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CONJUGACY INVARIANTS OF SUBSHIFTS: AN APPROACH FROM PROFINITE SEMIGROUP THEORY

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Abstract: It is given a structural conjugacy invariant in the set of pseudowords whose finite factors are factors of a given subshift. Some profinite semigroup tools are developed for this purpose. With these tools a shift equivalence invariant of sofic subshifts is obtained, improving an invariant introduced by Béal, Fiorenzi and Perrin using different techniques. This new invariant is used to prove that some irreducible almost finite type subshifts with the same entropy and the same zeta function are not shift equivalent.

shift, subshift, free profinite semigroup, sofic, conjugacy, shift equivalence
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1. Introduction

There is a very strong and decidable conjugacy invariant of sofic subshifts, the class of shift equivalence, but the major problem of knowing if there is an algorithm that decides if two subshifts of finite type are conjugate or not remains open [16, 18]. The study of conjugacy problems about finite type subshifts is usually made in terms of matrices of non-negative integers, and in the more general case of sofic subshifts a certain kind of symbolic matrices is used [21]. A different approach using the algebraic theory of languages with profinite semigroup techniques was suggested to the author by Jorge Almeida. It is well known that a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is determined by the language $L(\mathcal{X})$ of A^+ whose elements are the finite factors of $\mathcal{X}$. As has been observed by Jorge Almeida, $\mathcal{X}$ is also determined by the infinite elements in the closure of $L(\mathcal{X})$ in the free profinite A-generated semigroup $\widehat{A^+}$. The algebraic-topological structure of $\widehat{A^+}$ is very rich, but it remains quite unexplored; see [5, 6] for some results; in [6] a dynamical parameter - entropy - is extensively used to achieve results about the minimal ideal of $\widehat{A^+}$. Almeida also established in [4] a strong link between minimal subshifts and the structure of free profinite semigroups. So it seems that the exploration of a profinite semigroup approach in symbolic dynamics deserves attention.

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Developing some tools for this purpose and showing their potentiality is one of the objectives of this paper.

Section 2 provides preliminary definitions and results about symbolic dynamics and free profinite semigroups. Our main reference for symbolic dynamics is the book of Lind an Marcus [21]. For background on classical semigroup theory, rational languages and finite automata see for example [20]. For profinite semigroups see the introductory text [3]. It is necessary basic background about Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$, and the quasi-orders $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$, as well about the notions of regular element and stable semigroup [20, 2].

We study in Section 3 the effect of a subshift conjugacy in the set of pseudowords whose finite factors are in the language of a given subshift. We call this set the *mirage* of the subshift. The tools that we develop for this study allow us to arrive at a structural conjugacy invariant in the mirage. A corollary of this result is the conjugacy invariance of the Schützenberger group in the $\mathcal{J}$-class associated with a minimal subshift, a result announced by Almeida without proof in [3]; these groups have been computed for some classes of minimal subshifts, like Sturmian subshifts [4]. Our methods rely a lot on the pseudowords that are simultaneously $\mathcal{R}$-below and $\mathcal{L}$-below idempotents; we say that such words are *idempotent-bound*.

In Section 4 we introduce an effectively computable shift equivalence invariant of sofic subshifts, using the tools of Section 3. This invariant is obtained from the syntactic image of the language of the sofic subshift: it is the labeled poset of their idempotent-bound $\mathcal{J}$-classes, labeled with their Schützenberger groups. This improves the invariant introduced in [9] by Béal, Fiorenzi and Perrin, who only considered regular $\mathcal{J}$-classes. The results from [9] were an inspiring source, but our methods are substantially different. We use profinite syntactic methods, manipulating pseudowords, while in [9] symbolic matrices where the main ingredient. Béal, Fiorenzi and Perrin continued in [8] the search of conjugacy invariants in the structure of the syntactic semigroup of the language of a sofic subshift. There they established a hierarchy of classes of irreducible sofic subshifts closed for taking shift equivalent subshifts, and they proved that the important class of almost finite type subshifts. This class has pratical interest for coding with constrained channels [7].

In Section 5 we prove that the conjugacy invariant introduced in Section 4 is a shift equivalence invariant. The proof depends on the conjugacy invariance.

2. Preliminaries

2.1. Subshifts and codes. Let A be an alphabet. All alphabets in this paper are assumed to be finite. The semigroup of finite non-empty words (or blocks) on letters of A is as usually denoted by A^+; the empty word is denoted by 1 and A^* is the monoid $A^+ \cup \{1\}$. The length of $u \in A^*$ is denoted by $|u|$. Let $A^{\mathbb{Z}}$ be the set of sequences of letters of A indexed by $\mathbb{Z}$. We define in $A^{\mathbb{Z}}$ the *shift* function σ_A by the following formula:

$$\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}.$$

Note that σ_A is bijective. In general we denote σ_A simply by σ. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology of A. Note that $A^{\mathbb{Z}}$ is a compact Hausdorff space. From here on *compact* will mean both compact and Hausdorff.

A *shift dynamical system* or *subshift* is a non-empty closed topological subspace $\mathcal{X}$ of $A^{\mathbb{Z}}$ (for some alphabet A) that is invariant under the action of σ and σ^{-1} (that is, $\sigma(\mathcal{X}) \subseteq \mathcal{X}$ and $\sigma^{-1}(\mathcal{X}) \subseteq \mathcal{X}$). A factor of $(x_i)_{i\in\mathbb{Z}}$ is a finite sequence $x_{[i,i+n]} = x_i x_{i+1} \cdots x_{i+n-1} x_{i+n}$, where $i \in \mathbb{Z}$ and $n \geq 0$. If $\mathcal{X}$ is a non-empty subset of $A^{\mathbb{Z}}$ then we denote by $L(\mathcal{X})$ the set of factors of elements of $\mathcal{X}$. The set of words over A with length n is A^n; the set $L(\mathcal{X}) \cap A^n$ is denoted by $L_n(\mathcal{X})$. A language L of A^+ is *factorial* if it is closed for taking factors, and it is *prolongable* if for every word u in L there are letters a and b such that aub belongs to L. For every subshift $\mathcal{X}$, the language $L(\mathcal{X})$ is factorial and prolongable. In fact, it is easy to prove that the correspondence $\mathcal{X} \mapsto L(\mathcal{X})$ is a bijection between the subshifts of $A^{\mathbb{Z}}$ and the non-empty factorial prolongable languages of A^+.

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if for all $u, v \in L(\mathcal{X})$ there is $w \in A^*$ such that $uwv \in L(\mathcal{X})$.

A *code* G between the subshifts $\mathcal{X}$ of $A^{\mathbb{Z}}$ and $\mathcal{Y}$ of $B^{\mathbb{Z}}$ is a continuous function $G : \mathcal{X} \to \mathcal{Y}$ such that $G \circ \sigma_A = \sigma_B \circ G$. Note that the identity transformation of a subshift is a code and that the composition of two codes is a code. Note also that the inverse of a bijective code is a code. A bijective code is called a *conjugacy*. Two subshifts are *conjugate* if there is a conjugacy between them. A *conjugacy invariant* is a property of subshifts that is preserved for taking conjugate subshifts. Irreducibility is a conjugacy invariant. See [21] for the definition and computation of ordinary conjugacy invariants like the zeta function and the entropy.

It is well known [13] that a map $G : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ is a code between subshifts if and only if there are $k, l \geq 0$ and a map $g : A^{k+l+1} \to B$ such that $G(x) = (g(x_{[i-k,i+l]}))_{i\in\mathbb{Z}}$. We say that g is a *block map* of G with *memory* k and *anticipation* l. The code G depends only on the restriction of g to $L_{k+l+1}(\mathcal{X})$. We use the notation $G = g^{[-k,l]} : \mathcal{X} \to \mathcal{Y}$, or simply $G = g^{[-k,l]}$ when $\mathcal{X}$ and $\mathcal{Y}$ are known. Note that if $n \geq l$ and $m \geq k$ then $G = h^{[-m,n]}$, where $h(a_{[-m,n]}) = g(a_{[-k,l]})$, for all $a = a_{-m}a_{-m+1} \ldots a_{n-1}a_n \in A^{m+n+1}$ ($a_i \in A$). In particular, one can always choose $k = l$.

Given an alphabet A and $k \geq 1$, consider the alphabet A^k. To avoid ambiguities, we represent an element $w_1 \ldots w_n$ of $(A^k)^+$ (with $w_i \in A^k$) by $\langle w_1, \ldots, w_n\rangle$. For $k \geq 0$ let Φ_k the function from A^+ to $(A^{k+1})^*$ defined by

$$\Phi_k(a_1 \ldots a_n) = \begin{cases} 1 & \text{if } n \leq k, \\ \langle a_{[1,k+1]}, a_{[2,k+2]}, \ldots a_{[n-k-1,n-1]}, a_{[n-k,n]}\rangle & \text{if } n > k, \end{cases}$$

where $a_i \in A$ and $a_{[i,j]} = a_ia_{i+1} \ldots a_{j-1}a_j$. It is easy to see that, if $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$ and $i, j \geq 0$ are such that $i + j = k$, then the restriction of the code $\Phi_k^{[-i,j]}$ to $\mathcal{X}$ is a conjugacy between $\mathcal{X}$ and $\Phi_k^{[-i,j]}(\mathcal{X})$, whose inverse is the code with memory and anticipation 0 given by the block map that sends an element w of A^k into its $(i + 1)$-th letter as an element of A^+. A *one-block code* is a code having a block map with memory and anticipation 0. A *one-block conjugacy* is a conjugacy that is a one-block code.

Proposition 2.1. *For every code G there are one-block codes G_1 and G_2 such that G_1 is a conjugacy and $G = G_2 \circ G_1^{-1}$.*

Proof: For a code $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq A^{\mathbb{Z}}$ let G_1 be the restriction of $\Phi_{2k+1}^{[-k,k]}$ to $\mathcal{X}$ and let $G_2 : \Phi_{2k+1}^{[-k,k]}(\mathcal{X}) \to \mathcal{Y}$ be defined by $G_2 = g^{[0,0]}$. ■

Proposition 2.1 is well known and it is very useful because if we want to prove that some property is a conjugacy invariant it suffices to consider one-block conjugacies. This simplification will be used later in this paper.

2.2. Sofic subshifts. A subshift $\mathcal{X}$ is *sofic* if $L(\mathcal{X})$ is rational. We call *graph-automaton* to an automaton such that all states are initial and final. An automaton is said to be *essential* if all states lie in a bi-infinite path of the automaton. One can see that $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is recognized by an essential finite graph-automaton. We say that a finite graph-automaton *presents* the subshift $\mathcal{X}$ if it recognizes $L(\mathcal{X})$. A sofic subshift is irreducible if and only if it is presented by a strongly connected finite graph-automaton [12].

A subshift of *finite type* is a subshift $\mathcal{X}$ such that $L(\mathcal{X}) = A^+ \setminus A^*FA^*$ for some finite set F. Therefore finite type subshifts are sofic. A subshift presented by a finite graph-automaton in which every letter acts in at most one state is called an *edge subshift*. An edge subshift is a subshift of finite type, and every subshift of finite type is conjugate with an edge subshift.

We are going to use the usual definition of a *deterministic* automaton: letters act on states as partial functions, and there is a single initial state. By the *minimal* automaton of a rational language we mean its minimal deterministic automaton. The *Krieger cover* of $\mathcal{X}$ is the essential graph-automaton $\mathfrak{K}(\mathcal{X})$ obtained from the minimal automaton of $L(\mathcal{X})$ by deleting states that do not lie in bi-infinite paths. Call *Krieger edge subshift* of $\mathcal{X}$ the edge subshift obtained from $\mathfrak{K}(\mathcal{X})$ by labeling with different letters different arrows in the graphical representation of $\mathfrak{K}(\mathcal{X})$. Krieger showed in [19] that if $\mathcal{X}$ and $\mathcal{Y}$ are conjugate sofic subshifts, then their Krieger edge subshifts are also conjugate. If the sofic subshift $\mathcal{X}$ is irreducible then $\mathfrak{K}(\mathcal{X})$ has a unique terminal strongly connected component which is a graph-automaton $\mathfrak{F}(\mathcal{X})$ presenting $\mathcal{X}$ [10]. This graph-automaton is named the *Fischer cover* of $\mathcal{X}$. The right context of a state q in an automaton is the set of words that label paths starting at q. A *reduced* automaton is an automaton such that different states have different right contexts. The Fischer cover of $\mathcal{X}$ is the unique graph-automaton presenting $\mathcal{X}$ that is strongly connected, reduced and such that every letter acts on the states as a partial function [12].

2.3. Free profinite semigroups. A *compact semigroup* is a semigroup endowed with a compact topology for which the semigroup operation is continuous. Finite semigroups are assumed to be endowed with the discrete topology. A *profinite semigroup* is a compact semigroup T such that, for every pair of distinct elements u and v of T, there is a continuous homomorphism φ from T into a finite semigroup F such that $\varphi(u) \neq \varphi(v)$. Note that finite semigroups are profinite.

Let $u, v \in A^+$. If $u \neq v$ then there is some finite semigroup S that does not satisfy the identity $u = v$. Let $r(u, v)$ be the smallest possible cardinal for such an S. Let $d(u, v)$ be $2^{-r(u,v)}$ if $u \neq v$, and let $d(u, u) = 0$. Then d is a metric.

Let $\widehat{A^+}$ be the completion of A^+ with respect to d. We call its elements *pseudowords*. The topological space $\widehat{A^+}$ has a continuous semigroup operation that extends the concatenation product in A^+. The resulting semigroup $\widehat{A^+}$ is profinite. In fact we call $\widehat{A^+}$ the *free A-generated profinite semigroup* because for every map φ from A into a profinite semigroup S, there is a unique continuous homomorphism $\hat{\varphi} : \widehat{A^+} \to S$ whose restriction to A is φ. The elements of A^+ are isolated points of $\widehat{A^+}$. The elements of $\widehat{A^+} \setminus A^+$ can not be the limit of a sequence $(u_n)_n$ of elements of A^+ such that the sequence $(|u_n|)_n$ is bounded. For this reason we say that the elements of $\widehat{A^+} \setminus A^+$ have *infinite length*.

