \theta(X_1, \cdots, NX_i, X_{i+1}, \cdots, X_p). \tag{15}$$

The new structure leads also to a new Schouten bracket:

Proposition 4.3 ([14]). *Let* N *be a Nijenhuis operator on a Lie algebroid* A*. The Schouten bracket defined on* $\Gamma(\bigwedge A)$ *by extension of the deformed bracket* $[\cdot,\cdot]_N$ *satisfies*

$$[Q, Q']_N = [i_{N^*}Q, Q'] + [Q, i_{N^*}Q'] - i_{N^*}[Q, Q'], \tag{16}$$

where i_{N^*} *is the superderivation of degree zero on the sections of* $\bigwedge A$ *defined as:*

$$i_{N^*}\Psi(\theta_1, \cdots, \theta_p) = \sum_{i=1}^{p} \Psi(\theta_1, \cdots, N^*\theta_i, \theta_{i+1}, \cdots, \theta_p). \tag{17}$$

Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid with a 1-cocycle ϕ and N a Nijenhuis operator on A. Then,

Lemma 4.4. $N^*\phi$ *is a* 1*-cocycle for the Lie algebroid* $(A, [\cdot,\cdot]_N, \hat{N})$*.*

Proof: Let X and Y be any sections of A. Then, using (1) we know that

$$\begin{aligned}
\langle N^*\phi, [X,Y]_N\rangle &= \langle \phi, N[X,Y]_N\rangle = \langle \phi, [NX, NY]\rangle \\
&= \rho(NX)\langle \phi, NY\rangle - \rho(NY)\langle \phi, NX\rangle \\
&= \hat{N}(X)\langle N^*\phi, Y\rangle - \hat{N}(Y)\langle N^*\phi, X\rangle,
\end{aligned}$$

which means that $N^*\phi$ is a 1-cocycle for the Lie algebroid $(A, [\cdot,\cdot]_N, \hat{N})$.

■

Regarding the Lie algebroid $(A, [\cdot,\cdot]_N, \hat{N})$ and the 1-cocycle $\Phi = N^*\phi$, we can proceed as in Section 2. Hence, we can define:

- the deformed Schouten bracket with cocycle. For $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$, it is defined as
  $$[P,Q]_N^\Phi = [P,Q]_N + (p-1)P \wedge (i_\Phi Q) + (-1)^p (q-1)(i_\Phi P) \wedge Q,$$
- the deformed Lie derivative $\mathcal{L}^\Phi_{NX}(\beta) = \mathcal{L}^N_X \beta + \langle \Phi, X\rangle \beta$,
- and the deformed exterior differential $d_N^\Phi(\beta) = d_N\beta + \Phi \wedge \beta$.

The next step is to consider the action of a Nijenhuis transformation on a generalized Lie bialgebroid. Our objective is to apply these concepts to the deformation of Dirac structure in the following sections, therefore we consider now just those situations that may be interesting later.

Consider then a generalized Lie bialgebroid $((A,\phi),(A^*,W))$. In principle, we can consider two different transformations applied on each factor, i.e.:

$$\begin{cases} N : A \to A \\ \Upsilon : A^* \to A^*, \end{cases} \tag{18}$$

such that they define two new Lie algebroid structures on the factors, i.e. $(A, [\cdot,\cdot]_N, \hat{N})$ and $(A^*, [\cdot,\cdot]_\Upsilon, \hat{\Upsilon})$ are Lie algebroids with $\Phi = N^*\phi$ and $\Xi = \Upsilon^* W$ 1-cocycles for A and A^*, respectively. We must verify now whether they define a new generalized Lie bialgebroid. As we know, the transformation on the Lie algebroid structures implies a transformation of the Lie algebroid cohomologies, i.e. there are two new exterior differentials d_N^Φ and $d_{*\Upsilon}^\Xi$ which are the ones to consider.

