ON THE CORNERS OF CERTAIN DETERMINANTAL RANGES

NATÁLIA BEBIANO, JOÃO DE PROVIDÊNCIA AND ALEXANDER KOVAČEC

Abstract: Let $A$ be a complex $n \times n$ matrix and let $SO(n)$ be the group of real orthogonal matrices of determinant one. Define $\Delta(A) = \{ \det(A \ast Q) : Q \in SO(n) \}$, where $\ast$ denotes the Hadamard product of matrices. For a permutation $\sigma$ on $\{1, \ldots, n\}$, define $z_\sigma = d_\sigma(A) = \prod_{i=1}^{n} a_{i\sigma(i)}$. It is shown that if the equation $z_\sigma = \det(A \ast Q)$ has in $SO(n)$ only the obvious solutions ($Q = (\varepsilon_i \delta_{\sigma(i),j})$, $\varepsilon_i = \pm 1$ such that $\varepsilon_1 \cdots \varepsilon_n = \text{sgn} \sigma$), then the local shape of $\Delta(A)$ in a vicinity of $z_\sigma$ resembles a truncated cone whose opening angle equals $\hat{z}_\sigma 1 \hat{z}_\sigma 2$, where $\sigma_1, \sigma_2$ differ from $\sigma$ by transpositions. This lends further credibility to the well known de Oliveira Marcus Conjecture (OMC) concerning the determinant of the sum of normal $n \times n$ matrices. We deduce the mentioned fact from a general result concerning multivariate power series and also use some elementary algebraic topology.

1. Introduction

a. Notation. Our notation is standard where advisable. Here are listed in telegram style the notations and definitions that may need clarification.

$\mathbb{R}_{\geq 0}; \mathbb{R}_{>0}; \hat{\mathbb{R}}$, etc. reals $\geq 0$; $(\mathbb{R}_{>0})^n$; extended reals: $\mathbb{R} \cup \{\infty\}$; etc.

$S_n; T; i \in \tau$ symmetric group on $\{1, \ldots, n\}$; set $T = \{(i, j) : 1 \leq i < j \leq n\}$ often identified with the set of transpositions in $S_n$; $i \in \tau = \langle k, l \rangle \in T$ means $i = k$ or $i = l$.

so$(n), su(n)$ the Lie-algebras of (real) skew-symmetric and (complex) skew-hermitian $n \times n$ matrices of trace 0.

SO$(n), SU(n)$ Lie-groups of orthogonal and unitary $n \times n$ matrices of determinant 1.

$A; Q$ an arbitrary $n \times n$ complex matrix mostly fixed, a matrix in $SO(n)$ respectively.

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the diagonal product of matrix $M$ associated to permutation $\sigma$. $d_\sigma(M) = \prod_{i=1}^{n} m_{i\sigma(i)}$; in particular $d_{id}(M) = m_{11}m_{22}\ldots m_{nn}$. For the particular matrix $A$ mentioned before, we sometimes use $z_\sigma := d_\sigma(A)$.

Mostly the norm of an element $u$ in a normed space; $\mathbb{R}^n$, $\mathbb{C}$ carry euclidean norm.

$B(z, \rho)$, $B(x, \rho)$ open balls of radius $\rho > 0$ centers $z$ or $x$, in $\mathbb{C}$ or $\mathbb{R}^n$ respectively.

$|B|$, $P_\sigma$, $P_\sigma$ the matrix $([|b_{ij}|])$; for $\sigma \in S_n$ the matrix $([\delta_{\sigma i, j}])$; the set $\{Q \in SO(n) : |Q| = P_\sigma\}$.

$A \ast B$ the Hadamard product of matrices $A, B$ of same size: $(A \ast B)_{ij} = a_{ij}b_{ij}$.

lhs(.), rhs(.), mid(.) left hand side, right hand side, mid of an equation.

$l^+$; $px^+$; $px$ a ray; for points $p, x$, the ray with origin $p$ containing $x$; segment joining $p$ to $x$.

$f \simeq g$; $X \approx Y$ homotope maps; homoeomorphic spaces.

$clX$, or $\overline{X}$ the topological closure of a subset $X$ of the plane.

$p, x, 0$; $z, 0$ points $p, x, 0$ in the complex plane; a point in $\mathbb{R}^n$, dimension $n$ will follow from context; the zero of $\mathbb{R}^n$.

$\min b$; $\max b$ minimum/maximum of entries of real $n$-tuple $b = (b_1, \ldots, b_n)$.

[SW, p45c-3] example of reference to book or article: see [SW] page 45, about 3cm from last text row.

$cone Z$; $co Z$ for a set $Z \subseteq \mathbb{C}$, the set (cone) $\{\sum_{i=1}^{k} r_i z_i : k \in \mathbb{Z}_{\geq 1}, r_i \geq 0, z_i \in Z\}$; the similarly constructed set (convex hull) with additional restriction $\sum_i r_i = 1$. 
monomial \( c_i x_i^{|i|} \) an expression of the form \( c_{i_1 i_2 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \).

\(|i| = i_1 + \ldots + i_n\) is its degree.

powerseries a sum of possibly infinitely many monomials formally summed in any order.

b. Content and outline of results. Let \( A = (a_{ij}) \) be a complex \( n \times n\)-matrix. Since \( \text{SO}(n) = \text{Lie group of unitary } n \times n \text{ matrices of determinant } 1 \) is a compact connected set [SW, pp104c-4, 147c-1], the region \( \Delta(A) = \{ \det(A \cdot Q) : Q \in \text{SO}(n) \} \) is a compact connected set in the complex plane. Let \( z_\sigma = z_\sigma(A) = \prod_{i=1}^{n} a_{i \sigma_i} \) be the (unsigned) diagonal product of \( A \) associated to \( \sigma \in S_n \). The following formulation of a slightly weakened form of the Oliveira Marcus Conjecture [dO] appears first in [QK]; OMC itself claims the same thing to be true even if \( \Delta(A) \) is defined using \( \text{SU}(n) \) instead of \( \text{SO}(n) \).

Conjecture (OMC for \( \text{SO}(n) \)). If \( A \) is a rank 2 matrix, then

\[ \Delta(A) \subseteq \text{co}\{z_\sigma(A) : \sigma \in S_n\}. \]

Example. Although experiments indicate that the inclusion seems to remain true in many cases in which \( \text{rank} A > 2 \), this is not so in general: consider the case \( A = \text{diag}(1,1,1) \) and choose \( Q \) as the matrix at the left.