The definition of the *free A-generated profinite monoid* $\widehat{A^*}$ is similar to that of $\widehat{A^+}$: we just substitute "semigroups" by "monoids", and "semigroup homomorphisms" by "monoid homomorphisms". Considering the empty word as an isolated point of $\widehat{A^+} \cup \{1\}$, we may see $\widehat{A^+} \cup \{1\}$ as being $\widehat{A^*}$.

A *clopen* subset of a topological space is a subset that is open and closed. The following proposition [1] makes the connection between rational languages and the topology of $\widehat{A^+}$. For a proof see [2, Theorem 3.6.1].

Proposition 2.2. *If L is a subset of A^+ then L is rational if and only if the closure of L in $\widehat{A^+}$ is clopen. Moreover, the topology of $\widehat{A^+}$ is generated by the closures of the rational subsets of A^+.*

We say that a subset T of a semigroup S is *factorial* if it is closed for taking factors. This extends the notion of factorial language.

Lemma 2.3. *If L is a factorial rational language of A^+ then the closure of L in $\widehat{A^+}$ is factorial.*

Proof: Suppose that $xvy \in \overline{L}$, where $x, y \in \widehat{A^*}$ and $v \in \widehat{A^+}$. Let $(x_n)_n$, $(y_n)_n$ and $(v_n)_n$ be sequences of elements of A^+ converging to x, y and v, respectively. The set $\overline{L}$ is an open neighborhood of xvy by Proposition 2.2. Since $(x_n v_n y_n)_n$ converges to xvy, there is N such that $n \geq N$ implies $x_n v_n y_n \in \overline{L} \cap A^+$. The elements of A^+ are isolated points of $\widehat{A^+}$, thus $\overline{L} \cap A^+ = L$. Since L is factorial, if $n \geq N$ then $v_n \in L$, thus $v \in \overline{L}$. ■

A subset K of $\widehat{A^+}$ is *prolongable* if for all $u \in K$ there are $a, b \in A$ such that $aub \in K$.

Lemma 2.4. *If L is a prolongable language of A^+ then the closure of L in $\widehat{A^+}$ is prolongable.*

Proof: Let $u \in \overline{L}$ and let $(u_n)_n$ be a sequence of elements of L converging to u. Since L is prolongable, for each n there are letters a_n and b_n such that $a_n u_n b_n \in L$. Since there is a finite number of letters, the sequence $(a_n, b_n)_n$ has a constant subsequence equal to (a, b). Then $aub \in \overline{L}$. ■

A *prefix* (respectively, *suffix*) of a pseudoword w of $\widehat{A^+}$ is a pseudoword u of $\widehat{A^*}$ such that $w = u\pi$ (respectively, $w = \pi u$) for some π in $\widehat{A^*}$.

Lemma 2.5 ([2, Exercise 10.2.10]). *Let $\pi, \rho \in \widehat{A^+}$ and $u, v \in A^+$ be such that $u\pi = v\rho$ or $\pi u = \rho v$. If $|u| = |v|$ then $u = v$ and $\pi = \rho$.*

For every positive integer n and for every pseudoword w of $\widehat{A^+}$, let $\mathrm{i}_n(w)$ (respectively $\mathrm{t}_n(w)$) denote the unique longest prefix (respectively suffix) of w with length less or equal to n. The maps i_n and t_n are continuous.

The next lemma is a particular case of Lemma 7.2 of [6]:

Lemma 2.6. *Let $p, q, f \in \widehat{A^+}$ with f idempotent. If u is a finite factor of pfq then u is a factor of pf or is a factor of fq.*

2.4. Idempotent-bound semigroup elements. If s is an element of a compact semigroup S, then the closure of the subsemigroup generated by s has a unique idempotent [14], which we denote by s^ω. For profinite semigroups we have $s^\omega = \lim s^{n!}$. Let e and f be idempotents of S. We say that an element u of S is *bounded* by e and f (by this order) if $u = euf$. An element is *idempotent-bound* if it is bounded by some pair of idempotents. Regular elements are idempotent-bound.

Lemma 2.7. *If S is a compact semigroup, then an element u of S is idempotent-bound if and only if $u \in SuS$.*

Proof: The "only if" part is trivial. Conversely, let $x, y \in S$ be such that $u = xuy$. Then $u = x^n u y^n$ for all positive integer n. Hence $u = x^\omega u y^\omega$, since x^ω and y^ω are adherent points of $(x^n)_n$ and $(y^n)_n$, respectively. ■

Lemma 2.8. *A $\mathcal{J}$-class J of a compact semigroup S contains an idempotent-bound element if and only if all elements of J are idempotent-bound.*

Proof: Let u be an element of J bounded by the idempotents e and f. If $v \in J$ then $u = xvy$ and $v = zut$ for some $x, y, z, t \in S^1$. Then $v = zut = zeuft = zexvyft \in SvS$, and the result follows from Lemma 2.7. ∎

For compact semigroups, Lemma 2.8 allow us to refer to an *idempotent-bound $\mathcal{J}$-class.* The elements of an idempotent-bound $\mathcal{J}$-class are not necessarily bounded by the same idempotents; on the other hand, the elements of an $\mathcal{H}$-class are bounded by the same idempotents.

Lemma 2.9. *Let π be an element of $\widehat{A^+}$ such that $\pi = e\pi f$ for some idempotents e and f of $\widehat{A^+}$. If s is a suffix of e and p is a prefix of f then the pseudowords π and $s\pi p$ are $\mathcal{J}$-equivalent.*

Proof: We want to prove that π is a factor of $s\pi p$. Let e_0 and f_0 be such that $e = e_0 s$ and $f = pf_0$. Then $\pi = e_0(s\pi p)f_0$. ∎

Lemma 2.10. *Let π be an element of $\widehat{A^+}$ such that $\pi = e\pi f$ for some idempotents e and f of $\widehat{A^+}$. Let ρ be an element of $\widehat{A^+}$ such that $\pi = x\rho y$ for some $x, y \in A^*$. Then π and ρ are $\mathcal{J}$-equivalent.*

Proof: We want to show that π is a factor of ρ. Since $e\pi f = x\rho y$, we have $\mathrm{i}_n(e) = x$ and $\mathrm{t}_n(f) = y$. Hence there are e_0 and f_0 such that $e = xe_0$ and $f = f_0 y$, thus $xe_0\pi f_0 y = x\rho y$. Therefore $e_0\pi f_0 = \rho$ by Lemma 2.5. ∎

Of course, every homomorphic image of an idempotent(-bound) element is also idempotent(-bound). We have a sort of converse in the compact case:

Lemma 2.11. *Let $\varphi : T \to S$ be an onto continuous homomorphism between compact semigroups. If e is an idempotent of S then $\varphi^{-1}(e)$ contains an idempotent. If s is an element of S bounded by the idempotents e_1 and e_2, then there are $t \in \varphi^{-1}(s)$ and idempotents $f_i \in \varphi^{-1}(e_i)$ such that $t = f_1 t f_2$.*

Proof: Let $x \in \varphi^{-1}(e)$. Since T is compact we can consider the idempotent x^ω, which is an adherent point of the sequence $(x^n)_n$. Since φ is continuous and e is idempotent, we have $x^\omega \in \varphi^{-1}(e)$.

Let $t_0 \in \varphi^{-1}(s)$. As we have proved, the set $\varphi^{-1}(e_i)$ contains some idempotent f_i. Let $t = f_1 t_0 f_2$. Then $\varphi(t) = e_1 s e_2 = s$ and $t = f_1 t f_2$. ∎

3. Infinite pseudowords and subshifts

3.1. Coding of infinite pseudowords. One of the main reasons to investigate links between profinite semigroups and symbolic dynamics is that if $\mathcal{X}$

and $\mathcal{Y}$ are subshifts of $A^{\mathbb{Z}}$ such that $\overline{L(\mathcal{X})} \setminus A^+ = \overline{L(\mathcal{Y})} \setminus A^+$ then $\mathcal{X} = \mathcal{Y}$. In other words, we can recover $\mathcal{X}$ from $\overline{L(\mathcal{X})} \setminus A^+$. So it seems a good idea to investigate the set $\overline{L(\mathcal{X})} \setminus A^+$. This program has been initiated by Almeida [4]. Our goal is to find algebraic-topological conjugacy invariants in $\widehat{A^+}$ related to $\overline{L(\mathcal{X})} \setminus A^+$. We want to understand the changes of $\overline{L(\mathcal{X})} \setminus A^+$ when a conjugacy is applied to $\mathcal{X}$.

In [2, Lemma 10.6.1] it is proved that $\Phi_k : A^+ \to (A^{k+1})^*$ has a unique continuous extension $\widehat{A^+} \to \widehat{(A^{k+1})^*}$, which we also denote by Φ_k. Let $G : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a code having block map $g : A^{2k+1} \to B$, and memory and anticipation k. Let $\hat{g}$ be the unique continuous monoid homomorphism from $\widehat{(A^{2k+1})^*}$ into $\widehat{B^*}$ that extends g, and consider the map $\bar{g} = \hat{g} \circ \Phi_{2k}$. The map $\bar{g}$ extends the coding process described by g to every pseudoword of $\widehat{A^+}$. For all $u, v \in \widehat{A^+}$ we have:

$$\bar{g}(uv) = \bar{g}(u)\bar{g}(\mathrm{t}_{2k}(u)v) = \bar{g}(u\,\mathrm{i}_{2k}(v))\bar{g}(v) = \bar{g}(u\,\mathrm{i}_k(v))\bar{g}(\mathrm{t}_k(u)v).$$

This property is easily seen to be true when we have Φ_{2k} instead of $\bar{g}$, which suffices to prove the general case since A^+ is dense in $\widehat{A^+}$. The following properties are also inherited from the properties of Φ_{2k}: we have $\bar{g}(u) = 1$ if and only if $|u| < 2k+1$; if $u \in A^+$ and $|u| \geq 2k+1$ then $|\bar{g}(u)| = |u| - 2k$; and $u \in \widehat{A^+} \setminus A^+$ if and only if $\bar{g}(u) \in \widehat{B^+} \setminus B^+$. In general $\bar{g}$ is not a homomorphism (but it is so if $k = 0$). However it shares some nice properties with homomorphisms:

Lemma 3.1. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a code. Then:*

(1) For $\mathcal{K} \in \{\mathcal{J}, \mathcal{R}, \mathcal{L}\}$ and $u, v \in \widehat{A^+}$, if $u \leq_{\mathcal{K}} v$ then $\bar{g}(u) \leq_{\mathcal{K}} \bar{g}(v)$.

(2) If w is a regular element of $\widehat{A^+}$ then $\bar{g}(w)$ is a regular element of $\widehat{B^+}$.

(3) If w is an idempotent-bound element of $\widehat{A^+}$ then $\bar{g}(w)$ is an idempotent-bound element of $\widehat{B^+}$.

Proof: Suppose that $u \leq_{\mathcal{J}} v$. Then $u = xvy$ for some $x, y \in \widehat{A^*}$. Hence

$$\bar{g}(u) = \bar{g}(xv) \cdot \bar{g}(\mathrm{t}_{2k}(xv)y) = \bar{g}(x\mathrm{i}_{2k}(v)) \cdot \bar{g}(v) \cdot \bar{g}(\mathrm{t}_{2k}(xv)y).$$

Thus $\bar{g}(u) \leq_{\mathcal{J}} \bar{g}(v)$. The proof for the relations $\mathcal{R}$ and $\mathcal{L}$ is similar.

If w is a regular element of $\widehat{A^+}$ then $w = wxw$ for some x. Hence

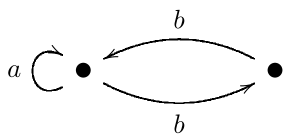
$$\bar{g}(w) = \bar{g}(wxw) = \bar{g}(w) \cdot \bar{g}(\mathrm{t}_{2k}(w)xw) = \bar{g}(w) \cdot \bar{g}(\mathrm{t}_{2k}(w)x\,\mathrm{i}_{2k}(w)) \cdot \bar{g}(w),$$

which proves that $\bar{g}(w)$ is regular.

Let e and f be idempotents such that $w = ewf$. Then $\bar{g}(w) = \bar{g}(e\mathrm{i}_{2k}(w)) \cdot \bar{g}(w) \cdot \bar{g}(\mathrm{t}_{2k}(w)f)$. Since $e\mathrm{i}_{2k}(w)$ and $\mathrm{t}_{2k}(w)f$ are infinite pseudowords, we conclude that $\bar{g}(e\mathrm{i}_{2k}(w))$ and $\bar{g}(\mathrm{t}_{2k}(w)f)$ are different from 1 (in fact they are infinite). Hence $\bar{g}(w)$ is idempotent-bound by Lemma 2.7. ■

3.2. The mirage. Given a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, let $\mathrm{Mir}(\mathcal{X})$ be the set of pseudowords of $\widehat{A^+}$ whose finite factors belong to $L(\mathcal{X})$. We call $\mathrm{Mir}(\mathcal{X})$ the *mirage* of $\mathcal{X}$ in $\widehat{A^+}$. Note that $\mathrm{Mir}(\mathcal{X})$ is a union of $\mathcal{J}$-classes. It is not surprising that in order to understand the changes of $\overline{L(\mathcal{X})} \setminus A^+$ when a conjugacy is applied to $\mathcal{X}$, one is lead to the mirage, because block maps operate "locally" on elements of $A^{\mathbb{Z}}$, and the definition of the mirage reflects this "locality". In general the sets $\mathrm{Mir}(\mathcal{X})$ and $\overline{L(\mathcal{X})}$ do not coincide.