Definition 4.5. *Consider a generalized Lie bialgebroid $((A,\phi),(A^*,W))$ and a transformation of the type (18), such that N and Υ are Nijenhuis operators on A and A^*, respectively. We say that the pair (N,Υ) defines a trivial deformation of the generalized Lie bialgebroid structure if and only if the pair $\Big(((A,[\cdot,\cdot]_N,\hat{N}),\Phi),((A^*,[\cdot,\cdot]_\Upsilon,\hat{\Upsilon}),\Xi)\Big)$ is a new generalized Lie bialgebroid, i.e. for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge A)$,*

$$d^\Xi_{*\Upsilon}[P,Q]^\Phi_N = [d^\Xi_{*\Upsilon}P,Q]^\Phi_N + (-1)^{p+1}[P, d^\Xi_{*\Upsilon}Q]^\Phi_N. \tag{19}$$

Now we want to consider the deformation of the double of a generalized Lie bialgebroid. In [2] this problem was studied in the case of a Lie bialgebroid. Consider then a generalized Lie bialgebroid $((A,\phi),(A^*,W))$. We saw in [24] that $(A \oplus A^*, \phi + W)$ is a generalized Courant algebroid. We are going to consider transformations of the skew-symmetric structure, and define the notion of Nijenhuis operator for this framework.

As it happens in [2], we do not consider the most general deformations of $A \oplus A^*$, but some particular types which are of interest in the next section to deform Dirac structures.

Let us consider the skew-symmetric bracket (6) on the sections of $A \oplus A^*$. Even if it does not define a Lie algebra structure it is still a skew-symmetric one, and we can still consider a deformation of the corresponding structure, and ask it to be trivial in the sense of being homomorphic to the original one.

Definition 4.6. *Consider the double of a generalized Lie bialgebroid* $(B = A \oplus A^*, \phi + W)$ *and the bracket (6). Consider a deformation of the structure in the form:*

$$[\![b_1, b_2]\!]_\lambda = [\![b_1, b_2]\!] + \lambda [\![b_1, b_2]\!]_{\mathcal{N}} \qquad \forall b_1, b_2 \in \Gamma B$$

where $[\![\cdot, \cdot]\!]_{\mathcal{N}} = [\![\mathcal{N}\cdot, \cdot]\!] + [\![\cdot, \mathcal{N}\cdot]\!] - \mathcal{N}[\![\cdot, \cdot]\!]$ *and* $\mathcal{N} : \Gamma B \to \Gamma B$ *is a linear operator. Consider also a deformation of the set of sections of* B *in the form:*

$$T_\lambda = \mathrm{Id}_{\Gamma B} + \lambda \mathcal{N}.$$

Then, we say that the deformation is trivial if

$$T_\lambda [\![b_1, b_2]\!]_\lambda = [\![T_\lambda b_1, T_\lambda b_2]\!] \quad \forall b_1, b_2 \in \Gamma B.$$

We also can consider a certain Nijenhuis-like property for the operator $\mathcal{N}$. Formally, we can still define the Nijenhuis torsion in the way we did above:

$$\mathcal{T}_{\mathcal{N}}(b_1, b_2) = [\![\mathcal{N}(b_1), \mathcal{N}(b_2)]\!] - \mathcal{N}([\![b_1, \mathcal{N}(b_2)]\!]) - \mathcal{N}([\![\mathcal{N}(b_1), b_2]\!]) + \mathcal{N}^2([\![b_1, b_2]\!]) \qquad \forall b_1, b_2 \in \Gamma B. \quad (20)$$

To ask this quantity to vanish identically, is equivalent to ask the operator $\mathcal{N}$ to satisfy:

$$\mathcal{N}[\![b_1, b_2]\!]_{\mathcal{N}} = [\![\mathcal{N} b_1, \mathcal{N} b_2]\!]. \quad (21)$$

In the following, we will also call Nijenhuis operator to an operator $\mathcal{N} : \Gamma B \to \Gamma B$ which satisfies this condition. Of course, it is clear that many of the properties of the pure Nijenhuis operator will not be shared by these, but the formal definition is still possible. It is still true, though, that these operators define trivial deformations of the skew-symmetric structure of B, as the usual Nijenhuis operators define trivial deformations of Lie structures.

