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BOUNDEDNESS OF THE EXTREMAL SOLUTION OF SOME p−LAPLACIAN PROBLEMS

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ABSTRACT: In this article we consider the p−Laplace equation $-\Delta_p u = \lambda f(u)$ on a smooth bounded domain of $\mathbb{R}^N$ with zero Dirichlet boundary conditions. Under adequate assumptions on f we prove that the extremal solution of this problem is in the energy class $W_0^{1,p}(\Omega)$ independently of the domain. Moreover, we prove its boundedness for some range of dimensions depending on the nonlinearity f. We also obtain L^q and $W^{1,q}$ estimates for such a solution.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and $p > 1$. We consider the following problem for the p−Laplacian operator $-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$,

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1_{\lambda,p})$$

where λ is a positive parameter and f satisfies the following assumptions:

$$f \text{ is an increasing } C^2 \text{ function such that } f(0) > 0,\ f(t)^{1/(p-1)} \text{ is superlinear at infinity } (i.e.,\ f(t)/t^{p-1} \to +\infty \text{ as } t \to +\infty), \qquad (2)$$

and

$$(f(t) - f(0))^{1/(p-1)} \text{ is convex in } [0, +\infty). \qquad (3)$$

We say that $u \in W_0^{1,p}(\Omega)$ is a *solution* of $(1_{\lambda,p})$ if $f(u) \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \int_\Omega f(u)\varphi \, dx, \quad \text{for all } \varphi \in C_0^1(\Omega). \qquad (4)$$

This kind of solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions.

On the other hand, we say that $u \in W_0^{1,p}(\Omega)$ is a *regular solution* of $(1_{\lambda,p})$ if $f(u) \in L^\infty(\Omega)$ and satisfies (4). Using regularity results for degenerate

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elliptic equations, one has that every regular solution belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ (see [7], [22], and [17]).

Under assumption (2), Cabré and the author [5] proved the existence of an extremal parameter $\lambda^* \in (0, \infty)$ such that: if $\lambda < \lambda^*$ then problem $(1_{\lambda,p})$ admits a regular solution u_λ which is minimal among all other possible solutions, and if $\lambda > \lambda^*$ then problem $(1_{\lambda,p})$ admits no regular solution. Moreover, minimal solutions are semi–stable in the sense that the second variation of the energy functional associated to $(1_{\lambda,p})$ is nonnegative definite (see Definition 8 below). Using this property [5] establishes that

$$u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda \tag{5}$$

is a solution of $(1_{\lambda^*,p})$ whenever the nonlinearity $f(u)$ makes its growth comparable to u^m; u^* is called the extremal solution. As a particular case, the power nonlinearity $f(u) = (1+u)^m$ with $m > p-1$ is studied, obtaining that u^* is a bounded (and hence regular) solution if

$$N < G(m,p) := \frac{p}{p-1}\left(1 + \frac{mp}{m-(p-1)} + 2\sqrt{\frac{m}{m-(p-1)}}\,\right). \tag{6}$$

Ferrero [9] also obtained (independently of [5]) the boundedness of the extremal solution when $N < G(m,p)$ and proved using phase plane techniques that u^* is unbounded if $N \geq G(m,p)$ and the domain Ω is the unit ball of $\mathbb{R}^N$.

García–Azorero, Peral, and Puel [11, 12] studied in detail problem $(1_{\lambda,p})$ when $f(u) = e^u$. They proved that u^* is a solution independently of Ω, and that u^* is a bounded solution if in addition

$$N < F(p) := p + \frac{4p}{p-1}. \tag{7}$$

Moreover, if $N \geq p + 4p/(p-1)$ and the domain Ω is the unit ball of $\mathbb{R}^N$ then u^* is unbounded.

All these results were first obtained for the Laplacian problem $(1_{\lambda,2})$. Crandall and Rabinowitz [6] obtained the existence of the branch of minimal solutions $\{(\lambda, u_\lambda) : \lambda \in (0, \lambda^*)\}$ and proved that u^* is a solution of the extremal problem $(1_{\lambda^*,2})$ for the exponential and power nonlinearities. Moreover, they proved the boundedness of the extremal solution in the range of dimensions commented before (for $p = 2$). Joseph and Lundgren [14] made a detailed analysis for both nonlinearities when the domain is the unit ball of $\mathbb{R}^N$. Using phase plane techniques, they obtained that u^* is an unbounded solution if $N \geq G(m,2)$ for $f(u) = (1+u)^m$, and if $N \geq F(2) = 10$ for $f(u) = e^u$, where G and F are defined in (6) and (7), respectively. Brezis *et al.* [2] proved, under assumptions (2) and (3), that u^* is a weak solution of $(1_{\lambda^*,2})$. Moreover, they proved nonexistence results for $\lambda > \lambda^*$. Brezis and Vázquez [3] gave a characterization of singular semi–stable solutions and, as consequence, obtained the results in [14] using variational methods instead of phase plane techniques. In [19], [10], [18], [23], and [8] other results can be found about the extremal solution of problem $(1_{\lambda^*,2})$.