Example 3.2. *Consider the sofic subshift $\mathcal{Z}$ with the following presentation:*



The finite factors of $ab^{\omega+1}a$ are a, ab^n, b^n and b^na, with $n \geq 1$. Hence $ab^{\omega+1}a \in \mathrm{Mir}(\mathcal{Z})$. If $n \geq 2$ then $ab^{n!+1}a \notin L(\mathcal{Z})$ because $n!+1$ is odd. Hence $ab^{\omega+1}a \notin \overline{L(\mathcal{Z})}$ because, by Proposition 2.2, the set $\overline{L(\mathcal{Z})}$ is open.

The *shadow* of $\mathcal{X}$ in $\widehat{A^+}$ is the set $\mathrm{Sha}(\mathcal{X}) = \bigcup_{u \in \overline{L(\mathcal{X})}} [u]_{\mathcal{J}}$. If $\mathcal{X}$ is sofic then $\overline{L(\mathcal{X})} = \mathrm{Sha}(\mathcal{X})$ by Lemma 2.3.

Let $\mathrm{Mir}_n(\mathcal{X})$ be the set of pseudowords whose finite factors of length n belong to $L(\mathcal{X})$. Note that $\mathrm{Mir}(\mathcal{X}) = \bigcap_{n \geq 1} \mathrm{Mir}_n(\mathcal{X})$. We have

$$\mathrm{Mir}_n(\mathcal{X}) = \widehat{A^+} \setminus \Big(\bigcup_{w \in A^n \setminus L_n(\mathcal{X})} \widehat{A^*} w \widehat{A^*} \Big).$$

The set $\widehat{A^*} w \widehat{A^*}$ is clopen because it is the closure in $\widehat{A^+}$ of the rational set $A^* w A^*$. Hence $\mathrm{Mir}_n(\mathcal{X})$ is clopen and $\mathrm{Mir}(\mathcal{X})$ is closed. It is clear that $L(\mathcal{X}) \subseteq \mathrm{Mir}(\mathcal{X})$, thus $\overline{L(\mathcal{X})} \subseteq \mathrm{Mir}(\mathcal{X})$, and so $\overline{L(\mathcal{X})} \subseteq \mathrm{Sha}(\mathcal{X}) \subseteq \mathrm{Mir}(\mathcal{X})$. We do not know if $\overline{L(\mathcal{X})} = \mathrm{Sha}(\mathcal{X})$ implies that $\mathcal{X}$ is sofic. If $\mathcal{X}$ is of finite type then for all sufficiently large n we have $L(\mathcal{X}) = \mathrm{Mir}_n(\mathcal{X}) \cap A^+$. Since A^+ is dense in $\widehat{A^+}$ and $\mathrm{Mir}_n(\mathcal{X})$ is clopen, this implies $\overline{L(\mathcal{X})} = \mathrm{Mir}_n(\mathcal{X}) = \mathrm{Mir}(\mathcal{X})$ for all sufficiently large n.

Lemma 3.3. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a code. Then*

(1) $\bar{g}(\overline{L_n(\mathcal{X})}) \subseteq \overline{L_{n-2k}(\mathcal{Y})}$ *for all* $n \geq 2k+1$*;*
(2) $\bar{g}(\overline{(L(\mathcal{X})}) \subseteq \overline{L(\mathcal{Y})} \cup \{1\}$*;*
(3) $\bar{g}(\mathrm{Sha}(\mathcal{X})) \subseteq \mathrm{Sha}(\mathcal{Y}) \cup \{1\}$*;*
(4) $\bar{g}(\mathrm{Mir}_n(\mathcal{X})) \subseteq \mathrm{Mir}_{n-2k}(\mathcal{Y}) \cup \{1\}$ *for all* $n \geq 2k+1$*;*
(5) $\bar{g}(\mathrm{Mir}(\mathcal{X})) \subseteq \mathrm{Mir}(\mathcal{Y}) \cup \{1\}$*.*

Proof: If $u \in L_n(\mathcal{X})$, then $u = x_{[1,n]}$ for some $x \in \mathcal{X}$. If $n < 2k+1$ then $\bar{g}(u) = 1$. Let $y = G(x)$. If $n \geq 2k+1$ then $y_{[k+1,n-k]} = \bar{g}(u)$, thus $\bar{g}(u) \in L_{n-2k}(\mathcal{Y})$. This proves (1), from which we deduce that

$$\bar{g}(L(\mathcal{X})) = \bigcup_{n\geq 1} \bar{g}(L_n(\mathcal{X})) \subseteq \bigcup_{n\geq 2k+1} L_{n-2k}(\mathcal{Y}) \cup \{1\} = L(\mathcal{Y}) \cup \{1\}.$$

Then (2) follows from the continuity of $\bar{g}$, and (3) follows from (2) and from Lemma 3.1 (1).

Let $x = x_1 \dots x_m \in \mathrm{Mir}_n(\mathcal{X}) \cap A^+$, with $x_i \in A$. If $m < n$ then $\bar{g}(x)$ is a word of length less than $n-2k$, and all such words belong to $M_{n-2k}(\mathcal{Y}) \cup \{1\}$. We suppose that $m \geq n$. Let w be a factor of length $n-2k$ of $\bar{g}(x)$. Then $\bar{g}(x) = pwq$ for some $p, q \in B^*$. Let $|p| = r$, $|q| = s$. We have

$$pwq = \bar{g}(x) = \bar{g}(x_{[1,2k+r]})\bar{g}(x_{[r+1,m-s]})\bar{g}(x_{[m-s+1-2k,m]}).$$

Since $|\bar{g}(x_{[1,2k+r]})| = r$ and $|\bar{g}(x_{[m-s+1-2k,m]})| = s$, we have $p = \bar{g}(x_{[1,2k+r]})$, $q = \bar{g}(x_{[m-s-1-2k,m]})$ and $w = \bar{g}(x_{[r+1,m-s]})$. Hence $|x_{[r+1,m-s]}| = |w| + 2k = n$ and therefore $x_{[r+1,m-s]} \in L_n(\mathcal{X})$. From item (1) it follows that $w \in L_{n-2k}(\mathcal{Y})$. Hence $\bar{g}(x) \in \mathrm{Mir}_{n-2k}(\mathcal{Y})$. This proves that $\bar{g}(\mathrm{Mir}_n(\mathcal{X}) \cap A^+) \subseteq \mathrm{Mir}_{n-2k}(\mathcal{Y}) \cup \{1\}$. Hence $\bar{g}(\mathrm{Mir}_n(\mathcal{X})) \subseteq M_{n-2k}(\mathcal{Y}) \cup \{1\}$ because A^+ is dense in the compact space $\widehat{A^+}$, and $\mathrm{Mir}_n(\mathcal{X})$ and $\mathrm{Mir}_{n-2k}(\mathcal{Y}) \cup \{1\}$ are clopen.

From item (4) we deduce the following:

$$\bar{g}(\mathrm{Mir}(\mathcal{X})) \subseteq \bigcap_{n\geq 1} \bar{g}(\mathrm{Mir}_n(\mathcal{X})) \subseteq \bigcap_{n\geq 2k+1} \mathrm{Mir}_{n-2k}(\mathcal{Y}) \cup \{1\} = \mathrm{Mir}(\mathcal{Y}) \cup \{1\},$$

which proves (5). ■

The following lemma and its two corollaries are crucial since they are the basic tools to be used in this article.

Lemma 3.4. *Let* $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ *be a code. Suppose that* $G = g_1^{[-k,k]} = g_2^{[-l,l]}$ *and that* $k \geq l$*. Let* u *be an element of* $\widehat{A^+}$ *with length greater or equal to* $2k+1$*. Let* $r = \mathrm{i}_{k-l}(\bar{g}_2(u))$ *and* $s = \mathrm{t}_{k-l}(\bar{g}_2(u))$*. If* $u \in \mathrm{Mir}_{2k+1}(\mathcal{X})$ *then* $\bar{g}_2(u) = r\bar{g}_1(u)s$*.*

Proof: If $x \in \mathcal{X}$ then $g_1(x_{[-k,k]}) = g_2(x_{[-l,l]})$. Every element of $L_{2k+1}(\mathcal{X})$ is of the form $x_{[-k,k]}$ for some $x \in \mathcal{X}$. Therefore,

$$\text{if } pwq \in L_{2k+1}(\mathcal{X}) \text{ and } |p| = |q| = k - l \text{ then } g_1(pwq) = g_2(w). \tag{3.1}$$

Since u belongs to the open set $\mathrm{Mir}_{2k+1}(\mathcal{X})$ and $|u| \geq 2k+1$, there is a sequence $(u_n)_n$ of elements of $A^+ \cap \mathrm{Mir}_{2k+1}(\mathcal{X})$ converging to u all of whose terms have length greater than $2k$. Let $u_n = z_1 \dots z_t$, with $z_i \in A$. Then,

$$\begin{aligned}
\bar{g}_2(u_n) &= \underbrace{\prod_{i=1}^{k-l} g_2(z_{[i,i+2l]})}_{r_n} \prod_{i=k-l+1}^{t-k-l} g_2(z_{[i,i+2l]}) \underbrace{\prod_{i=t-k-l+1}^{t-2l} g_2(z_{[i,i+2l]})}_{s_n} \\
&= r_n \prod_{i=1}^{t-2k} g_2(z_{[i+k-l,i+k+l]}) s_n \\
&= r_n \prod_{i=1}^{t-2k} g_1(z_{[i,i+2k]}) s_n \quad \text{(by (3.1))} \\
&= r_n \bar{g}_1(u_n) s_n.
\end{aligned}$$

We have $r_n = \mathrm{i}_{k-l}(\bar{g}_2(u_n))$ and $s_n = \mathrm{t}_{k-l}(\bar{g}_2(u_n))$. Hence $\bar{g}_2(u) = r\bar{g}_1(u)s$, since i_{k-l}, t_{k-l} and $\bar{g}_i$ are continuous. ∎

Corollary 3.5. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy, and let $H = h^{[-l,l]} : \mathcal{Y} \to \mathcal{X}$ be its inverse. Consider an element u of $\widehat{A^+}$ with length greater or equal to $2k+2l+1$. If $u \in \mathrm{Mir}_{2k+2l+1}(\mathcal{X})$ then $u = \mathrm{i}_{k+l}(u)\bar{h}\bar{g}(u)\mathrm{t}_{k+l}(u)$.*

Proof: Let f the map that associates to each element of $A^{2k+2l+1}$ the letter $h\bar{g}(u)$. Then $H \circ G = f^{[-(k+l),k+l]} = \mathrm{Id}^{[0,0]}$ and $\bar{f} = \bar{h}\bar{g}$. Applying Lemma 3.4 with $g_1 = f^{[-(k+l),k+l]}$ and $g_2 = \mathrm{Id}^{[0,0]}$ we deduce the result. ∎

Corollary 3.6. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy and let $H = h^{[-l,l]} : \mathcal{Y} \to \mathcal{X}$ be its inverse. Consider an element v of $\widehat{A^+}$. If r and s are words of length $k+l$ such that $rvs \in \mathrm{Mir}_{2k+2l+1}(\mathcal{X})$ then $v = \bar{h}\bar{g}(rvs)$.*

Proof: By Corollary 3.5 we have $rvs = r\bar{h}\bar{g}(rvs)s$, thus $v = \bar{h}\bar{g}(rvs)$ by Lemma 2.5. ∎

The following lemma provides a tool for dealing with idempotent-bound pseudowords in the mirage.

Lemma 3.7. *Let $G = g^{[0,0]} : \mathcal{X} \to \mathcal{Y}$ be a one-block conjugacy with inverse $H = h^{[-k,k]}$. Let e and f be idempotents of* $\mathrm{Mir}(\mathcal{X})$ *and let $\varepsilon = \bar{g}(e)$ and $\phi = \bar{g}(f)$. Suppose that v is an element of* $\mathrm{Mir}(\mathcal{Y})$ *such that $v = \varepsilon v \phi$. Consider the pseudoword $w = \bar{h}\big[\mathrm{t}_k(\varepsilon) v\, \mathrm{i}_k(\phi)\big]$. Then w is an element of* $\mathrm{Mir}(\mathcal{X})$ *such that $\bar{g}(w) = v$ and $w = ewf$.*

Proof: The map $\bar{g}$ is a homomorphism, hence ε and ϕ are idempotents. By Lemma 2.9, $\mathrm{t}_k(\varepsilon) v\, \mathrm{i}_k(\phi)$ is $\mathcal{J}$-equivalent to v, and therefore it is also an element of $\mathrm{Mir}(\mathcal{Y})$. Thus $w \in \mathrm{Mir}(\mathcal{X})$ by Lemma 3.3. Hence, by Corollary 3.6, $\bar{g}(w) = v$. We have the following chain of equalities:

$$\begin{aligned} w &= \bar{h}\big[\mathrm{t}_k(\varepsilon)\, \varepsilon \cdot \varepsilon v \phi \cdot \phi\, \mathrm{i}_k(\phi)\big] \\ &= \bar{h}\big[\mathrm{t}_k(\varepsilon)\, \varepsilon\, \mathrm{i}_k(\varepsilon)\big] \cdot \bar{h}\big[\mathrm{t}_k(\varepsilon)\, \varepsilon v \phi\, \mathrm{i}_k(\phi)\big] \cdot \bar{h}\big[\mathrm{t}_k(\phi)\, \phi\, \mathrm{i}_k(\phi)\big] \\ &= \bar{h}\big[\mathrm{t}_k(\varepsilon)\, \varepsilon\, \mathrm{i}_k(\varepsilon)\big] \cdot w \cdot \bar{h}\big[\mathrm{t}_k(\phi)\, \phi\, \mathrm{i}_k(\phi)\big]. \end{aligned}$$

Since $\bar{g}$ is a homomorphism, we have

$$w = \bar{h}\bar{g}\big[\mathrm{t}_k(e) e\, \mathrm{i}_k(e)\big] \cdot w \cdot \bar{h}\bar{g}\big[\mathrm{t}_k(f) f\, \mathrm{i}_k(f)\big]. \tag{3.2}$$

Again by Lemma 2.9, the pseudowords e and $\mathrm{t}_k(e) e \mathrm{i}_k(e)$ are $\mathcal{J}$-equivalent, and therefore they are both in $\mathrm{Mir}(\mathcal{X})$. For the same reason $\mathrm{t}_k(f) f\, \mathrm{i}_k(f)$ belongs to $\mathrm{Mir}(\mathcal{X})$. We have $\bar{h}\bar{g}\big[\mathrm{t}_k(e) e\, \mathrm{i}_k(e)\big] = e$ and $\bar{h}\bar{g}\big[\mathrm{t}_k(f) f\, \mathrm{i}_k(f)\big] = f$ by Corollary 3.6. Hence by (3.2) we conclude that $w = ewf$. ■

3.3. An invariant partially ordered set defined by the mirage. The expression *partially ordered set* will be abbreviated by the term *poset*. Let S be a semigroup. If T is a union of $\mathcal{J}$-classes of S, then we denote by $T^\dagger$ the poset defined as follows: the elements of $T^\dagger$ are the idempotent-bound $\mathcal{J}$-classes of S that are contained in T; the partial order of $T^\dagger$ is the one induced by the quasi-order $\leq_{\mathcal{J}}$.