If we consider a deformation of a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ by Nijenhuis operators, in the sense discussed above, we know that such a deformation provides new Lie structures on ΓA and ΓA^*, new Lie derivatives and new exterior differentials. This implies that we can consider the analogue of the bracket (6), for the new generalized Courant algebroid structure, i.e.

$$\begin{aligned}[\![X_1+\alpha_1, X_2+\alpha_2]\!]_{\mathcal{N}} &= \left([X_1,X_2]_N^{\Phi} + \mathcal{L}^{\Xi}_{*\Upsilon\alpha_1}X_2 - \mathcal{L}^{\Xi}_{*\Upsilon\alpha_2}X_1 - \frac{1}{2}d^{\Xi}_{*\Upsilon}(e_1,e_2)_-\right) \\ &\quad + \left([\alpha_1,\alpha_2]^{\Xi}_{\Upsilon} + \mathcal{L}^{\Phi}_{NX_1}\alpha_2 - \mathcal{L}^{\Phi}_{NX_2}\alpha_1 + \frac{1}{2}d^{\Phi}_N(e_1,e_2)_-\right), \end{aligned} \tag{22}$$

with $e_1 = X_1+\alpha_1$ and $e_2 = X_2+\alpha_2$, and where the brackets $[\cdot,\cdot]^{\Phi}_N$ and $[\cdot,\cdot]^{\Xi}_{\Upsilon}$ are the brackets associated (via the corresponding cocycles $\Phi = N^*\phi$ and $\Xi = \Upsilon^* W$) to the Lie algebroids $(A, [\cdot,\cdot]_N, \hat{N})$ and $(A^*, [\cdot,\cdot]_{\Upsilon}, \hat{\Upsilon})$.

But we can also try to see this new bracket as the result of a deformation of the original generalized Courant algebroid structure of $B = A\oplus A^*$, in the sense of Definition 4.6 using the bracket above as the linear term of the deformation. Then, the condition for the deformation $(T_\lambda, [\cdot,\cdot]_\lambda)$ to be trivial will be that the operator $\mathcal{N}$ in T_λ is a Nijenhuis operator for the original bracket. This implies:

$$\begin{aligned}&[\![X_1+\alpha_1, X_2+\alpha_2]\!]_{\mathcal{N}} = \\ &[\![\mathcal{N}(X_1+\alpha_1), X_2+\alpha_2]\!] + [\![X_1+\alpha_1, \mathcal{N}(X_2+\alpha_2)]\!] - \mathcal{N}[\![X_1+\alpha_1, X_2+\alpha_2]\!]\end{aligned}$$

and

$$\mathcal{N}[\![X_1+\alpha_1, X_2+\alpha_2]\!]_{\mathcal{N}} = [\![\mathcal{N}(X_1+\alpha_1), \mathcal{N}(X_2+\alpha_2)]\!].$$

Surprisingly, it turns out that not all deformations of the Lie algebroids A and A^* define also a deformation for the double. For instance, if we consider the usual case ($\phi = 0 = W$) some conditions arise from the double structure:

Theorem 4.7 ([2]). *Consider the double of a Lie bialgebroid $A\oplus A^*$ and consider two Nijenhuis operators N for A and Υ for A^*. Then, if*

- $N^* + \Upsilon = 2\lambda_1$ *where* $\lambda_1 \in \mathbb{R}$,
- $N^2 = \lambda_2$ *where* $\lambda_2 \in \mathbb{R}$,

then $\mathcal{N} = N \times \Upsilon : A\oplus A^* \to A\oplus A^*$ *is a Nijenhuis operator for* $A\oplus A^*$.