In [21] it is proved, assuming only (2), (3), and $p \geq 2$, that u^* is a solution of $(1_{\lambda^*,p})$ if $N < p(1+p')$, where $p' = p/(p-1)$. Moreover, if $p+p' \leq N < p(1+p')$ then $u^* \in L^q(\Omega)$, for all $1 \leq q < \bar{q}_0$, and $u^* \in W_0^{1,q}(\Omega)$, for all $1 \leq q < \bar{q}_1$, where

$$\bar{q}_0 := (p-1)\frac{N}{N-(p+p')} \quad \text{and} \quad \bar{q}_1 := (p-1)\frac{N}{N-(1+p')}.$$

It is also proved that $u^* \in L^\infty(\Omega)$ if $N < p+p'$. These results extend a work due to Nedev [20] for $p = 2$, establishing that u^* is a solution if $N \leq 5$, and that u^* is bounded if $N \leq 3$. It is still an open problem to prove the boundedness (or not) of the extremal solution when $p(1+p') \leq N < F(p) = p + 4p'$ even for $p = 2$ (note that when $f(u) = e^u$ and the domain Ω is the unit ball of $\mathbb{R}^N$, u^* is an unbounded solution if $N \geq F(p)$).

The main results of this work use the semi–stability property of minimal solutions to establish the boundedness of the extremal solution for a large class of nonlinearities. The first one applies to every convex f when $1 < p < 2$ and to some convex f when $p = 2$.

Theorem 1. *Assume* (2) *and* (3). *Let* u^* *be the function defined in* (5). *The following assertions hold:*
(i) *If* f *is a convex function,* $1 < p < 2$, *and*

$$N \leq H(p) := p + \frac{2p}{p-1}(1+\sqrt{2-p}), \tag{8}$$

then u^* *is a regular solution of* $(1_{\lambda^*,p})$. *In particular,* $u^* \in L^\infty(\Omega)$.
(ii) *Let*

$$\tau_- := \liminf_{t\to+\infty} \frac{(f(t)-f(0))f''(t)}{f'(t)^2}. \tag{9}$$

If $p = 2$, $0 < \tau_-$, and $N \le 6$, then u^* is a regular solution of $(1_{\lambda^*,2})$. *In particular, $u^* \in L^\infty(\Omega)$.*

First, we note that part (ii) extends the main result in [20] under an additional assumption on f: $0 < \tau_-$. Second, as we said before, if $N \ge F(p)$, where F is defined in (7), then the extremal solution u^* is not necessarily bounded. Since $1 < F(p) - H(p) < 4$, for all $1 < p < 2$, the optimal or larger dimension ensuring the boundedness will differ from (8) at most by four.

The next result extends Theorem 1, and give L^q and $W_0^{1,q}$ estimates for the extremal solution of $(1_{\lambda^*,p})$. Its proof uses some of the arguments appearing in [20] and [21].

Theorem 2. *Assume* (2) *and* (3). *Let u^* and τ_- be defined in* (5) *and* (9), *respectively. If*

$$\frac{p-2}{p-1} < \tau_- \tag{10}$$

then u^* is a solution of $(1_{\lambda^*,p})$. *Moreover the following assertions hold:*
(i) *If in addition*

$$N < N(p) := p + \frac{2p}{p-1}\left(1 + \sqrt{1-(p-1)(1-\tau_-)}\right), \tag{11}$$

then $u^* \in L^\infty(\Omega)$.
(ii) *If in addition $N \ge N(p)$ then $u^* \in L^q(\Omega)$, for all $1 \le q < q_0$, and $u^* \in W_0^{1,q}(\Omega)$, for all $1 \le q < q_1$, where*

$$q_0 := \frac{\left(p + 2\sqrt{1-(p-1)(1-\tau_-)}\ \right)N}{N - N(p)}$$

and

$$q_1 := \frac{(p-1)\left(p + 2\sqrt{1-(p-1)(1-\tau_-)}\ \right)N}{(p-1)N - 2\left(p + \sqrt{1-(p-1)(1-\tau_-)}\ \right)}. \tag{12}$$

For $f(u) = e^u$ we have that $\tau_- = 1$ and hence $N(p) = F(p)$, where F defined in (7). Therefore, Theorem 2 (i) recovers the boundedness of the extremal solution for the exponential nonlinearity. It also extends the main results in [21] under the assumption (10). However, $(p-2)/(p-1) \le \tau_-$ whenever (3) holds. Indeed, defining $h(t) := (f(t) - f(0))^{1/(p-1)}$ and using

(3) one obtains that $h''(t) \geq 0$ for all $t \geq 0$, or equivalently,

$$\frac{(f(t)-f(0))f''(t)}{f'(t)^2} \geq \frac{p-2}{p-1} \quad \text{for all} \quad t \geq 0.$$

Finally, it is easy to check that (10) implies the existence of positive constants c and $m > p-1$ such that $f(t) \geq c(1+t)^m$ for all $t \geq 0$. Hence, we are assuming more than the superlinarity of $f(t)^{1/(p-1)}$ at infinity.