Let G be a code between the subshifts $\mathcal{X}$ of $A^{\mathbb{Z}}$ and $\mathcal{Y}$ of $B^{\mathbb{Z}}$. Let g be a block map of G. If J is a $\mathcal{J}$-class of $\widehat{A^+}$ then all the elements of $\bar{g}(J)$ lie in the same $\mathcal{J}$-class by Lemma 3.1. Moreover, if J is idempotent-bound and is contained in $\mathrm{Mir}(\mathcal{X})$, then $[\bar{g}(J)]_{\mathcal{J}}$ is also idempotent-bound and is contained in $\mathrm{Mir}(\mathcal{Y})$, by Lemmas 3.1 and 3.3. We can therefore define the map $G^\dagger$ from $\mathrm{Mir}(\mathcal{X})^\dagger$ to $\mathrm{Mir}(\mathcal{Y})^\dagger$ that sends J to $[\bar{g}(J)]_{\mathcal{J}}$. The map $G^\dagger$ is order-preserving by Lemma 3.1.

Remark 3.8. *The map $G^\dagger$ depends only on the code G.*

Proof: Suppose that $G = g_1^{[-k,k]} = g_2^{[-l,l]}$ and that $k \geq l$. Let u be an idempotent-bound element of $\mathrm{Mir}(\mathcal{X})$. Then $\bar{g}_2(u)$ is idempotent-bound. By Lemma 3.4, there are words r and s of length $k-l$ such that $\bar{g}_2(u) = r\bar{g}_1(u)s$. By Lemma 2.10, the pseudowords $\bar{g}_1(u)$ and $\bar{g}_2(u)$ are $\mathcal{J}$-equivalent. ■

We denote by Id_Q the identity function on a set Q.

Proposition 3.9. *Let $G : \mathcal{X} \to \mathcal{Y}$ and $H : \mathcal{Y} \to \mathcal{Z}$ be codes between subshifts. Then $H^\dagger \circ G^\dagger = (H \circ G)^\dagger$. Moreover,* $(\mathrm{Id}_\mathcal{X})^\dagger = \mathrm{Id}_{\mathrm{Mir}(\mathcal{X})^\dagger}$.

Proof: It is clear that $(\mathrm{Id}_\mathcal{X})^\dagger = \mathrm{Id}_{\mathrm{Mir}(\mathcal{X})^\dagger}$.

Let $G = g^{[-k,k]} : \mathcal{X} \to \mathcal{Y}$ and $H = h^{[-l,l]} : \mathcal{Y} \to \mathcal{Z}$ be codes, with $\mathcal{X} \subseteq A^\mathbb{Z}$, $\mathcal{Y} \subseteq B^\mathbb{Z}$, and $\mathcal{Z} \subseteq C^\mathbb{Z}$. Let f be the map that associates to each element u of $A^{2k+2l+1}$ the letter $h\bar{g}(u)$. Then $f^{[-(k+l),k+l]}$ is a block map of $H \circ G$ with memory and anticipation $k+l$, and $\bar{f} = \bar{h} \circ \bar{g}$. Therefore if J is an idempotent-bound $\mathcal{J}$-class of $\mathrm{Mir}(\mathcal{X})$ then

$$H^\dagger(G^\dagger(J)) = H^\dagger([\bar{g}(J)]_\mathcal{J}) = [\bar{h}(\bar{g}(J))]_\mathcal{J} = [\bar{f}(J)]_\mathcal{J} = (H \circ G)^\dagger(J).$$

Hence $H^\dagger \circ G^\dagger = (H \circ G)^\dagger$. ■

Corollary 3.10. *The posets* $\mathrm{Mir}(\mathcal{X})^\dagger$ *and* $\mathrm{Sha}(\mathcal{X})^\dagger$ *are conjugacy invariants.*

Proof: For any code $G : \mathcal{X} \to \mathcal{Y}$ we have $G^\dagger(\mathrm{Sha}(\mathcal{X})^\dagger) \subseteq \mathrm{Sha}(\mathcal{Y})^\dagger$, by Lemma 3.3. By Proposition 3.9, if G is a conjugacy then $G^\dagger$ is an isomorphism of posets (with $(G^\dagger)^{-1} = (G^{-1})^\dagger$). ■

3.4. The Schützenberger groups in the mirage. Let H be an $\mathcal{H}$-class of a semigroup S. The *right stabilizer* of H is the submonoid of S^1 given by

$$T(H) = \{x \in S^1 : Hx = H\} = \{x \in S^1 : Hx \cap H \neq \emptyset\}.$$

In $T(H)$ we can consider the following equivalence relation:

$$x \approx y \Leftrightarrow \exists h \in H : hx = hy \Leftrightarrow \forall h \in H, hx = hy.$$

The relation $\approx$ is a monoid congruence on $T(H)$. Let $\Gamma(H) = T(H)/{\approx}$ and let ξ_H be the quotient homomorphism $T(H) \to \Gamma(H)$. The following theorem is proved in [20, Theorem 3.3] in a non-topological version; see [14] for the topological part.

Theorem 3.11. *Let H be an $\mathcal{H}$-class of a compact semigroup S. For the quotient topology, the monoid $\Gamma(H)$ is a compact group of permutations of H with the same cardinal as H. If H_1 and H_2 are two $\mathcal{H}$-classes contained* *in the same $\mathcal{J}$-class of S, then $\Gamma(H_1)$ and $\Gamma(H_2)$ are isomorphic compact groups. If H is a maximal subgroup of S, then H and $\Gamma(H)$ are isomorphic compact groups.*

The compact group $\Gamma(H)$ is called the *Schützenberger group* of H and of the $\mathcal{J}$-class of H. It is easy to prove that the Schützenberger groups of profinite semigroups are profinite groups. The non-topological version of Theorem 3.11 speaks about $\mathcal{D}$-classes, but in compact semigroups a $\mathcal{D}$-class is a $\mathcal{J}$-class.

Proposition 3.12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subshifts of $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ respectively. Let $G = g^{[0,0]} : \mathcal{X} \to \mathcal{Y}$ be a one-block conjugacy. Let U be an idempotent-bound $\mathcal{H}$-class of $\widehat{A^+}$ contained in $\mathrm{Mir}(\mathcal{X})$. Consider the $\mathcal{H}$-class V of $\widehat{B^+}$ that contains $\bar{g}(U)$. Then the Schützenberger groups of U and V are isomorphic compact groups.*

Proof: The reader should have present within this prove that $\bar{g}$ is a homomorphism. Let $u \in U$ and let e and f be idempotents of $\widehat{A^+}$ such that $u = euf$. Consider the set

$$P = \{x \in \widehat{A^+} : x = fxf \text{ and } ux \in U\}.$$

Every element of U has the form ux for some $x \in P$. Let η be the map from U into V that maps an element of the form ux with $x \in P$ to $\bar{g}(ux)$. Let h be a block map such that $G^{-1} = h^{[-k,k]}$. By Corollary 3.5 we have $ux = \mathrm{i}_k(e)\bar{h}\bar{g}(ux)\mathrm{t}_k(f)$. Therefore η is injective.

Let $v \in V$. Then there is $y \in \widehat{B^+}$ such that $v = \bar{g}(u)y$ and $y = \bar{g}(f)\, y\, \bar{g}(f)$. Since $y \in \mathrm{Mir}(\mathcal{Y})$, by Lemma 3.7 there is $x \in \mathrm{Mir}(\mathcal{X})$ such that $\bar{g}(x) = y$ and $x = fxf$. Thus $\bar{g}(ux) = v$, and $ux \in \mathrm{Mir}(\mathcal{X})$ by Lemma 2.6. Dually, there is x' such that $x' = ex'e$, $x'u \in \mathrm{Mir}(\mathcal{X})$ and $\bar{g}(x'u) = v$. Since ux and $x'u$ are bounded by the idempotents e and f, by Corollary 3.5 we have

$$ux = \mathrm{i}_k(e)\bar{h}\bar{g}(ux)\mathrm{t}_k(f) = \mathrm{i}_k(e)\bar{h}(v)\mathrm{t}_k(f) = \mathrm{i}_k(e)\bar{h}\bar{g}(x'u)\mathrm{t}_k(f) = x'u.$$

We have $G^\dagger[ux]_{\mathcal{J}} = [\bar{g}(u)]_{\mathcal{J}} = G^\dagger[u]_{\mathcal{J}}$. Therefore ux and u are $\mathcal{J}$-equivalent, since $G^\dagger$ is bijective. Moreover, ux and u are $\mathcal{H}$-equivalent because compact semigroups are stable and $ux = x'u$. Hence $ux \in U$, thus $x \in P$. This proves that η is bijective, since $v = \eta(ux)$.

Every element of $\Gamma(U)$ has the form $\xi_U(z)$ for some $z \in P$. The correspondence

$$\begin{aligned} \psi : \Gamma(U) &\to \Gamma(V) \\ \xi_U(x) &\mapsto \xi_V(\bar{g}(x)),\ x \in P. \end{aligned}$$

is a well-defined map. An element of $\Gamma(V)$ has the form $\xi_V(y)$ for some $y \in T(V)$. Since $\bar{g}(u)y \in V$ and η is onto, there is $z \in P$ such that $\bar{g}(u)y = \bar{g}(uz)$. Therefore $\xi_V(y) = \xi_V(\bar{g}(z))$, thus ψ is onto. The fact that the map ψ is one-to-one also follows easily from the fact that η is one-to-one. ■

A *labeled poset* is a poset in which every element is labeled by an element of a certain class. A morphism between two labeled posets $\mathscr{A}$ and $\mathscr{B}$ is an order-preserving map $\varphi : \mathscr{A} \to \mathscr{B}$ leaving the labels unchanged.

For a $\mathcal{J}$-class J of a compact semigroup S, we define the ordered pair $(\epsilon_J, \mathscr{G}_J)$ as follows: the symbol ϵ_J is equal to 1 if J is regular, and is equal to 0 otherwise; in both cases $\mathscr{G}_J$ is the isomorphism class (in the algebraic-topological sense) of the Schützenberger group of J. If T is a union of $\mathcal{J}$-classes of S then we denote by $T^{\dagger}_{\times}$ the labeled poset that results from $T^{\dagger}$ by labeling each $\mathcal{J}$-class J with the ordered pair $(\epsilon_J, \mathscr{G}_J)$.

Theorem 3.13. *The labeled posets* $\mathrm{Mir}(\mathcal{X})^{\dagger}_{\times}$ *and* $\mathrm{Sha}(\mathcal{X})^{\dagger}_{\times}$ *are conjugacy invariants.*

Proof: Let $G : \mathcal{X} \to \mathcal{Y}$ be a conjugacy of subshifts. By Proposition 3.9 the map $G^{\dagger} : \mathrm{Mir}(\mathcal{X})^{\dagger} \to \mathrm{Mir}(\mathcal{Y})^{\dagger}$ is an isomorphism of posets. Moreover, $G^{\dagger}(\mathrm{Sha}(\mathcal{X})) = \mathrm{Sha}(\mathcal{Y})$. It remains to prove that $G^{\dagger}$ preserves labels.

By Lemma 3.1, the maps $G^{\dagger}$ and $(G^{-1})^{\dagger}$ send regular $\mathcal{J}$-classes into regular $\mathcal{J}$-classes. Moreover, $(G^{-1})^{\dagger} = (G^{\dagger})^{-1}$, by Proposition 3.9. Therefore $G^{\dagger}$ sends non-regular $\mathcal{J}$-classes into non-regular $\mathcal{J}$-classes. Hence $G^{\dagger}$ preserves the labels ϵ_J.

By Proposition 2.1 there are one-block conjugacies G_1 and G_2 such that $G = G_2 \circ G_1^{-1}$. Let K be an element of $\mathrm{Mir}(\mathcal{X})^{\dagger}$. By Proposition 3.12 the Schützenberger group of $(G_1^{-1})^{\dagger}(K)$ is isomorphic to the Schützenberger groups of $G_1^{\dagger} \circ (G_1^{-1})^{\dagger}(K)$ and $G_2^{\dagger} \circ (G_1^{-1})^{\dagger}(K)$. By Proposition 3.9, we have $G_1^{\dagger} \circ (G_1^{-1})^{\dagger}(K) = K$ and $G_2^{\dagger} \circ (G_1^{-1})^{\dagger}(K) = G^{\dagger}(K)$. ■

Notice that $\mathrm{Mir}(\mathcal{X})^{\dagger}_{\times} = (\mathrm{Mir}(\mathcal{X}) \setminus A^{+})^{\dagger}_{\times}$, since idempotent-bound pseudowords are infinite.