5. Generalized Dirac-Nijenhuis structures and Jacobi-Nijenhuis manifolds

5.1. Dirac-Nijenhuis structures for generalized Lie bialgebroids. In [2] we studied the concept of Dirac-Nijenhuis structures in the context of usual Lie bialgebroids. We considered them associated with deformations of the Lie algebroid structure that Dirac bundles are endowed with. We introduced two slightly different types of these structures depending on whether or not the Courant algebroid structure was affected by the deformation. In this paper, we intend to consider the analogue notion in the case of generalized Courant algebroids. As most of the results are proven in a completely analogous way, we refer the interested reader to [2] for a more detailed exposition.

At a general level, we will consider the analogue [2] for a Dirac structure D defined on the double of a generalized Lie bialgebroid $((A, \phi), (A^*, W))$. We can consider two different frameworks:

- A transformation that defines a trivial deformation of the skew-symmetric operation (6) on the double of the generalized Lie bialgebroid and defines a new generalized Courant algebroid structure on it, and, as a consequence, transforms the Lie algebroid structure of D into a new one.
- A transformation that deforms the skew-symmetric operation (6) in such a way that it defines a trivial deformation of the Lie algebroid structure of D. In this case, we do not care about the transformation of the generalized Courant algebroid structure of the double.

Therefore, we can define:

Definition 5.1. *Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid over a differentiable manifold M. Then, $(D, \mathcal{N})$ is said to be a* **generalized Dirac-Nijenhuis structure of type I** *(or M to be a* **generalized Dirac-Nijenhuis manifold of type I***) if D is a Dirac structure for the generalized Lie bialgebroid $((A, \phi), (A^*, W))$ and the operator $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ preserves D (i.e. $\mathcal{N}(D) \subset D$) and defines:*

- *a trivial deformation of the skew-symmetric algebra of $A \oplus A^*$ whose Nijenhuis torsion vanishes (with respect to the skew-symmetric bracket defined on $A \oplus A^*$) and*

- *a new skew-symmetric operation* $[\![\cdot,\cdot]\!]_{\mathcal{N}}$ *for which* D *is also a Dirac structure with respect to the generalized Lie bialgebroid* $((A, N^*\phi), (A^*, \Upsilon^* W))$.

But we also saw that we have the possibility of deforming the Lie algebroid structure of D without paying attention to the structure of the double. Hence, it makes sense to define:

Definition 5.2. *Let* $((A,\phi),(A^*,W))$ *be a generalized Lie bialgebroid over a differentiable manifold* M. *Then,* $(D,\mathcal{N})$ *is said to be a* **generalized Dirac-Nijenhuis structure of type II** *(or* M *to be a* **generalized Dirac-Nijenhuis manifold of type II***) if* D *is a Dirac structure with respect to the generalized Lie bialgebroid* $((A,\phi),(A^*,W))$ *and the operator* $\mathcal{N} : A\oplus A^* \to A\oplus A^*$ *defines a trivial deformation of the Lie algebroid structure of* D, *i.e. such that the Nijenhuis torsion of* $\mathcal{N}$ *with respect to the bracket (6) vanishes identically on* ΓD. *Of course we also assume that the transformation* $\mathcal{N}$ *preserves* D.

From a formal point of view, the definitions of Dirac-Nijenhuis manifolds are obviously not affected by the introduction of cocycles in the skew-symmetric operation of the double. These changes do affect the bundles which are closed under them and that satisfy the definition, but not the formal definition itself. Being interested only in Jacobi manifolds, only the second definition is relevant for us in the present paper and henceforth we will postpone the analysis of type I manifolds to a future paper.

5.2. Jacobi-Nijenhuis manifolds. The original definition of Jacobi-Nijenhuis manifold was given by Marrero *et al.* in [22]. In [23], we introduced a stricter definition and called it *strict Jacobi-Nijenhuis manifold*, which is the one we are going to consider in this paper. For simplicity, we will omit the word strict.