Theorem 2 (i) applied to $f(u) = (1+u)^m$ with $m > p-1$, does not recovers the results commented before. Using Lemma 3.2 in [5] we improve Theorem 2 for some reaction terms $f(u)$ that make its growth comparable to a power of u.

Theorem 3. *Assume* (2), (3), *and that there exist positive constants* m *and* c *such that*

$$0 \leq f(t) \leq c(1+t)^m, \quad \textit{for all} \quad t \geq 0. \tag{13}$$

Let u^* *and* τ_- *be defined in* (5) *and* (9), *respectively. If* $(p-2)/(p-1) < \tau_-$ *and*

$$N < \frac{p}{p-1}\left(1 + \frac{mp}{m-(p-1)} + \frac{2m\sqrt{1-(p-1)(1-\tau_-)}}{m-(p-1)}\right), \tag{14}$$

then u^* *is a regular solution of* $(1_{\lambda^*,p})$. *In particular,* $u^* \in L^\infty(\Omega)$.

For $f(u) = (1+u)^m$ with $m > p-1$, we have

$$\frac{p-2}{p-1} < \tau_- = \frac{m-1}{m}.$$

Therefore, by Theorem 3 applied to $f(u) = (1+u)^m$ with $m > p-1$, we obtain that $u^* \in L^\infty(\Omega)$ if $N < G(m,p)$, where G is defined in (6). As a consequence, this result is optimal for the pure power nonlinearity.

Finally, we give a consequence of Theorem 2 and Theorem 3. This result takes into account the relation between assumption (13) and

$$\tau_+ := \limsup_{t\to+\infty} \frac{(f(t)-f(0))f''(t)}{f'(t)^2} < 1. \tag{15}$$

Theorem 4. *Assume* (2) *and* (3). *Let* u^*, τ_-, *and* τ_+ *be defined in* (5), (9), *and* (15), *respectively. If* $\tau_- > (p-2)/(p-1)$ *then* u^* *is a solution of* $(1_{\lambda^*,p})$. *Moreover the following assertions hold:*

(i) *Assume* $\tau_+ < 1$. *If in addition*

$$N < \frac{p}{p-1}\left(1 + \frac{p}{1-(p-1)(1-\tau_+)} + \frac{2\sqrt{1-(p-1)(1-\tau_-)}}{1-(p-1)(1-\tau_+)}\right), \qquad (16)$$

then $u^* \in L^\infty(\Omega)$.

(ii) *Assume* $\tau_+ \geq 1$. *If in addition*

$$N < N(p) = p + \frac{2p}{p-1}\left(1 + \sqrt{1-(p-1)(1-\tau_-)}\,\right), \qquad (17)$$

then $u^* \in L^\infty(\Omega)$.

(iii) *Assume* $\tau_- = \tau_+$. *If in addition*

$$N < F(p) = p + \frac{4p}{p-1},$$

then $u^* \in L^\infty(\Omega)$.

We remark that part (iii) in this theorem is sharp in the sense that there exists a nonlinearity f and a domain Ω such that the extremal solution u^* is unbounded if $N \geq F(p)$. Recently, Cabré, Capella, and the author [4] proved, when Ω is the unit ball of $\mathbb{R}^N$ and f is a general locally Lipschitz function, the boundedness of the extremal solution if $N < F(p)$. As we said before, this fact remains open for general domains. Theorem 4 gives a positive answer to this question for some nonlineaties.

Finally, we note that in all our results we are assuming $(p-2)/(p-1) < \tau_-$. Using the *a priori* estimates obtained in [21] and Lemma 3.2 in [5], it is possible to obtain analogous regularity results when $\tau_- = (p-2)/(p-1)$ and (13) (or (15)) holds. For instance, it can be proved that u^* is bounded for all N if $\tau = \tau_- = \tau_+ = (p-2)/(p-1)$. By Theorem 4, one expects to obtain the last assertion, since the function appearing in the right-hand side of (16) tends to infinity as τ goes to $(p-2)/(p-1)$.

The paper is organized as follows. In section 2 we give some known results. In section 3, we prove the existence and regularity of the extremal solution under suitable hypotheses on f which include the assumptions in Theorems 1 and 2 (see Proposition 10 below). In section 4 we prove Theorems 1 and 2. Finally, in section 5, we prove Theorems 3 and 4.

2. Known results

We consider

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where $g \in L^q(\Omega)$ for some $q \geq 1$.

The following result can be found in [13] or in [1].

Lemma 5. *Assume that $g \in L^q(\Omega)$, for some $q \geq 1$, and that u is a solution of (18). The following assertions hold:*
(i) *If $q > N/p$ then $u \in L^\infty(\Omega)$. Moreover,*

$$\|u\|_\infty \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, q, and $|\Omega|$.
(ii) *If $q = N/p$ then $u \in L^r(\Omega)$ for all $1 \leq r < +\infty$. Moreover,*

$$\|u\|_r \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, r, and $|\Omega|$.
(iii) *If $1 \leq q < N/p$ then $|u|^r \in L^1(\Omega)$ for all $0 < r < r_1$, where $r_1 := (p-1)Nq/(N-qp)$. Moreover,*

$$\||u|^r\|_1^{1/r} \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, q, r, and $|\Omega|$.