Almeida proved in [4] that if $\mathcal{X}$ is a minimal subshift of $A^{\mathbb{Z}}$ then all infinite pseudowords in $\overline{L(\mathcal{X})}$ are in a common $\mathcal{J}$-class of $\widehat{A^+}$, which we denote by $J(\mathcal{X})$; furthermore $J(\mathcal{X})$ is regular and the correspondence $\mathcal{Y} \mapsto J(\mathcal{Y})$ is a bijection between the set of minimal subshifts of $A^{\mathbb{Z}}$ and the set of $\mathcal{J}$-classes of $\widehat{A^+}$ that are $\mathcal{J}$-maximal among $\mathcal{J}$-classes of infinite pseudowords over A. Therefore, if $\mathcal{X}$ is minimal, then $\mathrm{Mir}_\times^\dagger(\mathcal{X})$ has only one element, which is labeled by the isomorphism class of the Schützenberger group $G(\mathcal{X})$ of $J(\mathcal{X})$. The conjugacy invariance of $G(\mathcal{X})$ was announced in [3], but no proof was given. That result is now a particular instance of Theorem 3.13. The group $G(\mathcal{X})$ is computed in [4] for several classes of minimal subshifts. For example, if $\mathcal{X}$ is Sturmian (see [22] for background) then $G(\mathcal{X})$ is a free profinite group on two free generators.

We can generalize the results of this section to a large class of relatively free profinite semigroups. See [3] for background about relatively free profinite semigroups and pseudovarieties. Let V be a pseudovariety containing the pseudovariety of finite semilattices and such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$, where D is the pseudovariety of finite semigroups with a right zero. Denote by $\overline{\Omega}_A\mathsf{V}$ the free A-generated pro-V semigroup. For a subshift $\mathcal{X}$, one could consider the closure $\overline{L(\mathcal{X})}^{\mathsf{V}}$ of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{V}$, as well the closed set $\mathrm{Mir}(\mathcal{X};\mathsf{V})$ of elements of $\overline{\Omega}_A\mathsf{V}$ whose finite factors are in $L(\mathcal{X})$, and the set $\mathrm{Sha}(\mathcal{X};\mathsf{V}) = \bigcup_{u\in\overline{L(\mathcal{X})}^{\mathsf{V}}}[u]_{\mathcal{J}}$. The proof that $\mathrm{Mir}(\mathcal{X};\mathsf{V})_\times^\dagger$ and $\mathrm{Sha}(\mathcal{X};\mathsf{V})_\times^\dagger$ are conjugacy invariants can be treated in the same way as we did for Theorem 3.13, with the appropriate adaptations. For instance, one should prove that the map $\Phi_k : A^+ \to (A^{k+1})^*$ has a unique continuous extension $\overline{\Omega}_A\mathsf{V} \to (\overline{\Omega}_{A^{k+1}}\mathsf{V})^1$ and that Lemma 2.5 remains valid for elements of $\overline{\Omega}_A\mathsf{V}$. The proofs of these results are easy once we are familiarized with the results from Section 10.6 of [2] about semidirect products with D.

4. The sofic case

4.1. The syntactic semigroup of a sofic subshift. A binary relation $\mathcal{K}$ in a semigroup S is *stable* if $r \;\mathcal{K}\; s$ implies $tr \;\mathcal{K}\; ts$ and $rt \;\mathcal{K}\; st$ for all $r,s,t \in S$. The semigroup congruences are the stable equivalence relations. The following quasi-order is stable:

$$v \leq_L u \Leftrightarrow [\forall x,y \in A^*, xuy \in L \Rightarrow xvy \in L].$$

The equivalence relation generated by $\leq_L$ is a semigroup congruence, the *syntactic congruence* of L. The quotient of A^+ by the syntactic congruence of L is called the *syntactic semigroup* of L. We denote it by $\mathrm{Syn}(L)$. Let δ_L be the canonical homomorphism from A^+ into $\mathrm{Syn}(L)$. The relation in $\mathrm{Syn}(L)$ also denoted $\leq_L$ (or simply $\leq$) and given by

$$\delta_L(v) \leq_L \delta_L(u) \Leftrightarrow v \leq_L u,$$

is a well-defined partial order. The semigroup $\mathrm{Syn}(L)$ equipped with the partial order $\leq_L$ is an example of an *ordered semigroup*: a semigroup equipped with a partial order stable for multiplication. There is a generalization of the theory of finite semigroups to finite ordered semigroups [24]. The language L is rational if and only if $\mathrm{Syn}(L)$ is finite, and if $\mathrm{Syn}(L)$ is finite then δ_L has a unique extension to a continuous homomorphism $\hat{\delta}_L : \widehat{A^+} \to \mathrm{Syn}(L)$.

Lemma 4.1. *Let u and v be elements of $\widehat{A^+}$. If L is a rational language of A^+ then*

$$\hat{\delta}_L(v) \leq_L \hat{\delta}_L(u) \Leftrightarrow \big[\forall x, y \in \widehat{A^*},\ xuy \in \overline{L} \Rightarrow xvy \in \overline{L}\,\big].$$

Proof: Let $(u_n)_n$ and $(v_n)_n$ be sequences of elements of A^+ converging to u and v respectively. There is an integer p such that if $n \geq p$ then $\delta_L(u_n) = \hat{\delta}_L(u)$ and $\delta_L(v_n) = \hat{\delta}_L(v)$. We have then the following equivalence:

$$\hat{\delta}_L(v) \not\leq \hat{\delta}_L(u) \Leftrightarrow \big[\,\exists k : \forall n \geq k, \delta_L(v_n) \not\leq \delta_L(u_n)\,\big].$$

We want therefore to prove the equivalence of the condition

$$\exists k : \forall n \geq k, \exists x_n, y_n \in A^* : x_n u_n y_n \in L \wedge x_n v_n y_n \notin L \tag{4.1}$$

with the condition

$$\exists x, y \in \widehat{A^*} : xuy \in \overline{L} \wedge xvy \notin \overline{L}. \tag{4.2}$$

Since the elements of A^+ are isolated points, we have $\overline{L} \cap A^+ = L$. From the fact that L is a rational language, it follows from Proposition 2.2 that the set $\overline{L}$ is clopen in $\widehat{A^+}$. Moreover, $A^+ \setminus L$ is also rational and $\widehat{A^+} \setminus \overline{L} = \overline{A^+ \setminus L}$. These properties of L, together with the compactness of $\widehat{A^+}$, make an easy routine the verification of the equivalence between (4.1) and (4.2). ∎

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and let $\mathrm{Syn}(\mathcal{X})$ be the syntactic semigroup of $L(\mathcal{X})$. Note that $\mathcal{X}$ is sofic if and only if $\mathrm{Syn}(\mathcal{X})$ is finite. We denote respectively by $\delta_{\mathcal{X}}$ and $\hat{\delta}_{\mathcal{X}}$ the homomorphisms $\delta_{L(\mathcal{X})}$ and $\hat{\delta}_{L(\mathcal{X})}$. Knowing that the transition semigroup of the minimal automaton of a rational language is isomorphic to its syntactic semigroup, one can see that for a sofic subshift $\mathcal{X}$ the transition semigroup of its Krieger cover (and its Fischer cover, in case $\mathcal{X}$ is irreducible) is isomorphic to $\mathrm{Syn}(\mathcal{X})$. This fact is used in [9, 8].

If $\mathcal{X} = A^{\mathbb{Z}}$ then $\mathrm{Syn}(\mathcal{X})$ is the trivial semigroup. The subshift $A^{\mathbb{Z}}$ is usually named the *full shift* of $A^{\mathbb{Z}}$. Suppose that $\mathcal{X}$ is not the full shift. Then $\mathrm{Syn}(\mathcal{X})$ is a non-trivial semigroup with a zero (denoted by 0). One can easily prove that $\delta_{\mathcal{X}}(u) = 0 \Leftrightarrow u \notin L(\mathcal{X})$ for all $u \in A^+$ [11]. This implies that if $\mathcal{X}$ is sofic then $\hat{\delta}_{\mathcal{X}}(u) = 0 \Leftrightarrow u \notin \overline{L(\mathcal{X})}$ for all $u \in \widehat{A^+}$. The zero is the maximal element of $\mathrm{Syn}(\mathcal{X})$ for $\leq_{L(\mathcal{X})}$, because if $u \in A^+ \setminus L(\mathcal{X})$ then $xuy \notin L(\mathcal{X})$ for all $x, y \in A^*$.

4.2. Tools for dealing with the syntactic semigroup. For $u \in A^+$, $r, v, s \in A^*$, if $u = rvs$ and $|r| = k-1$, $|rv| = l$ then we denote v by any of the following notations: $u_{[k,l]}, u_{]k-1,l]}, u_{[k,l+1[}$.

Proposition 4.2. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. Let e and f be idempotents of $\widehat{A^+}$ and u an element of $\overline{L(\mathcal{X})}$ such that $u = euf$. If $\hat{\delta}_{\mathcal{X}}(u) \geq \hat{\delta}_{\mathcal{X}}(v)$ then $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \geq \hat{\delta}_{\mathcal{Y}}(\bar{g}(evf))$.*

Proof: Suppose that $x\bar{g}(u)y \in \overline{L(\mathcal{Y})}$. By Lemma 2.4 there are words r and s of length $k+l$ such that $rx\bar{g}(u)ys \in \overline{L(\mathcal{Y})}$. Let $p = \mathrm{i}_{4k+2l}(e)$ and $q = \mathrm{t}_{4k+2l}(f)$, and let $\varepsilon, \phi \in \widehat{A^+}$ be such that $e = p\varepsilon$ and $f = \phi q$. Then

$$\begin{aligned} rx\bar{g}(u)ys &= rx\bar{g}(p\varepsilon u\phi q)ys \\ &= rx\bar{g}(p_{[1,4k]}) \cdot \bar{g}(p_{]2k,4k+2l]}\varepsilon u\phi q_{[1,2k+2l]}) \cdot \bar{g}(q_{]2l,4k+2l]})ys \end{aligned}$$

Let $H = h^{[-l,l]}$ be the inverse conjugacy of G. Since the words $\bar{g}(p_{]2k,4k+2l]})$ and $\bar{g}(q_{[1,2k+2l]})$ have length $2l$, we have then

$$\begin{aligned} &\bar{h}(rx\bar{g}(u)ys) = \\ &= \bar{h}\big[rx\bar{g}(p_{[1,4k]})\bar{g}(p_{]2k,4k+2l]})\big] \cdot \bar{h}\bar{g}(p_{]2k,4k+2l]}\varepsilon u\phi q_{[1,2k+2l]}) \cdot \\ &\qquad \cdot \bar{h}\big[\bar{g}(q_{[1,2k+2l]})\bar{g}(q_{]2l,4k+2l]})ys\big] \\ &= \bar{h}\big[rx\bar{g}(p_{[1,4k+2l]})\big] \cdot \bar{h}\bar{g}(p_{]2k,4k+2l]}\varepsilon u\phi q_{[1,2k+2l]}) \cdot \bar{h}\big[\bar{g}(q_{[1,4k+2l]})ys\big] \\ &= \bar{h}\big[rx\bar{g}(p_{[1,4k+2l]})\big] \cdot p_{]3k+l,4k+2l]}\varepsilon \cdot u \cdot \phi q_{[1,k+l]} \cdot \bar{h}\big[\bar{g}(q_{[1,4k+2l]})ys\big], \end{aligned}$$

where the last equality is justified by Corollary 3.6. Let z be the pseudoword

$$\bar{h}\big[rx\bar{g}(p_{[1,4k+2l]})\big] \cdot p_{]3k+l,4k+2l]}\varepsilon \cdot v \cdot \phi q_{[1,k+l]} \cdot \bar{h}\big[\bar{g}(q_{[1,4k+2l]})ys\big].$$

Since $rx\bar{g}(u)ys \in \overline{L(\mathcal{Y})}$, we have $\bar{h}(rx\bar{g}(u)ys) \in \overline{L(\mathcal{X})}$, by Lemma 3.3. Hence, since $\hat{\delta}_{\mathcal{X}}(u) \geq \hat{\delta}_{\mathcal{X}}(v)$, from Lemma 4.1 we deduce that $z \in \overline{L(\mathcal{X})}$, thus $\bar{g}(z) \in \overline{L(\mathcal{Y})}$. Again by Corollary 3.6,

$$\bar{h}\bar{g}(p_{[1,4k+2l]}) = p_{]k+l,3k+l]}, \quad \bar{h}\bar{g}(q_{[1,4k+2l]}) = q_{]k+l,3k+l]}.$$

Hence the suffix of length $2k$ of $\bar{h}\big[rx\bar{g}(p_{[1,4k+2l]})\big]$ is $p_{]k+l,3k+l]}$, and the prefix of length $2k$ of $\bar{h}\big[\bar{g}(q_{[1,4k+2l]})ys\big]$ is $q_{]k+l,3k+l]}$. Therefore,

$$\bar{g}(z) = \bar{g}\bar{h}\big[rx\bar{g}(p_{[1,4k+2l]})\big] \cdot \bar{g}(p_{]k+l,4k+2l]}\varepsilon v\phi q_{[1,3k+l]}) \cdot \bar{g}\bar{h}\big[\bar{g}(q_{[1,4k+2l]})ys\big]. \tag{4.3}$$