In this article we prove a result, see theorem 11, related to the shape of \( \Delta(A) \) near points \( z_\sigma(A) \in \mathbb{C} \).

In section 2 we compute the first terms of the power series \( \det(A \cdot \exp S) \) in the real and imaginary parts of the entries of \( S \in \text{su}(n) \) around the zero matrix. The salient feature is that the nontrivial homogeneous component of lowest degree of this series is a linear combination of the squares of these parts with coefficients that are simple expressions in the \( d_\sigma(A) \). Section 3 defines the concept of a corner of a region in the plane. An archetypical corner is a disk-sector of angle measure \( < \pi \). We show that under natural restrictions a set valued map defined on such a sector and deviating from the identity by small enough a quantity as its argument approaches its vertex has as image region approximately the sector. The proof employs some elementary algebraic topology. Section 4 gives a lemma on power series of the type
encountered for $\det(A^* \exp S)$. It assures that such power series defines in a natural manner a set valued map of the type considered previously. This is used to deduce the main result, theorem 11, in section 5. We end with some remarks.

2. A Power Series

Recall that $\text{so}(n) = \text{Lie-algebra of real skew-symmetric } n \times n \text{ matrices } S \text{ of trace 0 is associated to } \text{SO}(n) \text{ via the exponential map: indeed, by } [\text{SW, p}147c-2] \text{ (or } [\text{BtD, p}165c4]), \text{ every } Q \in \text{SO}(n) \text{ can be written } Q = \exp(S) \text{ for some } S \in \text{so}(n). \text{ Hence}

$$\Delta(A) = \{\det(A^* Q) : Q \in \text{SO}(n)\} = \{\det(A^* \exp S) : S \in \text{so}(n)\}.$$  

For the proper understanding of the theory of absolutely summable series in a Banach space, and in particular function spaces and power series, as referred below, see [D, pp. 94-5, 127-8, 193-7]. For the formal background to these (of lesser importance here), see [ZS].

Note that the matrices $S \in \text{su}(n)$ are precisely the matrices of the form $S = A + iB$ where $A$ is a skew symmetric with zero diagonal and $B$ is symmetric of trace 0. Hence there enter $(n^2 - n)/2 + (n^2 - n)/2 + (n - 1) = n^2 - 1$ real variables. By a polynomial in the entries of $S$, we mean a polynomial in these real variables; in particular that the square of the modulus of such entries is a polynomial of degree 2 in these variables. Finally recall that if $\tau = \langle i, j \rangle \in T$, then we permit $s_\tau$ as a shorthand for $s_{ij}$, $i < j$.

**Theorem 1.** Let $A$ be a complex $n \times n$ matrix and let $S$ be a matrix in $\text{su}(n)$. For $\tau \in T$ put $d_\tau(A) = d_\tau(A) - d_{id}(A)$. Then we have a development

$$\det(A^* \exp(S)) = d_{id}(A) + \sum_{\tau \in T} d_\tau(A)|s_\tau|^2 + \sum_{k \geq 3} p_k(S).$$

Here each $p_k(S)$ as well as $|s_\tau|^2$ is either 0 or a homogeneous polynomial of degree $k$ respectively 2, in $\leq n^2 - 1$ real variables. There is for any neighbourhood $U_0$ of the zero (matrix) in $\text{su}(n) \approx \mathbb{R}^{n^2-1}$, a constant $M$, so that for every monomial $m(\cdot)$ occurring in this power series, and every $S \in U_0$, there holds $|m(S)| \leq M$.

**Proof.** Since the matrix $S = (s_{ij})$, satisfies for all $i, j \in \{1, \ldots, n\}$, the relations $s_{ij} = -\overline{s}_{ji}$, in particular $s_{ii} \in \sqrt{-1}\mathbb{R}$, we find that the $(i, i)$-entry of $S^2$ is given by
\[ \sum_{\nu=1}^{n} s_{iv} s_{vi} = -|s_{ii}|^2 - \sum_{\tau : i \in \tau} |s_{\tau}|^2. \]

Since \( \exp S = I + S + \frac{1}{2} S^2 + \ldots \), and since the nonzero entries of \( S^k \) are homogeneous polynomials of degree \( k \) in the \( s_{ij} \), we find

\[
(\exp S)_{ij} = \begin{cases} 
1 + s_{ii} - \frac{1}{2} |s_{ii}|^2 - \frac{1}{2} \sum_{\tau : i \in \tau} |s_{\tau}|^2 + p_{ii}(S) & \text{if } i = j \\
 s_{ij} + p_{ij}(S) & \text{if } i \neq j,
\end{cases}
\]

where the power series \( p_{ii}(S) \) has under-degree \( \geq 3 \), while for \( i \neq j \), \( p_{ij}(S) \) has under-degree \( \geq 2 \).

From this we extract information about the diagonal products \( d_{\sigma}(\exp S) \). First, using \( \sum_i s_{ii} = 0 \), and hence also

\[ 0 = (\sum_i s_{ii})^2 = 2 \sum_{l<k} s_{ll}s_{kk} - \sum_i |s_{ii}|^2, \]

we find

\[
d_{id}(\exp S) = \prod_{i=1}^{n} (1 + s_{ii} - \frac{1}{2} |s_{ii}|^2 - \frac{1}{2} \sum_{\tau : i \in \tau} |s_{\tau}|^2 + p_{ii}(S))
\]

\[ = 1 + \sum_{i} s_{ii} + \sum_{i \neq j} s_{ii}s_{jj} - \frac{1}{2} \sum_{i} |s_{ii}|^2 - \frac{1}{2} \sum_{i} \sum_{\tau : i \in \tau} |s_{\tau}|^2 + p_{id}(S) \]

\[ = 1 - \frac{1}{2} \sum_{i} \sum_{\tau : i \in \tau} |s_{\tau}|^2 + p_{id}(S) \]

\[ = 1 - \sum_{\tau \in T} |s_{\tau}|^2 + p_{id}(S), \]

where the power series \( p_{id}(S) \) has underdegree \( \geq 3 \).