To obtain the estimates for the gradient of the extremal solution we will use the following regularity result which follows from Theorem 1.6 in [15].

Lemma 6. *If $g \in L^q(\Omega)$ for some $q \geq \tilde{q}$, where*

$$\tilde{q} := \frac{Np}{(p-1)N+p}, \tag{19}$$

then there exists a unique solution u of (18). If in addition $q < N/p$, then $u \in W_0^{1,r}(\Omega)$, where $r = (p-1)Nq/(N-q)$.

Remark 7. *We note that the existence and uniqueness of a solution is well known if $f \in W^{-1,p'}(\Omega)$ (see [16]), and hence, if $f \in L^{\tilde{q}}(\Omega)$ (since $\tilde{q} = (p^*)'$, where $p^* = Np/(N-p)$ corresponds to the critical Sobolev embedding).*

Now, we recall the definition of semi–stable solution introduced in [5] and give a technical lemma that we will use to prove Theorem 3 (see Lemma 3.2 in [5]).

Definition 8. Let $u \in W_0^{1,p}(\Omega)$ be a solution of $(1_{\lambda,p})$. Define

$$A_u := W_0^{1,p}(\Omega) \qquad \text{if } p \geq 2,$$

and

$$\begin{aligned} A_u := \ & \{\psi \in W_0^{1,p}(\Omega) : |\psi| \leq Cu \text{ and } |\nabla\psi| \leq C|\nabla u| \\ & \text{in } \Omega, \text{ for some constant } C\} \qquad \text{if } 1 < p < 2. \end{aligned}$$

We say that u is *semi–stable* if

$$\int_{\{\nabla u \neq 0\}} |\nabla u|^{p-2} \left\{ (p-2)(\frac{\nabla u}{|\nabla u|} \cdot \nabla\psi)^2 + |\nabla\psi|^2 \right\} dx - \lambda \int_\Omega f'(u)\psi^2 dx \geq 0, \tag{20}$$

for all $\psi \in A_u$.

We note that the left-hand side of (20) is the second variation of the energy functional associated to $(1_{\lambda,p})$ and that it is well defined on the set of admissible functions A_u (see [5] for more comments).

Lemma 9. *Assume that there exist positive constants m and c such that*

$$0 \leq f(t) \leq c(1+t)^m, \quad \textit{for all} \quad t \geq 0.$$

Let u be a solution of $(1_{\lambda,p})$. If $f(u) \in L^q(\Omega)$ for some $q \geq 1$ satisfying

$$\left(1 - \frac{p-1}{m}\right) N < qp,$$

then

$$\|u\|_\infty \leq C,$$

where C is a constant depending only on N, m, p, q, $|\Omega|$, c, and $\|\lambda f(u)\|_q$.

3. Preliminaries

The proof of all the results stated in the introduction is based in the following proposition.

Proposition 10. *Assume* (2) *and* (3)*, and define $\tilde{f}(t) := f(t) - f(0)$. If there exists $\gamma \geq 1/(p-1)$ such that*

$$\limsup_{t\to+\infty} (p-1)\gamma^2 \frac{\displaystyle\int_0^t \tilde{f}(s)^{2\gamma-2} f'(s)^2 ds}{\tilde{f}(t)^{2\gamma-1} f'(t)} < 1, \tag{21}$$

then $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a solution of $(1_{\lambda^*,p})$. Moreover, the following assertions hold:

(i) *If* $N < (2\gamma+1)p$ *then* $u^* \in L^\infty(\Omega)$. *In particular* $f(u^*) \in L^\infty(\Omega)$.
(ii) *If* $N \geq (2\gamma+1)p$ *then* $u^* \in L^q(\Omega)$, *for all* $1 \leq q < \tilde{q}_0$, *and* $f(u^*) \in L^q(\Omega)$, *for all* $1 \leq q < \tilde{q}_1$, *where*

$$\tilde{q}_0 := \frac{((p-1)(2\gamma+1)-1)N}{N-(2\gamma+1)p} \quad \text{and} \quad \tilde{q}_1 := \frac{(2\gamma+1-1/(p-1))N}{N-p/(p-1)}.$$

Remark 11. *First, we note that for* $N = (2\gamma+1)p$, *we have* $\tilde{q}_0 = +\infty$ *and hence, in this case, one obtains that* $u^* \in L^q(\Omega)$ *for all* $1 \leq q < +\infty$.