For $\pi \in \{p, q\}$ we have $\bar{g}(\pi_{[1,4k+2l]}) = \bar{g}(\pi_{[1,3k+l]})\bar{g}(\pi_{]k+l,4k+2l]})$. With these factorizations and observing that the length of each of the words r, s, $\bar{g}(p_{]k+l,4k+2l]})$ and $\bar{g}(q_{[1,3k+l]})$ is $k+l$, Corollary 3.6 allows us to perform the following simplification of equality (4.3):

$$\begin{aligned} \bar{g}(z) &= x\bar{g}(p_{[1,3k+l]}) \cdot \bar{g}(p_{]k+l,4k+2l]}\varepsilon v\phi q_{[1,3k+l]}) \cdot \bar{g}(q_{]k+l,4k+2l]})y \\ &= x\bar{g}(p_{[1,4k+2l]}\varepsilon v\phi q_{[1,4k+2l]})y \\ &= x\bar{g}(evf)y. \end{aligned}$$

Therefore $x\bar{g}(u)y \in \overline{L(\mathcal{Y})} \Rightarrow x\bar{g}(evf)y \in \overline{L(\mathcal{Y})}$. Then, from Lemma 4.1 we deduce that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \geq \hat{\delta}_{\mathcal{Y}}(\bar{g}(evf))$. ∎

Theorem 4.3. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. Let e and f be idempotents of $\widehat{A^+}$. Let u and v be elements of $\widehat{A^+}$ such that $u = euf$, $v = evf$, $u \in \overline{L(\mathcal{X})}$ and $v \in \mathrm{Mir}(\mathcal{X})$. Then $\hat{\delta}_{\mathcal{X}}(u) \geq \hat{\delta}_{\mathcal{X}}(v)$ if and only if $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \geq \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$.*

Proof: The direct implication follows immediately from Proposition 4.2. Conversely, suppose that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \geq \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$. Then,

$$\begin{aligned} \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_k(e)u\,\mathrm{i}_k(f))] &= \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_k(e)\,\mathrm{i}_{2k}(e))] \cdot \hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \cdot \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_{2k}(f)\,\mathrm{i}_k(f))] \\ &\geq \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_k(e)\,\mathrm{i}_{2k}(e))] \cdot \hat{\delta}_{\mathcal{Y}}(\bar{g}(v)) \cdot \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_{2k}(f)\,\mathrm{i}_k(f))] \\ &= \hat{\delta}_{\mathcal{Y}}[\bar{g}(\mathrm{t}_k(e)v\,\mathrm{i}_k(f))]. \end{aligned} \tag{4.4}$$

For $w \in \{u, v\}$ let w_0 be $\mathrm{t}_k(e)w\,\mathrm{i}_k(f)$. By Lemma 2.9, the pseudoword w_0 is $\mathcal{J}$-equivalent to w, hence $u_0 \in \overline{L(\mathcal{X})}$ and $v_0 \in \mathrm{Mir}(\mathcal{X})$ since $\overline{L(\mathcal{X})}$ and $\mathrm{Mir}(\mathcal{X})$ are unions of $\mathcal{J}$-classes (for the former set see Lemma 2.3). Hence $\bar{g}(u_0) \in \overline{L(\mathcal{Y})}$ and $\bar{g}(v_0) \in \mathrm{Mir}(\mathcal{Y})$. The pseudowords $\varepsilon = \bar{g}(\mathrm{t}_k(e)e\,\mathrm{i}_k(e))$ and

$\phi = \bar{g}(\mathrm{t}_k(f) f \, \mathrm{i}_k(f))$ are idempotents such that $\varepsilon \bar{g}(w_0)\phi = \bar{g}(w_0)$. Therefore, if $H = h^{[-l,l]}$ is the inverse conjugacy of G, then by Proposition 4.2 we have:

$$\hat{\delta}_{\mathcal{X}}(\bar{h}\bar{g}(u_0)) \geq \hat{\delta}_{\mathcal{X}}(\bar{h}\bar{g}(v_0)). \tag{4.5}$$

By Corollary 3.5 we have $w_0 = \mathrm{i}_{k+l}(w_0)\bar{h}\bar{g}(w_0)\, \mathrm{t}_{k+l}(w_0)$. Note that $\mathrm{i}_{k+l}(w_0) = \mathrm{t}_k(e)\mathrm{i}_l(e)$ and $\mathrm{t}_{k+l}(w_0) = \mathrm{t}_l(f)\mathrm{i}_k(f)$. Let $r = e\, \mathrm{i}_l(e)$ and $s = \mathrm{t}_l(f) f$. Then $r\bar{h}\bar{g}(w_0)s = w$. Multiplying on the left both members of (4.5) by $\hat{\delta}_{\mathcal{X}}(r)$, and on the right by $\hat{\delta}_{\mathcal{X}}(s)$, we obtain $\hat{\delta}_{\mathcal{X}}(u) \geq \hat{\delta}_{\mathcal{X}}(v)$. ∎

Corollary 4.4. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. Let e and f be idempotents of $\widehat{A^+}$. Let u and v be elements of $\widehat{A^+}$ such that $u = euf$, $v = evf$, $u \in \overline{L(\mathcal{X})}$ and $v \in \mathrm{Mir}(\mathcal{X})$. Then $\hat{\delta}_{\mathcal{X}}(u) = \hat{\delta}_{\mathcal{X}}(v)$ if and only if $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$.*

Proof: Suppose that $\hat{\delta}_{\mathcal{X}}(u) = \hat{\delta}_{\mathcal{X}}(v)$. Then $v \in \overline{L(\mathcal{X})}$, since $u \in \overline{L(\mathcal{X})}$. Applying Proposition 4.2 twice, we conclude that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$. Conversely, suppose that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$. From Theorem 4.3, we deduce $\hat{\delta}_{\mathcal{X}}(u) \geq \hat{\delta}_{\mathcal{X}}(v)$. If $v \notin \overline{L(\mathcal{X})}$, then $\hat{\delta}_{\mathcal{X}}(v) = 0 \geq \hat{\delta}_{\mathcal{X}}(u)$. If $v \in \overline{L(\mathcal{X})}$ then we apply again Theorem 4.3, interchanging the roles of u and v. ∎

4.3. A syntactic conjugacy invariant of sofic subshifts. Let $\mathcal{X}$ be a sofic subshift. If $\mathcal{X}$ is the full shift then $\mathrm{Syn}(\mathcal{X})$ is the trivial semigroup, and if not then $\mathrm{Syn}(\mathcal{X}) \setminus \{0\} = \delta_{\mathcal{X}}(L(\mathcal{X}))$. Note that $\delta_{\mathcal{X}}(L(\mathcal{X}))$ is a union of $\mathcal{J}$-classes. The *labeled poset of the idempotent-bound $\mathcal{J}$-classes in the syntactic image* (LPIJSI) of $\mathcal{X}$ is the labeled poset $\delta_{\mathcal{X}}(L(\mathcal{X}))^{\dagger}_{\times}$. In this section we prove that the LPIJSI is a conjugacy invariant of sofic subshifts.

Proposition 4.5. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. If in what follows J is an idempotent-bound $\mathcal{J}$-class contained in $\overline{L(\mathcal{X})}$ then the correspondence*

$$\begin{aligned} G^s : \delta_{\mathcal{X}}(L(\mathcal{X}))^{\dagger} &\to \delta_{\mathcal{Y}}(L(\mathcal{Y}))^{\dagger} \\ [\hat{\delta}_{\mathcal{X}}(J)]_{\mathcal{J}} &\mapsto [\hat{\delta}_{\mathcal{Y}}(G^{\dagger}(J))]_{\mathcal{J}} \end{aligned}$$

is a well-defined order-preserving function.

Proof: By Lemma 2.11 and since $\hat{\delta}_{\mathcal{X}}^{-1}(\delta_{\mathcal{X}}(L(\mathcal{X}))) = \overline{L(\mathcal{X})}$, every element of $\delta_{\mathcal{X}}(L(\mathcal{X}))^{\dagger}$ is the $\mathcal{J}$-class of an idempotent-bound element of $\overline{L(\mathcal{X})}$. Let u be an element of $\overline{L(\mathcal{X})}$ such that $u = euf$ for some idempotents e and f of

$\widehat{A^+}$. Suppose that w is such that $\hat{\delta}_{\mathcal{X}}(u) \leq_{\mathcal{J}} \hat{\delta}_{\mathcal{X}}(w)$. Then there are elements p and q of A^* such that $\hat{\delta}_{\mathcal{X}}(u) = \hat{\delta}_{\mathcal{X}}(pwq)$. By Corollary 4.4, $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(epwqf))$, thus $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \leq_{\mathcal{J}} \hat{\delta}_{\mathcal{Y}}(\bar{g}(w))$, which proves that G^s is order preserving. In particular, if $u \mathrel{\mathcal{J}} v$ then $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \mathrel{\mathcal{J}} \hat{\delta}_{\mathcal{Y}}(\bar{g}(v))$, which proves that G^s is a well-defined function. ∎

The following is an immediate corollary of Proposition 3.9.

Proposition 4.6. *Let $G : \mathcal{X} \to \mathcal{Y}$ and $H : \mathcal{Y} \to \mathcal{Z}$ be conjugacies between sofic subshifts. Then $H^s \circ G^s = (H \circ G)^s$. Moreover, $(\mathrm{Id}_{\mathcal{X}})^s = \mathrm{Id}_{\delta_{\mathcal{X}}(L(\mathcal{X}))^\dagger}$.*

Corollary 4.7. *The poset $\delta_{\mathcal{X}}(L(\mathcal{X}))^\dagger$ is a conjugacy invariant of sofic subshifts.*

Proposition 4.8. *Let $G : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. If K is a $\mathcal{J}$-class of $\delta_{\mathcal{X}}(L(\mathcal{X}))$ then K is regular if and only if $G^s(K)$ is regular.*

Proof: By Lemma 2.11, if K is a regular $\mathcal{J}$-class of $\delta_{\mathcal{X}}(L(\mathcal{X}))$ then there is a regular $\mathcal{J}$-class J of $\widehat{A^+}$ such that $K = [\hat{\delta}_{\mathcal{X}}(J)]_{\mathcal{J}}$. We have $J \subseteq \overline{L(\mathcal{X})}$. Since $G^\dagger$ preserves the regularity of a $\mathcal{J}$-class, the $\mathcal{J}$-class $G^s(K) = [\hat{\delta}_{\mathcal{Y}}(G^\dagger(J))]_{\mathcal{J}}$ is regular. Hence G^s sends regular $\mathcal{J}$-classes to regular $\mathcal{J}$-classes. The same is true with $(G^{-1})^s$, which is the map $(G^s)^{-1}$, by Proposition 4.6. ∎

Proposition 4.9. *Let $G : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. If K is an idempotent-bound $\mathcal{J}$-class of $\delta_{\mathcal{X}}(L(\mathcal{X}))$ then the Schützenberger groups of K and $G^s(K)$ are isomorphic.*

Proof: By Propositions 2.1 and 4.6, we are reduced to the case where G is a one-block conjugacy. Suppose that g is a block map for G with memory and anticipation zero. Note that $\bar{g}$ is a homomorphism. Let U be an idempotent-bound $\mathcal{H}$-class of K. By Lemma 2.11 there is an element u of $\hat{\delta}_{\mathcal{X}}^{-1}(U)$ bounded by some idempotents e and f. Consider the set

$$P = \{z \in \widehat{A^+} : z = fzf \text{ and } \hat{\delta}_{\mathcal{X}}(z) \in T(U)\}.$$

By the definition of G^s, since u is idempotent-bound, we have $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) \in G^s(K)$. Let V be the $\mathcal{H}$-class of $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u))$. Every element of U has the form $\hat{\delta}_{\mathcal{X}}(uz)$ for some $z \in P$. Consider the correspondence:

$$\begin{aligned} \eta : U &\to \mathrm{Syn}(\mathcal{Y}) \\ \hat{\delta}_{\mathcal{X}}(uz) &\mapsto \hat{\delta}_{\mathcal{Y}}(\bar{g}(uz)), \ z \in P. \end{aligned}$$

Let z_1 and z_2 be elements of P. Note that $uz_i \in \overline{L(\mathcal{X})}$ and that $uz_i = euz_i f$. Therefore $\hat{\delta}_{\mathcal{X}}(uz_1) = \hat{\delta}_{\mathcal{X}}(uz_2)$ if and only if $\hat{\delta}_{\mathcal{Y}}(\bar{g}(uz_1)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(uz_2))$ by Corollary 4.4. This shows that η is a well-defined injective function.

Let $z \in P$. Then there is $z' \in \widehat{A^+}$ such that $\hat{\delta}_{\mathcal{X}}(uz) = \hat{\delta}_{\mathcal{X}}(z'u)$ and $z'u = ez'uf$. From Corollary 4.4 we deduce that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(uz)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(z'u))$. From this equality and the fact that, according to Proposition 4.5, $\hat{\delta}_{\mathcal{Y}}(\bar{g}(uz))$ is $\mathcal{J}$-equivalent to $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u))$, we deduce that $\hat{\delta}_{\mathcal{Y}}(\bar{g}(uz))$ is $\mathcal{H}$-equivalent to $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u))$ because finite semigroups are stable. Hence $\eta(U) \subseteq V$.

