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INTERPOLATIVE IDEAL PROCEDURES, INTERPOLATION, AND APPLICATIONS TO APPROXIMATION QUANTITIES

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Abstract: This paper deals with a generalization of a certain interpolative procedure introduced by U. Matter [9] by which from given Banach ideals $\mathcal{A}$ and $\mathcal{B}$ a new scale of Banach ideals $(\mathcal{A}, \mathcal{B})_\varphi$ is generated. In particular we elaborate the connection of our construction to interpolation theory. As an application we consider the ideal of (p, φ)-absolutely continuous operators which occurs when $\mathcal{A}$ is the class of p-summing operators and $\mathcal{B}$ is the class of all operators. We characterize (p, φ)-absolutely continuous operators by a special factorization property through a suitable interpolation space. We also give some applications to approximation quantities and entropy numbers.

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1. Introduction and Notation

This introductory section serves as a survey of known results on absolute continuity in the setting of operators between Banach spaces. The main purpose here is to give motivation for our further studies of this notion. Throughout the work, we shall use standard Banach space notation that may be found in [5].

Let Ω be a compact topological space. Consider a weakly compact operator T defined on the space $C(\Omega)$ consisting of all real valued continuous functions on Ω with values in some Banach space X. A result of R.G. Bartle, N. Dunford and J. Schwarz from [1] asserts that there exists a certain control measure μ for the operator T. More precisely, it can be shown that for every $\varepsilon > 0$ there exists a positive constant $N(\varepsilon)$ such that

$$\|Tf\| \leq N(\varepsilon) \int |f| d\mu + \varepsilon \|f\| \quad \text{for every} \quad f \in C(\Omega).$$

Motivated by the above property C. P. Niculescu introduced in his pioneering work [11] and [12] the class of absolutely continuous operators with respect

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to a certain seminorm. A systematic study of this notion was initiated in the work of H. Jarchow, U. Matter [9, 10, 7] and F. Räbiger [14].

In order to generalize the ideas of Niculescu the following definition was considered in [9]. Let $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(X,Z)$ and $R \in \mathcal{L}(X,W)$ be operators between Banach spaces X, Y, Z and W. An operator T is said to be *absolutely continuous with respect to* (S,R), denoted by $T \ll (S,R)$, if for arbitrary $\varepsilon > 0$ there is a constant $N(\varepsilon) \geq 0$ such that

$$\|Tx\| \leq N(\varepsilon)\|Sx\| + \varepsilon\|Rx\| \quad \text{for all } x \in X. \tag{1.1}$$

In the case $X = W$ and $R = I_X$, we call such an operator T absolutely continuous with respect to S and denote this by $T \ll S$. It was shown by C. P. Niculescu that $T \ll S$ if and only if $T'' \ll S''$.

Let us now consider the notion of absolute continuity of operators from the operator ideal point of view. For definitions and facts from operator ideal theory we refer the reader to the monograph [13]. Recall that for an operator ideal $\mathcal{A}$ its *injective hull* $\mathcal{A}^{\mathrm{inj}}$ consists of all operators $T \in \mathcal{L}(X,Y)$ that become a member of $\mathcal{A}$ by extending the codomain $X \xrightarrow{T} Y \xrightarrow{J} Y_0$. Here J denotes an injection into a suitable Banach space Y_0. Due to the extension property we may take $Y_0 = \ell_\infty(I)$ with an appropriate index set I. An ideal is called *injective* if $\mathcal{A} = \mathcal{A}^{\mathrm{inj}}$. Injectivity of $\mathcal{A}$ implies that the associated class of Banach spaces is stable when passing to subspaces. The following result of H. Jarchow and A. Pełczyński characterizes the closed injective hull $\overline{\mathcal{A}}^{\mathrm{inj}}$ of $\mathcal{A}$, see [6].