On the other hand, we want to explain the relation between assumptions (3) *and* (21). *Let* $h(t) = \tilde{f}(t)^{1/(p-1)}$. *By* (3), *h is a convex function in* $[0, +\infty)$. *In particular,* $h'(t) \geq h(t)/t$ *for all* $t > 0$, *or equivalently,*

$$f'(t) \geq (p-1)\frac{\tilde{f}(t)}{t}, \quad \text{for all } t > 0. \tag{22}$$

Therefore, under assumption (2), *we obtain that* $f'(t) > 0$ *for all* $t > 0$. *Moreover, since* $h'(s) \leq h'(t)$, *for all* $0 < s < t$, *we have*

$$f'(s) \leq \left(\frac{\tilde{f}(t)}{\tilde{f}(s)}\right)^{\frac{2-p}{p-1}} f'(t), \quad \text{for all } 0 < s < t.$$

From this inequality, we obtain

$$\int_0^t \tilde{f}(s)^{2\gamma-2} f'(s)^2 ds \leq \left(2\gamma - \frac{1}{p-1}\right)^{-1} \tilde{f}(t)^{2\gamma-1} f'(t), \quad \text{for all } t > 0,$$

and as a consequence, we get

$$\limsup_{t\to+\infty} (p-1)\gamma^2 \frac{\int_0^t \tilde{f}(s)^{2\gamma-2} f'(s)^2 ds}{\tilde{f}(t)^{2\gamma-1} f'(t)} \leq \frac{(p-1)\gamma^2}{2\gamma - 1/(p-1)}. \tag{23}$$

We note that the right-hand side of this inequality is one for $\gamma = 1/(p-1)$. *In this sense, hypothesis* (21) *is not very restrictive whenever* (3) *holds.*

Finally, we have to mention that hypothesis $(p-2)/(p-1) < \tau_-$ *in our main results may be replaced by the weakest assumption* (21) (*see Lemma* 12 *below*). *However, for the sake of clarity, it seems better to consider* $(p-2)/(p-1) < \tau_-$ *instead of* (21). *We also note that in Proposition* 10 *it is not necessary to assume that* f *is a* C^2 *function, but only* C^1. *Moreover, as a consequence of Proposition* 10 (i), *one obtains that* u^* *is bounded if* $N < p + 2p/(p-1)$, *since* $\gamma \geq 1/(p-1)$.

Proof of Proposition 10. Let $\tilde{f}(t) = f(t) - f(0)$, $\lambda \in (0, \lambda^*)$, and let u_λ be the minimal solution of $(1_{\lambda,p})$. Recalling that $u_\lambda \in C^{1,\alpha}(\bar{\Omega})$ and the definition of A_{u_λ} given in Definition 8, it is easy to check that $\psi := \tilde{f}(u_\lambda)^\gamma \in A_{u_\lambda}$, since $\gamma \geq 1/(p-1)$. Therefore, taking ψ in the semi–stability condition (20), we obtain

$$\lambda \int_\Omega \tilde{f}(u_\lambda)^{2\gamma} f'(u_\lambda)\, dx \leq (p-1)\gamma^2 \int_\Omega \tilde{f}(u_\lambda)^{2\gamma-2} f'(u_\lambda)^2 |\nabla u_\lambda|^p\, dx. \tag{24}$$

Let $g'(t) := \tilde{f}(t)^{2\gamma-2} f'(t)^2$. Taking $\varphi = g(u_\lambda)$ as a test function in (4), we have

$$\int_\Omega \tilde{f}(u_\lambda)^{2\gamma-2} f'(u_\lambda)^2 |\nabla u_\lambda|^p dx = \lambda \int_\Omega \tilde{f}(u_\lambda) g(u_\lambda)\, dx + \lambda f(0) \int_\Omega g(u_\lambda) dx. \tag{25}$$

From (24) and (25), we obtain

$$\int_\Omega \tilde{f}(u_\lambda)^{2\gamma} f'(u_\lambda) dx \leq (p-1)\gamma^2 \left(\int_\Omega \tilde{f}(u_\lambda) g(u_\lambda) dx + f(0) \int_\Omega g(u_\lambda) dx \right). \tag{26}$$

Using (21) and (23), we obtain that

$$\limsup_{t\to+\infty} (p-1)\gamma^2 \frac{\tilde{f}(t) g(t)}{\tilde{f}(t)^{2\gamma} f'(t)} < 1$$

and

$$\lim_{t\to+\infty} \frac{g(t)}{\tilde{f}(t)^{2\gamma} f'(t)} = 0.$$

From these limits and (26), it follows that

$$\int_\Omega \tilde{f}(u_\lambda)^{2\gamma} f'(u_\lambda)\, dx \leq C,$$

where C, here and in the rest of the proof, is a constant independent of λ. Moreover, by (22), we obtain

$$\int_\Omega \frac{\tilde{f}(u_\lambda)^{2\gamma+1}}{u_\lambda}\, dx \leq C, \tag{27}$$

and hence, since $f(t)^{1/(p-1)}$ is superlinear at infinity by assumption (2), $f(u_\lambda)$ is uniformly bounded in $L^{2\gamma+1-1/(p-1)}(\Omega)$.

If $N < (2\gamma+1-1/(p-1))p$ then, by Lemma 5 (i), u_λ is uniformly bounded in $L^\infty(\Omega)$. Therefore $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a regular extremal solution of $(1_{\lambda^*,p})$. This proves part of assertion (i).