The diagonal products corresponding to transpositions are given as follows.

\[
d_{(i,j)}(\exp S) = \left( \prod_{l \neq i,j}^{n} (1 + s_{ll} - \frac{1}{2} |s_{ll}|^2 - \frac{1}{2} \sum_{\tau : l \in \tau} |s_{\tau}|^2 + p_{ll}(S)) \right) \times (s_{ij} + p_{ij}(S))(-s_{ij} + p_{ji}(S))
\]

\[ = -|s_{ij}|^2 + p'_{ij}(S), \]
where \( p'_{ij}(S) \) has underdegree \( \geq 3 \). Finally, what concerns the diagonal products corresponding to \( \sigma \notin \{\text{id}\} \cup \mathcal{T} \), the set \( \{i : \sigma(i) \neq i\} \) contains at least three elements. It follows that an associated diagonal product yields a power series of underdegree \( \geq 3 \).

Consequently

\[
\det(A \cdot \exp S) = \sum_{\sigma \in S_n} \text{sgn} \, d_\sigma(A) d_\sigma(\exp S) \\
= d_{\text{id}}(A)(1 - \sum_{\tau} |s_\tau|^2 + p_{\text{id}}(S)) \\
- \sum_{\tau \in \mathcal{T}} d_\tau(A)(-|s_\tau|^2 + p'_\tau(S)) + \sum_{\sigma \notin \mathcal{T} \cup \{\text{id}\}} \text{sgn} \, d_\sigma(A) d_\sigma(\exp S).
\]

This formula and the degree properties of \( p_{\text{id}}(S), p'_\tau(S), d_\sigma(\exp S) \) imply the formal expression given for \( \det(A \cdot \exp S) \). Now each of the \( n^2 \) functions \( \text{su}(n) \ni S \mapsto (\exp S)_{ij}, i, j = 1, \ldots, n \), is a power series of complex coefficients in \( n^2 - 1 \) real variables. Since the exponential series converges absolutely on \( U_0 \) [SW, p25], the family of monomials in these variables occurring in the power series \( (\exp S)_{ij} \) is absolutely (or normally) summable on \( U_0 \) in the sense of [D, p95c7, p128]. Since \( \det(.) \) is a polynomial in the entries of a matrix, the claim concerning \( m(S) \) is easily inferred. \( \square \)

3. A set valued map

Definitions 2. a. Call a cone in the sense of the notation section degenerate if it is one of these: the plane \( \mathbb{C} \), a half plane, a ray, or a straight line.

b. A closed (convex) non-degenerate cone will be called a \textit{cnd-cone}, for short. It is an exercise in plane geometry to show that a cnd-cone can be uniquely written in the form \( C = \text{cone}\{e^{i\theta_1}, e^{i\theta_2}\} \) with \( \theta_1, \theta_2 \in ]-\pi, \pi] \), satisfying \( 0 < \alpha = \min\{2\pi - |\theta_1 - \theta_2|, |\theta_1 - \theta_2|\} < \pi \). The real \( \alpha \) is the usual measure of the angle the cone defines.

c. An angular region (or cone) at \( z \) is a set given by \( \text{ar} = z + C \), with \( C \) a cnd-cone.

d. The (disk-)sector of radius \( \rho \) given by this ar is \( S(\text{ar}, \rho) = \text{ar} \cap B(z, \rho) \).

e. Let \( \text{ar} \) be a (nondegenerate) angular region at \( z \) with angle \( \alpha > 0 \) and let \( \varepsilon > 0 \) be such that \( 0 < \alpha - 2\varepsilon < \alpha < \alpha + 2\varepsilon < \pi \). We call the two angular regions with the same vertex \( z \) and bissector as \( \text{ar} \), but by a small
angle $2\varepsilon > 0$ smaller/wider than $\alpha$ the $\varepsilon$-contraction $\text{ar}_-\varepsilon$ / $\varepsilon$-extension $\text{ar}_+\varepsilon$ of $\text{ar}$.

The central definition for this paper is that of a corner of a subset of the plane.

**Definition 3.** Let $\Delta$ be a subset of $\mathbb{C}$, and let $z \in \Delta$. The point $z$ is called a *corner* of $\Delta$, if there exists a nondegenerate angular region $\text{ar}$ at $z$ such that for every small $\varepsilon > 0$:

there exists a $\delta > 0$ so that $S(\text{ar}_-\varepsilon, \delta) \subseteq \Delta \cap B(z, \delta) \subseteq S(\text{ar}_+\varepsilon, \delta)$.

In this case we also may say $\Delta$ has in $z$ the corner $\text{ar}$.

**Example 4.** The idea of what a corner is, can be gleaned from the following series of pictures: the shaded regions (a) and (b) have in $z$ corners whose angular regions $\text{ar}$ are indicated by tangent lines. The region (c) has in $z$ no corner. Similarly region (d) has in $z$ no corner, since it has a sequence of ‘holes’ converging towards $z$. Assume a boundary curve of $\Delta$ near $z$ exists. If it is strictly convex (‘inward bounded’) then as $\varepsilon \to 0$, $\delta$ has to go to 0 to satisfy the first inclusion, while if it is concave, $\delta \to 0$ is required to satisfy the second inclusion.

![Figures a, b, c, d](image-url)

**Observations 5.** Let $\Delta, \Delta', \Delta''$ be subsets of the plane.

a. If $\Delta \subseteq \Delta' \subseteq \Delta''$ and $\Delta$ and $\Delta''$ have in $z$ the corner $\text{ar}$ then $\Delta'$ has in $z$ the corner $\text{ar}$.

b. $\Delta$ has in $z$ the corner $\text{ar}$ iff $\Delta \cap B(z, r)$ has for some small $r > 0$ the corner $\text{ar}$.

c. If $\Delta$ has in $z$ the corner $\text{ar}$, then $u + \Delta$ has in $u + z$ the corner $u + \text{ar}$.

**Proof.** The simple considerations necessary are left to the reader. \(\square\)
Let $\mathcal{P}(\mathbb{R}^2) =$family of subsets (i.e. powerset) of $\mathbb{R}^2$.

**Theorem 6.** Let $S = S(\ar, \rho)$ be a disk sector with vertex in 0 and let $F : S \to \mathcal{P}(\mathbb{R}^2)$ be a set valued map with the following further properties:

(i) For some function $r : S \to \mathbb{R}_{\geq 0}$, satisfying $\lim_{x \to 0} r(x)/|x| = 0$ and $r(0) = 0$, there holds $F(x) \subseteq B(x, r(x))$ for all $x \in S$.