Let $v \in V$. Then there are $p_1, p_2 \in \widehat{B^*}$ such that $v = \hat{\delta}_{\mathcal{Y}}(\bar{g}(u)p_1) = \hat{\delta}_{\mathcal{Y}}(p_2\bar{g}(u))$. Since $\hat{\delta}_{\mathcal{Y}}(\bar{g}(u)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(uf)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(eu))$ we can assume that $p_1 = \bar{g}(f)\,p\,\bar{g}(f)$ and $p_2 = \bar{g}(e)\,p\,\bar{g}(e)$. Since $p_i \in \overline{L(\mathcal{Y})}$, by Lemma 3.7 there is $q_i \in \operatorname{Mir}(\mathcal{X})$ such that $\bar{g}(q_i) = p_i$ and $q_1 = fq_1f$, $q_2 = eq_2e$. Then $uq_1, q_2u \in \operatorname{Mir}(\mathcal{X})$ by Lemma 2.6. Since $v = \hat{\delta}_{\mathcal{Y}}(\bar{g}(uq_1)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(q_2u))$, we have $\hat{\delta}_{\mathcal{X}}(uq_1) = \hat{\delta}_{\mathcal{X}}(q_2u)$ by Corollary 4.4. Let $w = \hat{\delta}_{\mathcal{X}}(uq_1)$. Note that w is $\mathcal{R}$-below and $\mathcal{L}$-below $\hat{\delta}_{\mathcal{X}}(u)$. On the other hand we have $G^s[w]_{\mathcal{J}} = [\hat{\delta}_{\mathcal{Y}}(\bar{g}(uq_1))]_{\mathcal{J}} = G^s(K)$. Therefore w is $\mathcal{J}$-equivalent to $\hat{\delta}_{\mathcal{X}}(u)$ since G^s is bijective. Thus $w \in U$, because $\operatorname{Syn}(\mathcal{Y})$ is stable. Hence $q_1 \in P$ and $\eta(w) = v$. This proves $\eta(U) = V$, thus $|U| = |V|$.

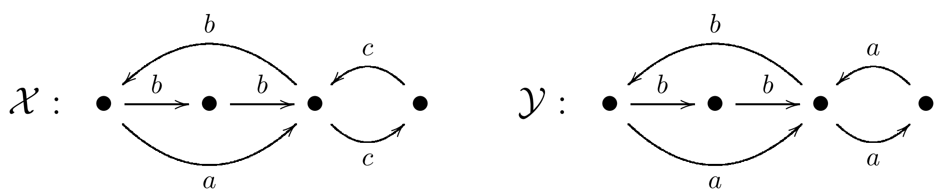
Note that every element of $\Gamma(U)$ has the form $\xi_U[\hat{\delta}_{\mathcal{X}}(z)]$ for some $z \in P$. Let $z_1, z_2 \in P$. Then $\xi_U[\hat{\delta}_{\mathcal{X}}(z_1)] = \xi_U[\hat{\delta}_{\mathcal{X}}(z_2)] \Leftrightarrow \hat{\delta}_{\mathcal{X}}(uz_1) = \hat{\delta}_{\mathcal{X}}(uz_2)$ and $\xi_V[\hat{\delta}_{\mathcal{Y}}(\bar{g}(z_2))] = \xi_V[\hat{\delta}_{\mathcal{Y}}(\bar{g}(z_2))] \Leftrightarrow \eta[\hat{\delta}_{\mathcal{X}}(uz_1)] = \eta[\hat{\delta}_{\mathcal{X}}(uz_2)]$. Therefore, since η is an injective map, the correspondence

$$\xi_U[\hat{\delta}_{\mathcal{X}}(z)] \mapsto \xi_V[\hat{\delta}_{\mathcal{Y}}(\bar{g}(z))],\ z \in P,$$

is a well-defined injective homomorphism from $\Gamma(U)$ into $\Gamma(V)$. And since $|\Gamma(U)| = |U| = |V| = |\Gamma(V)|$, it is an isomorphism. ∎

From Corollary 4.7 and Propositions 4.8 and 4.9, we finally conclude that the LPIJSI is a conjugacy invariant of sofic subshifts.

Example 4.10. *Consider the sofic subshifts $\mathcal{X}$ and $\mathcal{Y}$ with the following Fischer covers:*



Subshifts $\mathcal{X}$ and $\mathcal{Y}$ have the same zeta function and the same entropy. These invariants are also equal in their Krieger edge shifts. The LPIJSI of $\mathcal{X}$ and $\mathcal{Y}$ are not isomorphic (see Figure 1), hence $\mathcal{X}$ and $\mathcal{Y}$ are not conjugate.

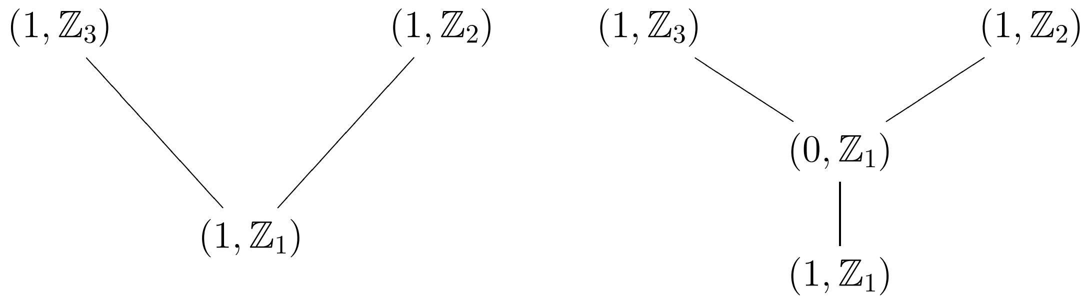


FIGURE 1. The LPIJSI of $\mathcal{X}$ and $\mathcal{Y}$.

The LPIJSI is of no use for detecting non-conjugate irreducible subshifts of finite type, since in such cases it is reduced to a single point labeled $(1,\mathbb{Z}_1)$ (see [9]). On the other hand, both subshifts of example 4.10 belong to the class of *almost finite type subshifts* [7].

4.4. Krieger and Fischer presentations. For $\alpha = (x_i)_{i\in\mathbb{Z}^-} \in A^{\mathbb{Z}^-}$, $\beta = (x_i)_{i\in\mathbb{Z}_0^+} \in A^{\mathbb{Z}_0^+}$ and $\gamma = z_1 \dots z_n \in A^n$ (where $n \in \mathbb{Z}^+$), we denote by $\alpha.\beta$ the sequence $(x_i)_{i\in\mathbb{Z}}$ and by $\alpha\gamma$ the sequence $\dots x_{-3}x_{-2}x_{-1}z_1 \dots z_n$ of $A^{\mathbb{Z}^-}$. Let $\mathcal{X}$ be a sofic subshift of $A^{\mathbb{Z}}$. Consider the set

$$\mathcal{X}^- = \{x \in A^{\mathbb{Z}^-} : x.y \in \mathcal{X} \text{ for some } y \in A^{\mathbb{Z}^+}\}.$$

The *future* in $\mathcal{X}$ of a sequence x of $\mathcal{X}^-$ is the set $\{y \in A^{\mathbb{Z}_0^+} : x.y \in \mathcal{X}\}$. The relation on $A^{\mathbb{Z}^-}$ that identifies sequences with the same future is an equivalence relation. We denote by $[x]$ the equivalence class of x. The Krieger cover of $\mathcal{X}$ is isomorphic to the graph-automaton whose states are the classes $[x]$ with $x \in \mathcal{X}^-$, and whose transition homomorphism τ is defined by $[x]\cdot a = [xa]$, for $a \in A$ such that $xa \in \mathcal{X}^-$ [19]. Let $\hat{\tau}$ be the unique continuous homomorphism extending τ to $\widehat{A^+}$. We have $\hat{\tau}(u) = \hat{\tau}(v)$ if and only if $\hat{\delta}_{\mathcal{X}}(u) = \hat{\delta}_{\mathcal{X}}(v)$, for all $u, v \in \widehat{A^+}$. We denote by $\mathrm{Im}_{\mathfrak{K}(\mathcal{X})}\, u$ the image of $\hat{\tau}(u)$ and by $\mathrm{Im}_{\mathfrak{F}(\mathcal{X})}\, u$ the image of the restriction of $\hat{\tau}(u)$ to the states of $\mathfrak{F}(\mathcal{X})$. It is well known that $\mathcal{J}$-equivalent elements in a semigroup of partial functions have the same rank (the rank is the cardinal of the image). For a $\mathcal{J}$-class K of

$\mathrm{Syn}(\mathcal{X})$ let $\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K$ (respectively $\mathrm{rank}_{\mathfrak{F}(\mathcal{X})} K$) be the rank of K considering $\mathrm{Syn}(\mathcal{X})$ as the transition semigroup of $\mathfrak{K}(\mathcal{X})$ (respectively $\mathfrak{F}(\mathcal{X})$).

Proposition 4.11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be sofic subshifts of $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$, respectively, and let $G : \mathcal{X} \to \mathcal{Y}$ be a conjugacy. If K is an idempotent-bound $\mathcal{J}$-class of $\delta_{\mathcal{X}}(L(\mathcal{X}))$ then* $\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K = \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} G^s(K)$. *If $\mathcal{X}$ and $\mathcal{Y}$ are irreducible then* $\mathrm{rank}_{\mathfrak{F}(\mathcal{X})} K = \mathrm{rank}_{\mathfrak{F}(\mathcal{X})} G^s(K)$.

Proof: If we prove the inequality $\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K \leq \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} G^s(K)$, then we just apply it to G^{-1}, deducing the converse inequality:

$$\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} G^s(K) \leq \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} (G^{-1})^s G^s(K) = \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K.$$

Let $G = g^{[-k,k]}$ and $G^{-1} = h^{[-l,l]}$. Let $u \in A^+$ and $e, f \in \widehat{A^+}$ be such that e and f are idempotents and $\hat{\delta}_{\mathcal{X}}(euf) \in K$. There are elements e_0 and f_0 of A^+ of length greater than $2k + 2l$ and such that

$$\begin{aligned}
&\delta_{\mathcal{X}}(e_0) = \hat{\delta}_{\mathcal{X}}(e), \quad \delta_{\mathcal{Y}}(\bar{g}(e_0)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(e)), \quad \mathrm{t}_{2k}(f) = \mathrm{t}_{2k}(f_0),\\
&\delta_{\mathcal{X}}(f_0) = \hat{\delta}_{\mathcal{X}}(f), \quad \delta_{\mathcal{Y}}(\bar{g}(f_0)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(f)), \quad \mathrm{i}_{2k}(e) = \mathrm{i}_{2k}(e_0).
\end{aligned}$$

Then

$$\begin{aligned}
\delta_{\mathcal{Y}}(\bar{g}(e_0 u f_0)) &= \hat{\delta}_{\mathcal{Y}}(\bar{g}(e_0)\bar{g}(\mathrm{t}_{2k}(e_0)u\,\mathrm{i}_{2k}(f_0))\bar{g}(f_0))\\
&= \hat{\delta}_{\mathcal{Y}}(\bar{g}(e)\bar{g}(\mathrm{t}_{2k}(e)u\,\mathrm{i}_{2k}(f)\bar{g}(f))\\
&= \hat{\delta}_{\mathcal{Y}}(\bar{g}(euf)).
\end{aligned}$$

For $x \in A^{\mathbb{Z}^-}$, let $\bar{g}(x) = (g(x_{i-k,i+k}))_{i<-k} \in A^{\mathbb{Z}^-}$; and for $x \in A^{\mathbb{Z}_0^+}$, let $\bar{g}(x) = (g(x_{i-k,i+k}))_{i \geq k} \in A^{\mathbb{Z}_0^+}$. Let $\mathcal{F} = (v_q)_{q \in \mathrm{Im}_{\mathfrak{K}(\mathcal{X})} e_0 u f_0}$ be a family of elements of $\mathcal{X}^-$ such that $q = [v_q] \cdot e_0 u f_0$. Consider the map

$$\begin{aligned}
\psi_{\mathcal{F}} : \mathrm{Im}_{\mathfrak{K}(\mathcal{X})} e_0 u f_0 &\to \mathrm{Im}_{\mathfrak{K}(\mathcal{Y})} \bar{g}(e_0 u f_0)\\
q &\mapsto [\bar{g}(v_q e_0 u f_0)].
\end{aligned}$$

This map is well defined, because $[\bar{g}(v_q e_0 u f_0)] = [\bar{g}(v_q\, \mathrm{i}_{2k}(e_0 u f_0))] \cdot \bar{g}(e_0 u f_0)$. Since $\delta_{\mathcal{Y}}(\bar{g}(e_0 u f_0)) = \hat{\delta}_{\mathcal{Y}}(\bar{g}(euf)) \in G^s(K)$, our goal of proving the condition $\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K \leq \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} G^s(K)$ will be achieved if we prove that $\psi_{\mathcal{F}}$ is injective. Suppose that q and r are distinct elements of $\mathrm{Im}_{\mathfrak{K}(\mathcal{X})} K$. Then, without loss of generality, we can suppose that there is $s \in A^{\mathbb{Z}^+}$ such that:

$$v_q e_0 u f_0.s \in \mathcal{X}, \quad v_r e_0 u f_0.s \notin \mathcal{X}. \tag{4.6}$$

Hence

$$\bar{g}(v_q e_0 u f_0).\bar{g}(\mathrm{t}_{2k}(v_q e_0 u f_0)s) \in \mathcal{Y}. \tag{4.7}$$

We want to prove that

$$\bar{g}(v_r e_0 u f_0).\bar{g}(\mathrm{t}_{2k}(v_q e_0 u f_0)s) \notin \mathcal{Y}. \tag{4.8}$$

Suppose the contrary. Then for all $i \geq 1$ we have

$$\bar{g}((v_r)_{[-i-k-l,-1]} e_0 u f_0 s_{[0,i+1+k+l]}) \in L(\mathcal{Y}). \tag{4.9}$$

Note that $(v_r)_{[-i-k-l,-1]} e_0 u f_0 \in L(\mathcal{X})$ (because $v_r e_0 u f_0 \in \mathcal{X}^-$) and that $f_0 s_{[0,i+1+k+l]} \in L(\mathcal{X})$ (because $v_q e_0 u f_0.s \in \mathcal{X}$). Hence, since f_0 has length greater than $2k+2l$, we have $(v_r)_{[-i-k-l,-1]} e_0 u f_0 s_{[0,i+1+k+l]} \in \mathrm{Mir}_{2k+2l+1}(\mathcal{X})$. From Corollary 3.6 we deduce that

$$(v_r)_{[-i,-1]} e_0 u f_0 s_{[0,i+1]} = \bar{h}\bar{g}((v_r)_{[-i-k-l,-1]} e_0 u f_0 s_{[0,i+1+k+l]}).$$

Therefore $(v_r)_{[-i,-1]} e_0 u f_0 s_{[0,i+1]} \in L(\mathcal{X})$ by (4.9). Since i can be taken arbitrarily large, this implies that $v_r e_0 u f_0.s \in \mathcal{X}$, which contradicts (4.6). Hence (4.8) is true. Comparing (4.7) and (4.8), we conclude that $\psi_{\mathcal{F}}(q) \neq \psi_{\mathcal{F}}(r)$. Hence $\psi_{\mathcal{F}}$ is injective, and in fact bijective, since we have proved that $\mathrm{rank}_{\mathfrak{K}(\mathcal{X})} K = \mathrm{rank}_{\mathfrak{K}(\mathcal{X})} G^s(K)$.