Theorem 1.1. *Let $\mathcal{A}$ be a quasinormed operator ideal. An operator $T \in \mathcal{L}(X,Y)$ belongs to the closed injective hull $\overline{\mathcal{A}}^{\mathrm{inj}}$ of $\mathcal{A}$ if and only if there exist a Banach space Z and an operator $S \in \mathcal{A}(X,Z)$ such that $T \ll S$.*

We conclude this section by stressing an important connection of this notion with interpolation theory. Let us consider for a fixed $r > 0$ the function $N(\varepsilon) = \varepsilon^{-r}$ as a function appearing in (1.1). More precisely, for a fixed $\theta \in (0,1)$ and $x \in X$ computing the minimum value of the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(t) = t^{\theta/(\theta-1)}\|Sx\| + t\|Rx\|$, which controls the right hand side of inequality (1.1), shows that the definition of absolute continuity of operators is equivalent to the following statement:

$$\|Tx\| \leq \|\widetilde{S}x\|^{1-\theta}\|Rx\|^{\theta} \quad \text{for all } x \in X. \tag{1.2}$$

Here $\widetilde{S}$ denotes a constant multiple of the operator S occurring in (1.1). The value of the underlying constant is $((1-\theta)/\theta)^\theta + (\theta/(1-\theta))^\theta$. The inequalities of type (1.2) play an important rôle in interpolation theory, see [3]. This problem will be discussed in detail in Section 3.

Let us now consider the function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by $\varphi(s,t) = s^{1-\theta}t^\theta$. Then the condition (1.2) reads as follows

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|) \quad \text{for all } x \in X. \tag{1.3}$$

For simplicity of notation, we write here S instead of $\widetilde{S}$. Interesting questions that arise from the above considerations are the following: For which functions $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ does (1.3) already imply $T \ll (S,R)$, and conversely, does the existence of a function φ such that (1.3) holds already follows from $T \ll (S,R)$?

To answer these questions let us define the class **AC** consisting of all continuous, positively homogeneous, concave functions $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $\varphi(s,0) = \varphi(0,t) = 0$ and $\varphi(1,1) = 1$. Taking into account the homogeneity of φ we often put $\varphi(s,t) = s\rho(t/s)$ with $\rho : [0,\infty) \to \mathbb{R}_+$. The function ρ is also non-decreasing, concave and continuous with $\rho(0) = 0$ and $\rho(1) = 1$. It was shown by F. Räbiger in [14] that the following statements are equivalent

- $T \ll (S,R)$.
- There exists a function $\varphi \in$ **AC** such that (1.3) holds.

Let us now present the contents of this paper in some detail. In the next section we present a generalization of a interpolative procedure introduced by U. Matter [9] by which from given Banach ideals $\mathcal{A}$ and $\mathcal{B}$ a new scale of Banach ideals $(\mathcal{A},\mathcal{B})_\varphi$ is generated. In Section 3 we stress an important connection of our construction to interpolation theory. As an application in Section 4 we characterize (p,φ)-absolutely continuous operators by a special factorization property through a suitable interpolation space. The last section is devoted to give some applications to approximation quantities and entropy numbers.

2. An interpolative ideal procedure

This section presents a procedure by which, from given operator ideals $\mathcal{A}$ and $\mathcal{B}$, a scale of new ideals $(\mathcal{A},\mathcal{B})_\varphi$ is generated. For definitions and basic facts on operator ideals we refer the reader to the monograph [13]. Recall that $\mathcal{L}$ denotes the ideal of all operators between arbitrary Banach spaces.

In what follows, let $\varphi \in \mathbf{AC}$. We start our considerations by showing the superadditivity of φ. This property will be frequently used in the sequel.