Let M be a differentiable manifold and $N : TM\times\mathbb{R} \to TM\times\mathbb{R}$ a Nijenhuis operator on the Lie algebroid $(TM\times\mathbb{R}, [\cdot,\cdot], \pi)$, with $[\cdot,\cdot]$ given by (2).

Suppose thus that M is equipped with a Jacobi structure (Λ, E) and a Nijenhuis operator N and consider a bivector field Λ_1 and a vector field E_1 on M, defined by

$$(\Lambda_1, E_1)^\# = N \circ (\Lambda, E)^\#.$$

Definition 5.3. *A* Jacobi-Nijenhuis manifold $(M,(\Lambda,E),N)$ *is a Jacobi manifold* (M,Λ,E) *with a Nijenhuis operator* N *satisfying the following compatibility conditions:*

$$N\circ(\Lambda,E)^{\#}=(\Lambda,E)^{\#}\circ N^{*} \tag{23}$$

and

$$\mathcal{C}((\Lambda,E),N)=0, \tag{24}$$

where

$$\begin{aligned}\mathcal{C}((\Lambda,E),N)((\alpha,f),(\beta,g)) &= [(\alpha,f),(\beta,g)]_{(\Lambda_1,E_1)}-[N^*(\alpha,f),(\beta,g)]_{(\Lambda,E)}\\ &- [(\alpha,f),N^*(\beta,g)]_{(\Lambda,E)}+N^*[(\alpha,f),(\beta,g)]_{(\Lambda,E)},\end{aligned}$$

for all $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$*, and* $[\cdot,\cdot]_{(\Lambda,E)}$ *(resp.* $[\cdot,\cdot]_{(\Lambda_1,E_1)}$*) is the bracket (4) associated with* (Λ,E) *(resp.* (Λ_1,E_1)*).*

We see that the structure is very similar to the case of Poisson-Nijenhuis structure which was proven to be a Dirac-Nijenhuis structure of type II in [2]. Therefore, as we had to move from Courant algebroids to generalized Courant algebroids to describe Jacobi manifolds as Dirac bundles defined on them, it is reasonable to expect that Jacobi-Nijenhuis structures should be seen as examples of generalized Dirac-Nijenhuis structures of type II. The next theorem establishes this fact.

Theorem 5.4. *Let* $(M,(\Lambda,E),N)$ *be a Jacobi-Nijenhuis manifold. Then,*

$$(\mathrm{graph}(\Lambda,E)^{\#},\mathcal{N}=N\times N^*)$$

is a generalized Dirac-Nijenhuis structure of type II for the generalized Courant algebroid $(TM\times\mathbb{R}\oplus T^*M\times\mathbb{R},(0,1)+(0,0))$*.*

Proof: Let (α,f) and (β,g) be two arbitrary sections of $T^*M\times\mathbb{R}$. Since

$$\begin{aligned}\mathcal{N}((\Lambda,E)^{\#}(\alpha,f)+(\alpha,f)) &= N((\Lambda,E)^{\#}(\alpha,f))+N^*(\alpha,f)\\ &\overset{(23)}{=} (\Lambda,E)^{\#}(N^*(\alpha,f))+N^*(\alpha,f),\end{aligned}$$

$\mathcal{N}$ preserves graph $(\Lambda,E)^{\#}$. Hence, we just need to prove that $\mathcal{N}$ is a Nijenhuis operator on graph$(\Lambda,E)^{\#}$.