Assume $N \geq (2\gamma+1-1/(p-1))p$. Using Lemma 5 (ii)–(iii), we have that u_λ is uniformly bounded in $L^r(\Omega)$ for all

$$1 \leq r < r_0 := \frac{(p-1)(2\gamma+1-1/(p-1))N}{N-(2\gamma+1-1/(p-1))p}. \tag{28}$$

We note that $r_0 \geq p$ since $\gamma \geq 1/(p-1)$.

We will do an iterative process starting with r_0. Assume that there exists $r_n \geq p$ such that u_λ is uniformly bounded in $L^r(\Omega)$ for all $1 \leq r < r_n$. Let

$$\alpha_n := \frac{2\gamma+1}{1+r_n}$$

and set $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 := \{x \in \Omega : \tilde{f}(u_\lambda)^{2\gamma+1}/u_\lambda > \tilde{f}(u_\lambda)^{2\gamma+1-\alpha_n}\}$$

and

$$\Omega_2 := \{x \in \Omega : \tilde{f}(u_\lambda) \leq u_\lambda^{1/\alpha_n}\}.$$

From (27) we have

$$\int_{\Omega_1} \tilde{f}(u_\lambda)^{2\gamma+1-\alpha_n}\, dx \leq C.$$

On the other hand,

$$\int_{\Omega_2} \tilde{f}(u_\lambda)^r\, dx \leq \int_{\Omega_2} u_\lambda{}^{\frac{r}{\alpha_n}}\, dx \leq C, \quad \text{for all } 1 \leq r < \alpha_n r_n.$$

Therefore,

$$f(u_\lambda) \in L^r(\Omega), \text{ for all } 1 \leq r < (2\gamma+1)\frac{r_n}{1+r_n} = 2\gamma+1-\alpha_n = \alpha_n r_n. \tag{29}$$

Using Lemma 5 again, the following assertions hold:

1. If $(1+r_n)N < (2\gamma+1)r_n p$ then u_λ is uniformly bounded in $L^\infty(\Omega)$. As a consequence, $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a solution of $(1_{\lambda^*,p})$.
2. If $(1+r_n)N \geq (2\gamma+1)r_n p$ then u_λ is uniformly bounded in $L^r(\Omega)$, for all
$$1 \leq r < r_{n+1} := \frac{(p-1)(2\gamma+1)r_n N}{(1+r_n)N-(2\gamma+1)r_n p}.$$

We start the bootstrap argument with r_0 given in (28). If $N < (2\gamma+1)p$ then assertion 1 holds for some n, and hence, part (i) in the proposition follows.

If $N \geq (2\gamma+1)p$ then we obtain, by assertion 2, an increasing sequence with limit

$$r_\infty = \frac{((p-1)(2\gamma+1)-1)N}{N-(2\gamma+1)p}.$$

From this, assertion 2, and (29), it follow

$$u^* \in L^q(\Omega) \text{ for all } 1 \leq q < \frac{((p-1)(2\gamma+1)-1)N}{N-(2\gamma+1)p} = \tilde{q}_0$$

and

$$f(u^*) \in L^q(\Omega) \text{ for all } 1 \leq q < \frac{(2\gamma+1-1/(p-1))N}{N-p/(p-1)} = \tilde{q}_1,$$

since all the estimates obtained for u_λ and $f(u_\lambda)$ are independent of λ.

Finally, we prove that $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a solution of $(1_{\lambda^*,p})$. Using $\gamma \geq 1/(p-1)$, we have

$$\tilde{q}_1 = \frac{(2\gamma+1-1/(p-1))N}{N-p/(p-1)} \geq \frac{p^*}{p^*-1},$$

where $p^* = Np/(N-p)$. Therefore, we obtain that $f(u_\lambda)$ converges to $f(u^*)$ as $\lambda \uparrow \lambda^*$ in $L^{p^*/(p^*-1)}(\Omega)$ and also in $W^{-1,p'}(\Omega)$, since $L^{p^*/(p^*-1)}(\Omega) \subset W^{-1,p'}(\Omega)$. The continuity of $(-\Delta_p)^{-1}$ from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$ gives that u_λ converges, strongly in $W_0^{1,p}(\Omega)$, to u^* as $\lambda \uparrow \lambda^*$. Hence, we conclude that for each $\varphi \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla\varphi \, dx &= \lim_{\lambda\uparrow\lambda^*} \int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla\varphi \, dx \\ &= \lim_{\lambda\uparrow\lambda^*} \lambda \int_\Omega f(u_\lambda) \, dx = \lambda^* \int_\Omega f(u^*)\varphi \, dx. \quad \blacksquare \end{aligned}$$

4. Proof of Theorem 1 and Theorem 2

In order to prove Theorem 2 we need the following technical lemma.

Lemma 12. *Assume* (2) *and* (3). *Let* τ_- *be defined in* (9). *If* $\tau_- > (p-2)/(p-1)$ *then every*

$$\gamma \in \left(\frac{1}{p-1}, \frac{1+\sqrt{1-(p-1)(1-\tau_-)}}{p-1}\right)$$

satisfies (21).