Lemma 2.1. *For any $a_i, b_i \geq 0$ the following inequality holds*

$$\sum_{i=1}^{n} \varphi(a_i, b_i) \leq \varphi\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right). \tag{2.1}$$

Proof: By the definition of concavity the inequality

$$\sum_{i=1}^{n} \lambda_i \varphi(a_i, b_i) \leq \varphi\left(\sum_{i=1}^{n} \lambda_i a_i, \sum_{i=1}^{n} \lambda_i b_i\right)$$

holds for every $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. The homogeneity of φ gives

$$\sum_{i=1}^{n} c\lambda_i \varphi(a_i, b_i) \leq \varphi\left(\sum_{i=1}^{n} c\lambda_i a_i, \sum_{i=1}^{n} c\lambda_i b_i\right) \text{ for every } c > 0.$$

Taking $c\lambda_i = 1$ for every $i = 1, \ldots, n$ gives the claim. ■

We are now in a position to present a generalization of the procedure introduced by U. Matter in [9]. An operator $T \in \mathcal{L}(X, Y)$ belongs to $(\mathcal{A}, \mathcal{B})_\varphi$ if there exist Banach spaces Z, W and operators $S \in \mathcal{A}(X, Z)$, $R \in \mathcal{B}(X, W)$ such that

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|) \quad \text{for all } \ x \in X. \tag{2.2}$$

It is easy to see that the above defined class possesses the ideal property. To show that $TV \in (\mathcal{A}, \mathcal{B})_\varphi(X_0, Y)$ for $V \in \mathcal{L}(X_0, X)$ and $T \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ we choose operators $S \in \mathcal{A}(X, Z)$, $R \in \mathcal{B}(X, W)$ according to (2.2). Then $SV \in \mathcal{A}(X_0, Z)$ and $RV \in \mathcal{B}(X_0, W)$. We check at once that

$$\|TVx\| \leq \varphi(\|SVx\|, \|RVx\|) \quad \text{for all } \ x \in X_0.$$

To deduce that $UT \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y_0)$ for $U \in \mathcal{L}(Y, Y_0)$ and $T \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ let us take the $\|U\|$-multiple of operators appearing in (2.2). By homogeneity of φ we obtain

$$\|UTx\| \leq \|U\|\|Tx\| \leq \|U\|\varphi\left(\|Sx\|, \|Rx\|\right) = \varphi\left(\|\|U\|Sx\|, \|\|U\|Rx\|\right)$$

for all $x \in X$. In order to see that $(\mathcal{A}, \mathcal{B})_\varphi$ is a linear space, we provide the following alternative characterization of operators belonging to this class.

Proposition 2.2. *An operator T belongs to $(\mathcal{A},\mathcal{B})_\varphi$ if and only if there exist some $n \in \mathbb{N}$, Banach spaces Z_i, W_i and operators $S_i \in \mathcal{A}(X, Z_i), R_i \in \mathcal{B}(X, W_i), i = 1, 2, \ldots, n$, such that*

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) \quad \textit{for all } \ x \in X. \tag{2.3}$$

Proof: The only if part is obvious. In order to prove the converse implication let

$$Z = \Big(\bigoplus_{i=1}^{n} Z_i\Big)_1 \text{ and } \ W = \Big(\bigoplus_{i=1}^{n} W_i\Big)_1.$$

We define operators $S : X \to Z$ and $R : X \to W$ by

$$S = \sum_{i=1}^{n} J_{Z_i}^{Z} S_i \qquad \text{and} \qquad R = \sum_{i=1}^{n} J_{W_i}^{W} R_i,$$

where $J_{Z_i}^{Z} : Z_i \to Z$ denotes the canonical injection. By lemma 2.1 we obtain

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) \leq \varphi\left(\sum_{i=1}^{n} \|S_i x\|, \sum_{i=1}^{n} \|R_i x\|\right) = \varphi(\|Sx\|, \|Rx\|),$$

which finishes our proof. ∎

Now it follows from Proposition 2.2 that with operators $T_1, T_2 \in (\mathcal{A},\mathcal{B})_\varphi$ and numbers λ_1, λ_2 also $\lambda_1 T_1 + \lambda_2 T_2 \in (\mathcal{A},\mathcal{B})_\varphi$. Hence $(\mathcal{A},\mathcal{B})_\varphi$ is an operator ideal.