(ii) There exists a continuous selection $S \ni x \mapsto f(x) \in F(x)$.

Then for all small $r' > 0$, the set $F(S(\ar, r'))$ has $\ar$ as a corner at 0.

**Proof.**

The left figure shows the boundaries $C_{r_1}, C_{r_2}$ of two disk-sectors which we think of being $\bar{I}_{r_1} = \text{cl} S(\ar_{-\varepsilon}, r_1), \bar{I}_{r_2} = \text{cl} S(\ar, r_2)$. Of $\varepsilon, r_1, r_2$ we require in the moment only that $\varepsilon$ be small enough so that $\ar_{-\varepsilon}$ is nontrivial, and that the radii are assumed to satisfy $0 < r_1 / \cos \varepsilon < r_2 \leq \rho$.

We dispense with proving that $C_{r_1}, C_{r_2}$ are rectifiable curves; that the Jordan curve theorem [M, p31] applies to them; that their respective Jordan-interiors [M, p36c-1; Enc. 93B& K] $I_{r_1}, I_{r_2}$, as well as $\bar{I}_{r_1}, \bar{I}_{r_2}$ are (convex) disk sectors; that $C_{r_2} \setminus \{0\}$ lies in the Jordan-exterior of $C_{r_1}$; and that we have a homeomorphism $\bar{I}_{r_2} \approx \text{closed unit disc}$, which induces a homeomorphism $C_{r_2} \approx S^1$.

Let $L =$perimeter of $C_{r_2}$ and parametrize $C_{r_2}$ by traversing it counterclockwise from 0 to 0 and defining $l : C_{r_2} \to [0, L]$ by $l(x) =$arc-length from 0 to $x$; also let $d(x) =$distance from $x \in C_{r_2}$ to $C_{r_1}$. Note that $l$ is a continuous bijection. Simple geometry, in particular the cosine theorem, yields the following:

$$d(x) = \begin{cases} 
\frac{l(x) \sin(\varepsilon)}{\sqrt{l(x)^2 + r_1^2 - 2l(x)r_1 \cos \varepsilon}} & \text{for } l(x) \in [0, r_1 / \cos \varepsilon], \\
\frac{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(1 + \varepsilon - (l(x)/r_2))}}{r_2 - r_1} & \text{for } l(x) \in [r_1 / \cos \varepsilon, r_2], \\
d(l^{-1}(L - l(x))) & \text{for } l(x) \in [r_2, r_2(1 + \varepsilon)], \\
r_2 - r_1 & \text{for } l(x) \in [r_2(1 + \varepsilon), \frac{L}{2}], \\
r_2 - r_1 & \text{for } l(x) \in [\frac{L}{2}, L]. 
\end{cases}$$
The graph $l(x)$-versus-$d(x)$ for the example shown above is this figure for $l(x) \leq L/2$. The requirement $r_1 / \cos \varepsilon < r_2$ (instead of simply $r_1 < r_2$) was made to simplify analysability of $d(x)$.

We define the function

$$[0, \rho] \ni t \mapsto \tilde{r}(t) := \sup \{r(x) : x \in S, |x| = t\} \in \mathbb{R}_{\geq 0}.$$

From the hypothesis on $r$ we get $*_{\varepsilon} : \lim_{t \downarrow 0} \tilde{r}(t)/t = 0$. Now fix an $\varepsilon$ satisfying $0 < \varepsilon \leq \min \{0.9, \alpha/2, (\pi - \alpha)/2\}$.

Fact 1. For small $r_2$, there exist $r_1 > 0$ so that for $x \in C_{r_2} \setminus \{0\}$, $r(x) < d(x)$.

[By $*_{\varepsilon}$ we find for small $r_2$ $\leq \rho$ that for all $0 < t \leq r_2$, $\tilde{r}(t) < \frac{\sin \varepsilon}{1 + \sin \varepsilon} \cdot t$. Choose such an $r_2$ and put $r_1 = r_2/(1 + \sin \varepsilon)$. Then from the hypothesis on $\varepsilon$ one checks that we have $r_2 > r_1 / \cos \varepsilon > r_1$. Note that for $x \in C_{r_2}$, $|x| = \min \{l(x), r_2\} \leq r_2$. Then from the formulae for $d(x)$ one finds by routine checks for $x \in C_{r_2} \setminus \{0\}$, that $r(x) \leq \tilde{r}(|x|) < \frac{\sin \varepsilon}{1 + \sin \varepsilon} |x| \leq d(x)$.

Let $r_1 < r_2$ be as in fact 1; it implies for $x \in C_{r_2} \setminus \{0\}$, that $F(x) \cap C_{r_1} = \emptyset$. Since, when connecting $x$ by a segment to a point $p \in I_{r_1}$ we cross $C_{r_1}$, it follows that $|x - p| > d(x)$. So $p \not\in F(x)$. This shows $*_{2} : I_{r_1} \cap F(C_{r_2}) = \{0\}$.

Fact 2. Every point in $I_{r_1} \setminus \{0\}$ lies in the image of $I_{r_2}$ under $F$: $I_{r_1} \setminus \{0\} \subseteq F(I_{r_2})$.

[Assume there exists a point $p \in I_{r_1} \setminus \{0\}$ so that $p \not\in F(I_{r_2})$. Then $p \neq f(x)$ for all $x \in I_{r_2}$. It is also clear by $*_{2}$ that $p \not\in f(C_{r_2})$. So we have a continuous map $f|_{I_{r_2}} : I_{r_2} \rightarrow \mathbb{R}^2 \setminus \{p\}$. Let $\beta : \mathbb{R}^2 \setminus \{p\} \rightarrow C_{r_2}$ be the standard retraction map that carries each $x \in \mathbb{R}^2 \setminus \{p\}$ to the unique intersection of the ray $px^+$ with $C_{r_2}$: $\beta(x) = px^+ \cap C_{r_2}$. Then we get a continuous map $\beta \cdot f|_{I_{r_2}} : I_{r_2} \rightarrow C_{r_2}$ extending $\beta \cdot f|_{C_{r_2}} : C_{r_2} \rightarrow C_{r_2}$. By Spanier [S, p27] this means that $\beta \cdot f|_{C_{r_2}}$ is nullhomotopic. Note that we can write $f(x) = x + \epsilon(x)$ for some continuous map $\epsilon(x)$ satisfying $|\epsilon(x)| \leq r(x)$. Since for $t \in [0, 1]$,
\[|te(x)| \leq |e(x)|,\] by fact 1 we have a homotopy

\[C_{r_2} \times [0, 1] \ni (x, t) \xrightarrow{H} x + te(x) \in \mathbb{R}^2 \setminus \{p\}\]

showing \(id_{C_{r_2}} \simeq f|C_{r_2}\) as \(t : 0 \not\to 1\). But since \(C_{r_2} \approx S^1\) and \(id_{S^1}\) is not nullhomotopic (as follows from the observations [S, pp25c-7, 56c4, 59c5, 23c6]), we get that \(id_{C_{r_2}}\) is not nullhomotopic. Now \(\beta^* H\) yields a homotopy \(id_{C_{r_2}} = \beta^* id_{C_{r_2}} \simeq \beta^* f|C_{r_2}\); so we get a contradiction, proving the claim. 