Proof. Let $\tau \in (0,1)$. We have the following equivalence:

$$(p-1)\gamma^2 \frac{\displaystyle\int_0^t \tilde{f}(s)^{2\gamma-2} f'(s)^2 ds}{\tilde{f}(t)^{2\gamma-1} f'(t)} < \tau$$

if and only if

$$G_{\gamma,\tau}(t) := (p-1)\gamma^2 \int_0^t \tilde{f}(s)^{2\gamma-2} f'(s)^2 ds - \tau \tilde{f}(t)^{2\gamma-1} f'(t) < 0.$$

We note that

$$G'_{\gamma,\tau}(t) = \left[(p-1)\gamma^2 - \tau(2\gamma-1) - \tau \frac{\tilde{f}(t) f''(t)}{f'(t)^2} \right] \tilde{f}(t)^{2\gamma-2} f'(t)^2.$$

Let $\epsilon_0 := \tau_- - (p-2)/(p-1) > 0$ and note that for every $\epsilon \in (0,\epsilon_0)$ there exists $t_0 = t_0(\epsilon) > 0$ such that

$$G'_{\gamma,\tau}(t) \le [(p-1)\gamma^2 - \tau(2\gamma - 1 + \tau_- - \epsilon)] \tilde{f}(t)^{2\gamma-2} f'(t)^2, \quad \text{for all } t \ge t_0. \tag{30}$$

Noting that $\tau_0 := (p-1)(1-\tau_- + \epsilon) < 1$ (for all $\epsilon \in (0,\epsilon_0)$), we obtain that

$$(p-1)\gamma^2 - (2\gamma - 1 + \tau_- - \epsilon)\tau < 0, \tag{31}$$

for all $\tau \in (\tau_0, 1)$ and

$$\gamma \in \left[\frac{\tau}{p-1}, \frac{\tau + \sqrt{\tau(\tau - (p-1)(1-\tau_- + \epsilon))}}{p-1} \right). \tag{32}$$

Moreover, since $f(t)^{1/(p-1)}$ is superlinear at infinity and (22), we have

$$\lim_{t\to+\infty} \tilde{f}(t)^{2\gamma-2} f'(t)^2 = +\infty, \quad \text{for all } \gamma \ge \frac{1}{p-1}.$$

Now, using the last limit, (31), and (30), we obtain

$$\lim_{t\to+\infty} G'_{\gamma,\tau}(t) = -\infty,$$

for all $\epsilon \in (0,\epsilon_0)$, $\tau \in (\tau_0, 1)$, and γ satisfying (32). In particular,

$$\lim_{t\to+\infty} G_{\gamma,\tau}(t) = -\infty$$

for the same range of parameters. The result follows from the last limit and the equivalence given at the beginning of the proof, since the arbitrariness of ϵ and τ. ■

As a consequence of Proposition 10 and Lemma 12 we prove Theorem 2.

Proof of Theorem 2. Assume $\tau_- > (p-2)/(p-1)$. By Lemma 12, every

$$\gamma \in \left(\frac{1}{p-1}, \frac{1+\sqrt{1-(p-1)(1-\tau_-)}}{p-1} \right)$$

satisfies (21). Therefore, u^* is a solution of $(1_{\lambda^*,p})$ by Proposition 10.

(i) If in addition $N < N(p)$, where $N(p)$ is defined in (11), then the boundedness of u^* follows from Proposition 10 (i) and the arbitrariness of γ.

(ii) If in addition $N \geq N(p)$, then Proposition 10 (ii) and the arbitrariness of γ give that $u^* \in L^q(\Omega)$, for all $1 \leq q < q_0$, and $f(u^*) \in L^q(\Omega)$, for all $1 \leq q < \bar{q}_1$, where

$$q_0 = \left(p + 2\sqrt{1-(p-1)(1-\tau_-)} \right) \frac{N}{N - N(p)}$$

and

$$\bar{q}_1 = \left(p + 2\sqrt{1-(p-1)(1-\tau_-)} \right) \frac{N}{(p-1)N - p}.$$

Let $\tilde{q} = (p^*)'$ be defined in (19). Noting that $\bar{q}_1 \leq N/p$ (since $N \geq N(p)$) and $\tilde{q} < \bar{q}_1$, we have $f(u^*) \in L^q(\Omega)$ for all $\tilde{q} \leq q < \bar{q}_1 \leq N/p$. Therefore, by Lemma 6, we obtain that $u^* \in W_0^{1,r}(\Omega)$ with $1 \leq r < (p-1)N\bar{q}_1/(N - \bar{q}_1)$. We conclude the proof by noting that the exponent q_1 given in (12) coincides with $(p-1)N\bar{q}_1/(N-\bar{q}_1)$. ■

Now, we prove Theorem 1 as a corollary of Theorem 2.

Proof of Theorem 1. (i) Assume f convex and $1 < p < 2$. Under these assumptions it is clear that

$$\frac{p-2}{p-1} < 0 \leq \tau_-.$$

Therefore, from Theorem 2, we obtain that u^* is a bounded solution of $(1_{\lambda^*,p})$ if

$$N < N(p) = p + \frac{2p}{p-1}\left[1 + \sqrt{1-(p-1)(1-\tau_-)} \right].$$

We conclude noting that

$$N(p) \geq H(p) = p + \frac{2p}{p-1}\left[1 + \sqrt{2-p} \right] > 6,$$

where H is given in (8).