Moreover, we know from Lemma 3.8 that the deformed skew-product of the generalized Courant algebroid factorizes. Then we have

$$\begin{aligned}
&[\![\mathcal{N}((\Lambda,E)^{\#}(\alpha,f)+(\alpha,f)),\mathcal{N}((\Lambda,E)^{\#}(\beta,g)+(\beta,g))]\!] = \\
&\quad = [\![(\Lambda,E)^{\#}(N^*(\alpha,f))+N^*(\alpha,f),(\Lambda,E)^{\#}(N^*(\beta,g))+N^*(\beta,g)]\!] \\
&\quad \overset{(10)}{=} [N(\Lambda,E)^{\#}(\alpha,f),N(\Lambda,E)^{\#}(\beta,g)] + [N^*(\alpha,f),N^*(\beta,g)]_{(\Lambda,E)}
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\mathcal{N}([\![(\Lambda,E)^{\#}(\alpha,f)+(\alpha,f),(\Lambda,E)^{\#}(\beta,g)+(\beta,g)]\!]_{\mathcal{N}}) = \\
&= \mathcal{N}([\![\mathcal{N}((\Lambda,E)^{\#}(\alpha,f)+(\alpha,f)),(\Lambda,E)^{\#}(\beta,g)+(\beta,g)]\!] \\
&\quad +[\![(\Lambda,E)^{\#}(\alpha,f)+(\alpha,f),\mathcal{N}((\Lambda,E)^{\#}(\beta,g)+(\beta,g))]\!] \\
&\quad -\mathcal{N}([\![(\Lambda,E)^{\#}(\alpha,f)+(\alpha,f),(\Lambda,E)^{\#}(\beta,g)+(\beta,g)]\!])) \\
&= \mathcal{N}([\![(\Lambda,E)^{\#}(N^*(\alpha,f))+N^*(\alpha,f),(\Lambda,E)^{\#}(\beta,g)+(\beta,g)]\!] \\
&\quad +[\![(\Lambda,E)^{\#}(\alpha,f)+(\alpha,f),(\Lambda,E)^{\#}(N^*(\beta,g))+N^*(\beta,g)]\!]) \\
&\quad -N^2([(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(\beta,g)]) - (N^*)^2([(\alpha,f),(\beta,g)]_{(\Lambda,E)}) \\
&= N([(\Lambda,E)^{\#}(N^*(\alpha,f)),(\Lambda,E)^{\#}(\beta,g)]) + N^*([(N^*(\alpha,f)),(\beta,g)]_{(\Lambda,E)}) \\
&\quad +N([(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(N^*(\beta,g))]) + N^*([(\alpha,f),(N^*(\beta,g))]_{(\Lambda,E)}) \\
&\quad -N^2([(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(\beta,g)]) - (N^*)^2([(\alpha,f),(\beta,g)]_{(\Lambda,E)}) \\
&= N([(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(\beta,g)]_N) + N^*([(\alpha,f),(\beta,g)]_{(\Lambda,E)})_{N^*}),
\end{aligned} \tag{26}$$

where, for the sake of simplicity we write $[\![\cdot,\cdot]\!]$ instead of $[\![\cdot,\cdot]\!]_{|\mathrm{graph}(\Lambda,E)^{\#}}$.

The fact that N is a Nijenhuis operator ensures the equality of the first terms of (25) and (26). On the other hand, the vanishing of the Nijenhuis torsion of N and of the concomitant $\mathcal{C}((\Lambda,E),N)$, guarantee that

$$[N^*(\alpha,f),N^*(\beta,g)]_{(\Lambda,E)} = N^*([(\alpha,f),(\beta,g)]_{(\Lambda,E)})_{N^*}.$$

So,

$$\begin{aligned}
&\mathcal{N}([\![(\Lambda,E)^{\#}(\alpha,f)+(\alpha,f),(\Lambda,E)^{\#}(\beta,g)+(\beta,g)]\!]_{\mathcal{N}}) = \\
&\quad = [\![\mathcal{N}((\Lambda,E)^{\#}(\alpha,f)+(\alpha,f)),\mathcal{N}((\Lambda,E)^{\#}(\beta,g)+(\beta,g))]\!]
\end{aligned}$$

and $\mathcal{N}_{|\mathrm{graph}(\Lambda,E)^{\#}}$ is a Nijenhuis operator for the Lie algebroid graph $(\Lambda,E)^{\#}$.

■

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