Fact 3. For all small \(r_2 > 0\) there exists \(r_1 > 0\) so that

\[*\text{3 : } S(ar_{-\varepsilon}, r_1) \subseteq F(S(ar, r_2)) \cap B(0, r_1) \subseteq S(ar_{+\varepsilon}, r_1).\]

Recall that \(\hat{I}_{r_1} = \text{cl}S(ar_{-\varepsilon}, r_1)\). Also, by i, \(F(0) = 0\). So for given \(\varepsilon\), as above, facts 1 and 2 yield that for all small \(r_2\) there exists an \(r_1 > 0\), so that \(S(ar_{-\varepsilon}, r_1) \subseteq F(S(ar, r_2))\). Intersecting both sides with \(B(0, r_1)\) yields the left of the inclusions. Next let \(u \in \text{mid}(\ast 3)\). Then \(u \in F(x)\) for some \(x \in S(ar, r_2)\). As in the proof of fact 1 we have observed that this means \(r(x) \leq \frac{\sin x}{1 + \sin x} |x| < |x| \sin \varepsilon\). Consequently \(u \in B(x, |x| \sin \varepsilon)\). Suppose \(u \not\in ar_{+\varepsilon}\). Since \(x \in ar \subseteq ar_{+\varepsilon}, u \not\in ar\). It follows that the segment \(ux\) has to contain a point in a side of \(ar\) and another in a side of \(ar_{+\varepsilon}\). These two sides define an angle \(\geq \varepsilon\) with vertex 0. Consequently \(|u - x| \geq |x| \sin \varepsilon\). Contradiction. Hence \(u \in ar_{+\varepsilon}\). Since also \(|u| \leq r_1\), we get \(u \in \text{rhs}(\ast 3)\). 

With fact 3 the theorem is proved. 

\[\square\]

4. A lemma on power series

**Lemma 7.** Let \(f(x) = \sum_{k \geq 2} f_k(x)\) be a power series over \(\mathbb{C}\) where every \(f_k\)

is either 0 or a homogeneous polynomial of degree \(k\). Assume that

(i) \(f_2(x) = \sum_{i=1}^{n} c_i x_i^2\), with coefficients satisfying \(0 \not\in \text{co}\{c_i : i = 1, \ldots, n\}\);

(ii) there exist \(M > 0, b \in \mathbb{R}_{>0}\), so that \(|c_i b^2| < M\) for all monomials \(c_i x_i^2\) of \(f(x)\).

For any real positive \(r < \min b\), we have a continuous function

\([-r, r]^n \ni x \mapsto f(x) \in \mathbb{C}\).

Furthermore, \(|f_2(x)| \to 0, x \in [-r, r]^n,\) implies \(\sum_{k \geq 3} f_k(x)/|f_2(x)| \to 0\).
Proof. That $f$ defines in the closed cube $[-r, r]^n$ a continuous function is a consequence of [D, p194c1..5]. From i we get that there exist

$$0 < \rho_1 < \rho_2 = \max\{|c_i| : i = 1, \ldots, n\}$$

such that

$$\rho_1 \leq \left| \sum_{j=1}^{n} c_j \frac{x_j^2}{x_1^2 + \ldots + x_n^2} \right|,$$

so:

$$\rho_1 (x_1^2 + \ldots + x_n^2) \leq |f_2(x)| \leq \rho_2 (x_1^2 + \ldots + x_n^2);$$

for the set of values the expression $\sum \ldots$ assumes as $x$ varies over any neighbourhood of 0 is just the convex hull of $c_1, \ldots, c_n$. Henceforth, we assume $f_k(x) = \sum_{|i|=k} c_ix_i^k$, $k = 3, \ldots$.

We put

$$L_k = \{i : |i| = k, i_\nu \leq 1 \text{ for all } \nu\}, Q_k = \{i : |i| = k, i_\nu \geq 2 \text{ for some } \nu\}.$$}

Case $i \in L_k$. Then exactly $k$ of the $i_\nu$s are 1, say $i_{\nu_1} = \ldots = i_{\nu_k} = 1$. We have the estimates

$$x_{i_{\nu_1}} \cdots x_{i_{\nu_k}} \leq \frac{1}{k} (|x_{i_{\nu_1}}|^k + \ldots + |x_{i_{\nu_k}}|^k); \text{ and } \frac{|x_i|^k}{x_1^2 + \ldots + x_n^2} \leq |x_i|^{k-2},$$

$i = 1, \ldots, n$, the first following from the arithmetic geometric mean inequality, the second being trivial. These inequalities imply

$$|c_i| \frac{x_i}{x_1^2 + \ldots + x_n^2} \leq \frac{1}{k} \sum_{i_{\nu_1}=1} |c_i||x_{i_\nu}|^{k-2}.$$}