(ii) Assume $0 < \tau_-$ and $p = 2$. By Theorem 2, we obtain that u^* is a bounded solution of $(1_{\lambda^*,p})$ if

$$N < N(2) = 2 + 4(1 + \sqrt{\tau_-}).$$

The assertion and the theorem follow noting that $N(2) > 6$. ■

5. Proof of Theorem 3 and Theorem 4

We start proving Theorem 3 as a consequence of Proposition 10 and Lemmas 12 and 9.

Proof of Theorem 3. Assume $\tau_- > (p-2)/(p-1)$ and let $N(p)$ be given in (11). If $N < N(p)$ then the assertion follows from Theorem 2 (i). Thus, we may assume $N \geq N(p)$. It follows from Lemma 12 and Proposition 10 that u^* is a solution of $(1_{\lambda^*,p})$ and

$$f(u^*) \in L^q(\Omega) \quad \text{for all } q < \bar{q}_1 = \left(p + 2\sqrt{1-(p-1)(1-\tau_-)}\right) \frac{N}{(p-1)N-p}.$$

By Lemma 9, we obtain that $u^* \in L^\infty(\Omega)$ if

$$(1 - \frac{p-1}{m})N < p\bar{q}_1,$$

or equivalently, if (14) holds. ■

In order to prove Theorem 4, we need the following technical result that states a relation between assumptions (13) and (15).

Lemma 13. *Let f be a positive C^2 function such that $f'(t) > 0$, for all $t > 0$. Let τ_+ be given in (15). If $\tau_+ < 1$ then, for every $\epsilon \in (0, 1-\tau_+)$, there exists a positive constant c depending in ϵ such that*

$$f(t) \leq c(1+t)^{\frac{1}{1-(\tau_+ + \epsilon)}}, \quad \textit{for all} \quad t \geq 0.$$

Proof. Let $\epsilon \in (0, 1-\tau_+)$. By definition of τ_+ there exists $t_0 = t_0(\epsilon)$ such that $\tilde{f}(t)f''(t) \leq (\tau_+ + \epsilon)f'(t)^2$, for all $t \geq t_0$. Therefore

$$[\ln f'(t)]' = \frac{f''(t)}{f'(t)} \leq (\tau_+ + \epsilon)\frac{f'(t)}{\tilde{f}(t)} = (\tau_+ + \epsilon)[\ln \tilde{f}(t)]', \quad \text{for all } t \geq t_0.$$

Integrating the last expression with respect to t, we obtain

$$\ln\left(\frac{f'(t)}{f'(t_0)}\right) \leq \ln\left(\frac{\tilde{f}(t)}{\tilde{f}(t_0)}\right)^{\tau_+ + \epsilon}, \quad \text{for all } t > t_0,$$

or equivalently,

$$\frac{f'(t)}{\tilde{f}(t)^{\tau_+ + \epsilon}} = \left(\frac{\tilde{f}(t)^{1-(\tau_+ + \epsilon)}}{1-(\tau_+ + \epsilon)}\right)' \le \frac{f'(t_0)}{\tilde{f}(t_0)^{\tau_+ + \epsilon}}, \quad \text{for all } t > t_0.$$

Integrating again, we obtain

$$f(t) \le \left[(1-(\tau_+ + \epsilon))\frac{f'(t_0)}{\tilde{f}(t_0)^{\tau_+ + \epsilon}}(t-t_0) + \tilde{f}(t_0)^{1-(\tau_+ + \epsilon)}\right]^{\frac{1}{1-(\tau_+ + \epsilon)}} + f(0),$$

for all $t \ge t_0$. The lemma follows easily from the last inequality. ■

Finally, we prove Theorem 4 as a consequence of the previous lemma, Theorem 2, and Theorem 3.

Proof of Theorem 4. Since $\tau_- > (p-2)/(p-1)$, u^* is a solution of $(1_{\lambda^*,p})$ by Theorem 2.

(i) Assume $\tau_+ < 1$. By Lemma 13, for every $\epsilon \in (0, 1-\tau_+)$, there exists a positive constant c (depending in ϵ) such that

$$f(t) \le c(1+t)^{\frac{1}{1-(\tau_+ + \epsilon)}}, \quad \text{for all } t \ge 0.$$

Therefore, from Theorem 3 with $m = 1/(1-(\tau_+ + \epsilon))$, it follows that $u^* \in L^\infty(\Omega)$ if

$$N < \frac{p}{p-1}\left(1 + \frac{p}{1-(p-1)(1-(\tau_+ + \epsilon))} + \frac{2\sqrt{1-(p-1)(1-\tau_-)}}{1-(p-1)(1-(\tau_+ + \epsilon))}\right).$$

Hence, we obtain the assertion by the arbitrariness of ϵ.

(ii) It is clear from Theorem 2 (i).

(iii) We conclude the proof noting that the right-hand side of inequalities (16) and (17) is bigger or equal than $F(p) = p + 4p/(p-1)$. ■

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