Again, to prove that $\mathrm{rank}_{\mathfrak{F}(\mathcal{X})} K = \mathrm{rank}_{\mathfrak{F}(\mathcal{X})} G^s(K)$ it suffices to show that $\mathrm{rank}_{\mathfrak{F}(\mathcal{X})} K \leq \mathrm{rank}_{\mathfrak{F}(\mathcal{Y})} G^s(K)$. Let r be a state of $\mathfrak{F}(\mathcal{Y})$. Since $\mathfrak{F}(\mathcal{Y})$ is strongly connected, there is $z \in B^+$ such that $r \cdot z = r$, which implies $r \cdot z^\omega = r$ and $z^\omega \in \overline{L(\mathcal{Y})}$. Let $z' = \mathrm{i}_{k+l}(z^\omega)\, z^\omega\, \mathrm{t}_{k+l}(z^\omega)$. The pseudowords z^ω and z' are $\mathcal{J}$-equivalent by Lemma 2.9, thus z' is an idempotent-bound element of $\overline{L(\mathcal{Y})}$ by Lemma 2.3. Hence $\bar{h}(z')$ is an idempotent-bound element of $\overline{L(\mathcal{X})}$ and $\bar{g}(\bar{h}(z')) = z^\omega$, by Corollary 3.6. We may therefore consider a map $\psi_{\mathcal{G}} : \mathrm{Im}_{\mathfrak{K}(\mathcal{X})}\, \bar{h}(z') \to \mathrm{Im}_{\mathfrak{K}(\mathcal{Y})}\, z^\omega$ for some family $\mathcal{G}$. Since $\psi_{\mathcal{G}}$ is onto, we have $r = [\bar{g}(x)]$ for some $x \in \mathcal{X}^-$. For each $q \in \mathrm{Im}_{\mathfrak{F}(\mathcal{X})} e_0 u f_0$, let $v_q \in A^{\mathbb{Z}^-}$ be such that $[v_q]$ is a state of $\mathfrak{F}(\mathcal{X})$ and $q = [v_q] \cdot e_0 u f_0$. Because $\mathfrak{F}(\mathcal{X})$ is the unique terminal strongly connected component of $\mathfrak{K}(\mathcal{X})$ there is $w_q \in A^+$ such that $[v_q] = [x] \cdot w_q = [x w_q]$. We have already proved that the map

$$\begin{aligned} \varphi : \mathrm{Im}_{\mathfrak{F}(\mathcal{X})}\, e_0 u f_0 &\to \mathrm{Im}_{\mathfrak{K}(\mathcal{Y})}\, \bar{g}(e_0 u f_0) \\ q &\mapsto [\bar{g}(x w_q e_0 u f_0)]. \end{aligned}$$

is injective. On the other hand, $\varphi(q) = [\bar{g}(x)] \cdot \bar{g}(\mathrm{t}_{2k}(x) w_q e_0 u f_0))$ and $[\bar{g}(x)]$ is a state of $\mathfrak{F}(\mathcal{Y})$. Since $\mathfrak{F}(\mathcal{Y})$ is the terminal component of $\mathfrak{K}(\mathcal{Y})$, this shows that $\varphi(q) \in \mathrm{Im}_{\mathfrak{F}(\mathcal{Y})}\, \bar{g}(e_0 u f_0)$. ■

Consider a sofic subshift $\mathcal{X}$. For an idempotent-bound $\mathcal{J}$-class J of $\delta_{\mathcal{X}}(L(\mathcal{X}))$ we add to the label $(\epsilon_J, \mathscr{G}_J)$ of J in the LPIJSI a third element, the rank of J in $\mathfrak{K}(\mathcal{X})$. We call the resulting labeled poset the *Krieger LPIJSI* of $\mathcal{X}$. If $\mathcal{X}$ is irreducible, then we add a fourth element, the rank of J in $\mathfrak{F}(\mathcal{X})$, and we call *Fischer LPIJSI* to this labeled poset. The invariance of the LPIJSI and Proposition 4.11 are be summarized in the following theorem:

Theorem 4.12. *The Krieger LPIJSI is a conjugacy invariant of sofic subshifts, and the Fischer LPIJSI is a conjugacy invariant of irreducible sofic subshifts.*

If we delete the non-regular $\mathcal{J}$-classes in the Krieger and Fischer LPIJSI we still have a conjugacy invariant. The existence of these weaker invariants was proved in [9]. They are ineffective to prove that the subshifts of Example 4.10 are not conjugate. The authors of [9] used a substantially different approach, one that used symbolic matrices, the main tool being Nasu's Classification Theorem for sofic subshifts [23].

It seems hard to find examples in which the Krieger and Fischer LPIJSI detect non conjugate sofic subshifts with the same syntactic LPIJSI, the same entropy and the same zeta function. The pair of of Example 1.4 in [15] is an example of this kind, but another invariant (of no use for irreducible sofic subshifts) is given there.

Note that in the proof of Proposition 4.11 the property of the Fischer cover that intervened was the fact that it is the unique strongly connected terminal component of the Krieger cover. Hence, if $\mathcal{X}$ and $\mathcal{Y}$ are conjugate sofic subshifts whose Krieger covers have unique strongly connected terminal components then Proposition 4.11 holds for the rank on such terminal components. Note that there are in fact non-irreducible subshifts with a unique strongly connected terminal component in its Krieger cover: the two subshifts of Example 1.4 in [15] are in such conditions.

5. Shift equivalence

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and let $l \geq 1$ be an integer. Consider the alphabet A^l of the elements in A^+ with length l and let γ_l be the map $\mathcal{X} \to (A^l)^{\mathbb{Z}}$ such that $(\gamma_l(x))_i = x_{[il,il+l-1]}$. The image of γ_l is a subshift of $(A^l)^{\mathbb{Z}}$ referred as the *l-th power shift* of $\mathcal{X}$ and denoted by $\mathcal{X}^l$. Two subshifts $\mathcal{X}$ and $\mathcal{Y}$ are *shift equivalent* if there is $l \geq 1$ such that $\mathcal{X}^l$ and $\mathcal{Y}^l$ are conjugate. If $\mathcal{X}^l$ and $\mathcal{Y}^l$ are conjugate then for all $k \geq l$ the subshifts $\mathcal{X}^k$ and $\mathcal{Y}^k$ are also conjugate.

Conjugate subshifts are shift equivalent, but the validity of the converse in the finite type case was a major open problem for a long time, until Kim and Roush found examples showing that the converse is false [17, 18]. There is an algorithm for deciding if two sofic subshifts are shift equivalent or not, but it is very complicated, even for finite type subshifts [16, 21].

Let $\mathcal{X}$ be a sofic subshift of $A^{\mathbb{Z}}$. Recall that $\delta_{\mathcal{X}}(u)$ is the equivalence class of u in A^+ for the syntactic congruence of $L(\mathcal{X})$. Consider an integer $l \geq 1$. We can naturally embed $(A^l)^+$ in A^+. It is easy to see that if $u \in (A^l)^+$ then $\delta_{\mathcal{X}^l}(u) = \delta_{\mathcal{X}}(u) \cap (A^l)^+$, so that the map that sends $\delta_{\mathcal{X}^l}(u)$ into $\delta_{\mathcal{X}}(u)$ is a well-defined one-to-one homomorphism from $\mathrm{Syn}(\mathcal{X}^l)$ into $\mathrm{Syn}(\mathcal{X})$, for which reason we can consider $\mathrm{Syn}(\mathcal{X}^l)$ as a subsemigroup of $\mathrm{Syn}(\mathcal{X})$. The following lemma isolates and generalizes an argument in the proof of the last theorem of [9]:

Lemma 5.1. *Let $\mathcal{X}$ be a sofic subshift of $A^{\mathbb{Z}}$ and consider an integer $l \geq 1$. Let E be the set of idempotents of $\mathrm{Syn}(\mathcal{X})$. For each e in E let w_e be an element of A^+ such that $\delta_{\mathcal{X}}(w_e) = e$. Let $l' = l \times \prod_{e \in E} |w_e| + 1$. Let s be an element of $\mathrm{Syn}(\mathcal{X})$ for which there is an idempotent f such that $s = sf$ or $fs = s$. Then $s \in \mathrm{Syn}(\mathcal{X}^{l'})$.*

Proof: Suppose that $s = sf$ (the other case is similar). Let $v \in A^+$ be such that $\delta_{\mathcal{X}}(v) = s$. Let $k = |v|\frac{(l'-1)}{|w_f|}$. Consider the word $u = v(w_f)^k$. Then $|u| = |v| \times l'$ and $\delta_{\mathcal{X}}(u) = \delta_{\mathcal{X}}(v)\delta_{\mathcal{X}}(w_f)^k = sf = s$. ∎

Proposition 5.2. *Let $\mathcal{X}$ be a sofic subshift. For an integer $l \geq 1$, let l' be as in Lemma 5.1. Then $\mathcal{X}$ and $\mathcal{X}^{l'}$ have the same LPIJSI.*

Proof: Let $P = \mathrm{Syn}(\mathcal{X})$ and $Q = \mathrm{Syn}(\mathcal{X}^{l'})$. By Lemma 5.1 the set of idempotent-bound elements of P equals the set of idempotent-bound elements of Q. We denote this set by $\mathscr{B}$. For $\mathcal{K} \in \{\mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{H}\}$ let $\mathcal{K}_S$ be the relation $\mathcal{K}$ in a semigroup S. Let $u, v \in \mathscr{B}$. Clearly if $u \leq_{\mathcal{L}_Q} v$ then $u \leq_{\mathcal{L}_P} v$. We prove the converse. Suppose that $u \leq_{\mathcal{L}_P} v$. Then $u = xv$ for some $x \in P^1$. Since v is idempotent-bound, there is an idempotent f in P such that $v = fv$. The element xf of P belongs to Q, by Lemma 5.1. Then $u \leq_{\mathcal{L}_Q} v$, since $u = (xf)v$. Similarly, $u \leq_{\mathcal{R}_Q} v$ if and only if $u \leq_{\mathcal{R}_P} v$. Hence the Green's relations $\mathcal{K}_P$ and $\mathcal{K}_Q$ coincide in $\mathscr{B}$. In particular $P^\dagger = Q^\dagger$.

Let U be an $\mathcal{H}$-class contained in $\mathscr{B}$. The proof will be complete if we show that the Schützenberger group Γ^P of U in P is isomorphic to the

Schützenberger group Γ^Q of U in Q. For $R \in \{P, Q\}$ let T^R be the right stabilizer of U in R, and let ξ^R be the quotient homomorphism $T^R \to \Gamma^R$. Suppose that U is $\mathcal{L}$-below the idempotent f of P. The set $Z = \{s \in T^P : s = fs\}$ is a subsemigroup of P. If $s \in T^P$ then $fs \in Z$ and $\xi^P(s) = \xi^P(fs)$, thus $\Gamma^P = \xi^P(Z)$. By Lemma 5.1 we have $Z \subseteq Q$, thus $Z \subseteq T^Q$. Therefore we can define the function from Γ^P to Γ^Q mapping an element of the form $\xi^P(s)$ with $s \in Z$ to $\xi^Q(s)$. This map is an isomorphism because it is clearly injective and Γ^P and Γ^Q have the same cardinal as H. ∎

Let $\mathfrak{G}$ be a graph-automaton with alphabet A and transition homomorphism τ. For an integer $l \geq 1$ denote by $\mathfrak{G}^l$ the graph-automaton with the same set of states as $\mathfrak{G}$, with alphabet A^l and whose transition homomorphism is the restriction of τ to A^l. Note that if $u \in (A^l)^+ \subseteq A^+$ then u has the same rank in $\mathfrak{G}$ and in $\mathfrak{G}^l$. It is not difficult to see that if $\mathcal{X}$ is a sofic subshift then $\mathfrak{K}(\mathcal{X}^l) = \mathfrak{K}(\mathcal{X})^l$ and $\mathfrak{F}(\mathcal{X}^l) = \mathfrak{F}(\mathcal{X})^l$, the latter case if $\mathcal{X}$ is also irreducible. From this fact and from Proposition 5.2 and Theorem 4.12, we deduce the following theorem:

Theorem 5.3. *The Krieger LPIJSI is a shift equivalence invariant of sofic subshifts, and so is the Fischer LPIJSI in the case of irreducible sofic subshifts.*

This result was proved in [9] for the invariant obtained from the Krieger and Fischer LPIJSI by removing the non-regular $\mathcal{J}$-classes.

We do not know if the structural conjugacy invariant in the mirage presented in Section 3 is a shift equivalence invariant.

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