Case $i \in Q_k$. Then, for a definite choice, we can define

$$j = j(i) = \min\{\nu : i_\nu = 2\},$$

and find

$$|c_i| \frac{x_i}{x_1^2 + \ldots + x_n^2} = |c_i| \frac{|x_j|^2}{x_1^2 + \ldots + x_n^2} |x_1|^{i_1} \cdots |x_j|^{i_j-2} \cdots |x_n|^{i_n} \leq |c_i||x_1|^{i_1} \cdots |x_j|^{i_j-2} \cdots |x_n|^{i_n}.$$
Now put \( m(x) = \max\{|x_1|, \ldots, |x_n|\} \). Then
\[
\left| \sum_{k \geq 3} f_k(x)/f_2(x) \right| \leq \frac{1}{\rho_1} \sum_{k \geq 3} \frac{1}{k} \sum_{i \in L_k} \frac{1}{q} \sum_{i/|i| = k} \left| c_i \right| |x_i|^k - 2 + \sum_{i \in Q_k} \left| c_i \right| |x_1|^{i_1} \cdots |x_2|^{i_2} \cdots |x_n|^{i_n}
\]
\[
\leq \frac{1}{\rho_1} \sum_{k \geq 3} \frac{1}{k} \sum_{i \in L_k} \left| c_i \right| (\max\{|x_1|, \ldots, |x_n|\})^k - 2
\]
\[
= \frac{1}{\rho_1} \sum_{k \geq 3} \sum_{\hat{\imath}/|\hat{\imath}| = k} \left| c_{\hat{\imath}} \right| m(x)^{k-2} = \frac{1}{\rho_1} \sum_{\hat{\imath}, |\hat{\imath}| \geq 3} \left| c_{\hat{\imath}} \right| m(x)^{|\hat{\imath}|-2}.
\]

The last equality sign is justified as follows: let \( b = \min\{b_1, \ldots, b_n\} \). By hypothesis ii we know \( |c_{\hat{\imath}}| b^{|\hat{\imath}|_-2} \leq M/b^2 \). Put \( q = r/b \). For all \( x \in ]-r, r[^n \), \( m(x)/b \leq q \), and so
\[
|c_{\hat{\imath}}| m(x)^{|\hat{\imath}|-2} \leq |c_{\hat{\imath}}| q^{|\hat{\imath}|-2} b^{|\hat{\imath}|-2} \leq M/b^2 q^{|\hat{\imath}|-2}.
\]

Now
\[
\sum_{\hat{\imath}, |\hat{\imath}| \geq 3} q^{|\hat{\imath}|-2} \leq 1/q^2 \sum_{\hat{\imath} \in \mathbb{Z}^n \geq 3} q^{|\hat{\imath}|} = (1-q)^{-n-2}.
\]

Therefore, by [D, p95c4..8], the denumerable family \( \{|c_{\hat{\imath}}| m(x)^{|\hat{\imath}|-2}\}_{\hat{\imath}, |\hat{\imath}| \geq 3} \) of bounded continuous functions on polycylinder \( ]-r, r[^n \) is absolutely summable. Furthermore, by [D, pp 128c7,129c3] it is continuous. Since \( m(0) = 0 \), we have that, as \( x \to 0 \), the right hand side converges to 0. This proves the lemma.

\textbf{Example 8.} Consider the polynomial \( f(x, y) = x^2 + y^3 \) as a power series in \( x, y \). Here, \( f_2(x) \to 0 \) does not imply \( f_3(x) \to 0 \). So hypothesis i of lemma 7 cannot be weakened to \( 0 \notin \text{co}\{c_i : c_i \neq 0, i = 1, \ldots, n\} \).

Note that if lemma 7 holds for a certain \( r > 0 \), then it holds also when formulated with a neighbourhood \( U \subseteq [-r,r]^n \) of \( 0 \) instead of \( [0,r]^n \).

\textbf{Corollary 9.} Assume the hypotheses and notation of lemma 7 in force and additionally that the \( c_i \) are not collinear. Then for all small neighbourhoods \( U \) of \( 0 \in \mathbb{R}^n \), \( f(U) \) has in 0 the angular region \( \text{ar} = \text{cone}\{c_1, \ldots, c_n\} \) as a corner.
Proof. The noncollinearity condition, ensures that ar obeys the nondegeneracy condition implicit in definition 2. We prove next two general facts.

Fact 1. For every neighbourhood $U$ of $0 \in \mathbb{R}^n$ we can find $0 < r_1 = r_1(U)$ and $0 < r_2 = r_2(U)$ such that $S(ar, r_1) \subseteq f_2(U) \subseteq S(ar, r_2)$ and so that diameter$(U) \to 0$ implies $r_2(U) \to 0$.

[>] Recall that according to inequality (*) in the proof of lemma 7 there exist two constants $0 < \rho_1 < \rho_2$ so that $\rho_1 |x|^2 \leq |f_2(x)| \leq \rho_2 |x|^2$. Choose balls $B(0, \rho) \subseteq U \subseteq B(0, \rho')$ with $\rho, \rho' = \text{diameter}(U) \in \mathbb{R}$. Define $r_1 = \rho_1 \rho^2$, $r_2 = \rho_2 \rho^2$. Let $x \in S(ar, r_1)$. Since from the very definition of a cone it follows that $f_2(\mathbb{R}^n) = ar$, there is an $x \in \mathbb{R}^n$ so that $x = f_2(x)$. Hence $\rho_1 |x|^2 \leq |x| \leq \rho_1$. Consequently $|x|^2 \leq \rho^2$. This shows $S(ar, r_1) \subseteq f_2(B(0, \rho)) \subseteq f_2(U)$. Next, assume $x \in f_2(U)$. Then there exists $x \in U$, hence $|x| \leq \rho'$, so that $x = f_2(x)$. So $|x| \leq \rho_2 \rho^2 = r_2$ and so we have $f_2(U) \subseteq S(ar, r_2)$. The remaining claim follows from the definitions of $r_2, \rho'$.

Now we define for any neighbourhood $U$ of $0 \in \mathbb{R}^n$ with $U \subseteq ]-r, r[ \ni$, for $x \in f_2(U)$:

$$C(x) = \{x \in U : f_2(x) = x\}, S(x) = \{\sum_{k \geq 3} f_k(x) : x \in C(x)\},$$

and $F(x) = x + S(x)$.

Fact 2. $f(U) = F(f_2(U))$.

[>] Choose any $x \in U$. Put $x = f_2(x)$. Then $x \in f_2(U)$, $x \in C(x)$, and

$$f(x) = f_2(x) + \sum_{k \geq 3} f_k(x) \in x + S(x) = F(x).$$

This shows $f(U) \subseteq F(f_2(U))$. Now choose any $x \in f_2(U)$. Next choose any $s \in S(x)$. Then $s = \sum_{k \geq 3} f_k(x)$ for some $x \in C(x)$; so that $x = f_2(x)$. Hence $x + s = f_2(x) + \sum_{k \geq 3} f_k(x) = f(x)$. Since $x \in U$, we have $x + s \in f(U)$. This shows $x + S(x) \subseteq f(U)$ and $F(f_2(U)) \subseteq f(U)$.

We emphasize that facts 1 and 2 hold for an arbitrary neighbourhood $U$ of $0 \in \mathbb{R}^n$ with $U \subseteq ]-r, r[ \ni$ and $f_2(U)$, $S(x), C(x)$, are conditioned by this choice.

We now fix $U$ to be a neighbourhood satisfying $U \subseteq ]-r, r[ \ni$, $r$ being chosen as in lemma 7. The set valued map $F$ can by fact 1 be restricted to a disc-sector $S$ of type $ar$ contained in $f_2(U)$: $\star_1$: $S \subseteq f_2(U)$.

Fact 3. $F : S \to \mathcal{P}(\mathbb{R}^2)$ satisfies the hypotheses of theorem 6.
Define for \( x \in S \) the function \( r(x) = 1.1 \cdot \sup\{|s| : s \in S(x)\} \). Then \( S(x) \subseteq B(0, r(x)) \). By lemma 7 we know
\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |f_2(x)| < \delta \Rightarrow \left| \sum_{k \geq 3} f_k(x) \right| \leq \varepsilon |f_2(x)|.
\]

Now fix an \( \varepsilon > 0 \), and choose an associated \( \delta > 0 \) accordingly. Let \( x \in S, |x| < \delta \). By \( *_1 \), \( x = f_2(x) \) for all \( x \in C(x) \). Hence \( |\sum_{k \geq 3} f_k(x)| \leq \varepsilon |x| \) for all \( x \in C(x) \). This means \( r(x) \leq \varepsilon |x| \). By the arbitrariness of \( \varepsilon > 0 \) we have shown, \( r(x)/|x| \rightarrow 0 \) as \( |x| \downarrow 0 \). Also, \( S(0) = \{0\} \). Since \( F(x) = x + S(x) \) we see \( F(x) \subseteq B(x, r(x)) \), so \( F \) satisfies hypothesis i of theorem 6. To see ii, we use that there exist two \( c_i, c_1 \) and \( c_2 \), say so that \( ar = \text{cone}\{c_1, c_2\} \). We can then write each \( x \in S \) in a unique way as \( x = c_1 x_1^2 + c_2 x_2^2 \). Clearly the coordinate functions \( x_1 = x_1(x), x_2 = x_2(x) \) depend continuously on \( x \). So
\[
S \ni x \mapsto f((x_1(x), x_2(x), 0_{n-2})) \in F(x)
\]
is a continuous selection, showing ii. \( \square \)

There exists, by theorem 6, an \( r_2 \leq \text{radius of } S \) so that for all \( 0 < r' \leq r_2 \) the set \( F(S(\text{ar}, r')) \) has in 0 a corner of type \( \text{ar} \). By (the arguments which proved) fact 1, we can choose a neighbourhood \( U' \subseteq U \) of 0, and an \( r_1 > 0 \) so that \( S(\text{ar}, r_1) \subseteq f_2(U') \subseteq S(\text{ar}, r_2) \). Upon applying \( F \), we get \( F(S(\text{ar}, r_1)) \subseteq F(f_2(U')) \subseteq F(S(\text{ar}, r_2)) \). The left and the right subsets of this inclusion are corners of type \( \text{ar} \). Hence, by observation 5a, \( F(f_2(U')) = f(U') \) also has \( \text{ar} \) as a corner in 0. This was to prove. \( \square \)

5. The main result

**Lemma 10.** Let \( A, Q, D, P_\sigma \) be \( n \times n \) matrices, \( D \) diagonal, \( \sigma, \rho \in S_n \), \( P_\sigma, P_\rho \) the associated permutation matrices. Then there hold the following computational rules.

\[
P_\rho \sigma = P_\sigma P_\rho, \quad d_\sigma(P_\rho A) = d_{\rho^{-1} \sigma}(A), \quad D(A \cdot Q) = A \cdot (DQ) = (DA) \cdot Q,
\]

\[
P_\sigma(A \cdot Q) = (P_\sigma A) \cdot (P_\sigma Q), \quad \det(A \cdot P_\sigma) = \text{sgn} \sigma d_\sigma(A).
\]

**Proof.** The easy proofs are left to the reader; see also [HJ, p304]. \( \square \)
Let \( P_\sigma = \{ Q \in \text{SO}(n) : |Q| = P_\sigma \} \). Clearly each \( Q \in P_\sigma \) can be written \( Q = DP_\sigma \), with \( D = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \), \( \varepsilon_i \in \{-1, +1\} \), \( \det(D) = \text{sgn} \sigma \). One consequence of lemma 10 is that if \( Q \in P_\sigma \), then \( \det(A^\top Q) = d_\sigma(A) \).

**Theorem 11.** Let \( A \) be a complex \( n \times n \) matrix, and let \( \sigma \in S_n \). Assume that the only matrices \( Q \in \text{SO}(n) \) for which \( \det(A^\top Q) = d_\sigma(A) \) are the matrices in \( P_\sigma \), and that the complex numbers \( \tilde{d}_{\sigma \tau}(A) = d_{\sigma \tau}(A) - d_{\sigma}(A) \), \( \tau \in T \), lie in an open half plane whose support contains the origin, and that they are not all collinear with 0. Then \( \Delta(A) = \{ \det(A^\top Q) : Q \in \text{SO}(n) \} \) has in \( d_\sigma(A) \) the corner \( d_\sigma(A) + \text{cone}\{\tilde{d}_{\sigma \tau}(A) : \tau \in T \} \).

**Proof.** Case \( \sigma = \text{id} \). The essentials lie in the proof for this case. By the theory of Lie-groups [SW, pp31c5, 145c4] we can choose small open neighbourhoods, \( U_0 \) of 0 in \( \text{so}(n) \) and \( U_I \) of \( I \in \text{SO}(n) \) so that the map \( U_0 \ni S \mapsto \exp(S) \in U_I \) delivers a bijection. Also, by [SW, p91c-5], if \( D = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \text{SO}(n) \), then, \( U_D = DU_I \) is a neighbourhood of \( D \). Let

\[
K = \text{SO}(n) \setminus \bigcup \{ U_D : D = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \text{SO}(n) \}.
\]

Then \( K \) is compact.

On \( \text{so}(n) \) and \( \text{SO}(n) \), respectively, define the maps \( f, \varphi \) by

\[
\text{so}(n) \ni S \xrightarrow{f} \det(A^\top \exp S) - d_{\text{id}}(A) \in \mathbb{C}; \quad \text{SO}(n) \ni Q \xrightarrow{\varphi} \det(A^\top Q) \in \mathbb{C}.
\]

From the hypothesis we find that \( \varphi K \) is a compact set not containing \( d_{\text{id}}(A) \). Since the distance between compact disjoint sets is positive [D, p61c-2], we can find a ball around \( d_{\text{id}}(A) \) having with \( \varphi K \) empty intersection. Now for every of the diagonal matrices \( D \) here present, and every \( Q \in \text{SO}(n) \), \( \varphi(DQ) = \varphi(Q) \).

So

\[
\Delta(A) = \varphi(\text{SO}(n)) = \varphi(K \cup \bigcup_D U_D) = \varphi K \cup \bigcup_D \varphi(DU_I)
\]

\[
= \varphi K \cup \varphi U_I = \varphi K \cup (\varphi \circ \exp U_0) = \varphi K \cup (f(U_0) + d_{\text{id}}(A)).
\]

For small \( r > 0 \), we now have

\[
\Delta(A) \cap B(d_{\text{id}}(A), r) = d_{\text{id}}(A) + (f(U_0) \cap B(0, r)).
\]

From theorem 1 we know that for \( S \in U_0 \),

\[
f(S) = \sum_{\tau \in T} |\tilde{d}_{\tau}(A)|s_{\tau}|^2 + \sum_{k \geq 3} p_k(S),
\]
and this can be rewritten as a real variable power series with complex coefficients, precisely in the form required in lemma 7. This yields by corollary 9 and the observation 5bc that $\Delta$ has in $d_{id}(A)$ the corner claimed.

Case $\sigma \in S_n$ arbitrary. As one may expect this case can be reduced to the previous one. Let $\tilde{A} = P_{\sigma^{-1}} A$ and let $Q \in SO(n)$. Choose a diagonal matrix $D$ so that $DP_{\sigma^{-1}} \in P_{\sigma^{-1}}$ and put $\tilde{Q} = DP_{\sigma^{-1}} Q$. Then

$$\det(\tilde{A} \cdot \tilde{Q}) = \det(P_{\sigma^{-1}} A \cdot (DP_{\sigma^{-1}} Q)) = \det(DP_{\sigma^{-1}}) \det(A \cdot Q) = \det(A \cdot Q),$$

and $d_\sigma(A) = d_{id}(\tilde{A})$. Now

$$\tilde{Q} \in P_{id} \quad \text{iff} \quad Q \in P_{\sigma} \quad \text{(easy)}$$

$$\text{iff} \quad \det(A \cdot Q) = d_\sigma(A) \quad \text{(by hypotheses)}$$

$$\text{iff} \quad \det(\tilde{A} \cdot \tilde{Q}) = d_{id}(\tilde{A}) \quad \text{(by the equations above)}$$

So we can apply the first case to the matrix $\tilde{A}$. So $\Delta(\tilde{A})$ has in $d_{id}(\tilde{A})$ the corner $ar = d_{id}(\tilde{A}) + \text{cone}\{d_\tau(\tilde{A}) : \tau \in S_n\}$. Now for any $Q \in SO(n)$,

$$\det(\tilde{A} \cdot Q) = \det((DP_{\sigma^{-1}} A) \cdot Q) = \det(A \cdot (P_\sigma DQ)).$$

Since $P_\sigma DSO(n) = SO(n)$, we can infer

$$\Delta(\tilde{A}) = \{\det(\tilde{A} \cdot Q) : Q \in SO(n)\} = \Delta(A).$$

Furthermore $d_{id}(\tilde{A}) = d_\sigma(A)$, and $d_\tau(\tilde{A}) = d_\tau(P_{\sigma^{-1}} A) = d_{\sigma\tau}(A)$. From this we get $ar = d_\sigma(A) + \text{cone}\{d_{\sigma\tau}(A) - d_\sigma(A) : \tau \in T\}$. The theorem is proved. \hfill \square

We end with three remarks.

**Remarks 12.** a. For technical reasons (in particular what concerns the reasoning employed in theorem 6, fact 2) we have restricted the formulation of the main result to the case that the $d_\tau(A)$ are not all collinear with 0. It seems to us that with obvious modifications it will also hold without this restriction (and indeed the proof will be easier).

$$Q(c, s) = \begin{bmatrix} c & 0 & s \\ -s & 0 & c \\ 0 & -1 & 0 \end{bmatrix}.$$  

b. For $c, s$ reals satisfying $c^2 + s^2 = 1$, define $Q = Q(c, s) \in SO(3)$, the matrix at the right.

Then $\det(I \cdot Q(c, s)) = 0 = d_\sigma(I)$ for all admissible $c, s$ and $\sigma \neq id$. So the hypothesis of theorem 11 usually is not satisfied.
At the other hand, the condition of theorem 11 is certainly not empty.
For example det(I \cdot Q) = 1 will happen only if Q \in SO(n) is a signed
identity matrix. Some proofs of the special cases of OMC already available
provide more examples; see e.g [F]. Indeed it seems to us that answering
the question for which pairs Q \in SO(n), and permutations \sigma \in S_n equations
det(A \cdot Q) = d_\sigma(A) can happen would mean - in case rankA = 2 at least - to
go a long way towards deciding OMC.

c. The reader may well ask why we have not formulated theorem 11 for
SU(n). The reason is that the diagonal entries of an S \in su(n) do not enter
in the homogeneous part of degree 2 in the real variable power series of complex
coefficients, f(S) = det(A \cdot \exp S). So in terms of lemma 7, see also example
8, we do not know whether f_2(S) \to 0 implies \sum_{k \geq 3} f_k(S)/f_2(S) \to 0; hence
we cannot apply our reasoning to these cases.

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Natália Bebiano  
Dep. Matemática, Univ. Coimbra, 3001 - 454 Coimbra, Portugal  
E-mail address: bebiano@mat.uc.pt

João de Providência  
Dep. Física, Univ. Coimbra, 3001 - 454 Coimbra, Portugal  
E-mail address: providencia@teor.fis.uc.pt

Alexander Kovačec  
Dep. Matemática, Univ. Coimbra, 3001 - 454 Coimbra, Portugal  
E-mail address: kovacec@mat.uc.pt