From now on, we assume that α, β are quasinorms on $\mathcal{A}, \mathcal{B}$, such that $\mathcal{A}$ and $\mathcal{B}$, respectively, become quasinormed Banach ideals. We now consider the following maps, which are connected to part (2.2) and (2.3), respectively: For each operator $T \in (\mathcal{A},\mathcal{B})_\varphi$ we put

(i)
$$\gamma(T) = \inf \varphi(\alpha(S), \beta(R)),$$
where the infimum ranges over all operators S, R such that inequality (2.2) holds,

(ii)
$$\overline{\gamma}(T) = \inf \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)),$$
where the infimum ranges over all $n \in \mathbb{N}$ and all operators S_i, R_i such that inequality (2.3) holds.

We recall that a mapping $\alpha : \mathcal{A} \to \mathbb{R}_+$ (in particular we can consider a quasinorm) is said to have the *ideal property* if for $V \in \mathcal{L}(X_0, X), T \in \mathcal{A}(X, Y)$ and $U \in \mathcal{L}(Y, Y_0)$ the following inequality holds

$$\alpha(UTV) \leq \|U\|\alpha(T)\|V\|.$$

Proposition 2.3. *Both maps γ and $\overline{\gamma}$ possess the ideal property. Moreover, the map $\overline{\gamma}$ is a norm on $(\mathcal{A}, \mathcal{B})_\varphi$.*

Proof: We prove the first statement only for γ. The proof for $\overline{\gamma}$ is similar. Let $V \in \mathcal{L}(X_0, X)$ and $U \in \mathcal{L}(Y, Y_0)$. Let S, R be operators such that

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|)$$

holds. Observe that

$$\|(UTV)x\| \leq \|U\|\|T(Vx)\| \leq \|U\|\varphi(\|S(Vx)\|, \|R(Vx)\|)$$
$$= \varphi(\|\|U\|SVx\|, \|\|U\|RVx\|).$$

So $\widetilde{S} = \|U\|SV$ and $\widetilde{R} = \|U\|RV$ are admissible operators in the definition of $\gamma(UTV)$. We obtain

$$\gamma(UTV) \leq \varphi(\alpha(\|U\|SV), \beta(\|U\|RV)) \leq \|U\|\varphi(\alpha(S), \beta(R))\|V\|.$$

Taking the infimum over all operators S and R gives the claim.

We only have to show the triangle inequality in the second statement. For that reason, let $T_1, T_2 \in (\mathcal{A}, \mathcal{B})_\varphi$ and assume that $S_1, \ldots, S_m; R_1, \ldots, R_m$ are such that

$$\|T_1x\| \leq \sum_{i=1}^{n} \varphi(\|S_ix\|, \|R_ix\|) \text{ and } \|T_2x\| \leq \sum_{i=n+1}^{m} \varphi(\|S_ix\|, \|R_ix\|)$$

for some $n < m$. Obviously

$$\|T_1x + T_2x\| \leq \sum_{i=1}^{m} \varphi(\|S_ix\|, \|R_ix\|).$$

We then have

$$\overline{\gamma}(T_1 + T_2) \leq \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)) + \sum_{i=n+1}^{m} \varphi(\alpha(S_i), \beta(R_i))$$

Turning to the infimum on the right hand side gives the claim. ∎

Theorem 2.4. *Let ρ be a submultiplicative function, i.e. there exists constant $c > 0$ such that*

$$\rho(st) \leq c\rho(s)\rho(t) \text{ for every } s,t \in \mathbb{R}_+.$$

Assume also that α, β are ideal norms. Then

$$\overline{\gamma}(T) \leq \gamma(T) \leq c\,\overline{\gamma}(T). \tag{2.4}$$

In other words, both maps are equivalent provided that the function ρ is submultiplicative.

Proof: The left hand inequality in (2.4) is obvious. First we prove that the following inequality holds for any $\xi, \eta, \tau \in \mathbb{R}_+$:

$$\varphi(\xi,\eta) \leq c\,\varphi\Big(\varphi(1,\tau)\xi, \varphi(1/\tau,1)\eta\Big). \tag{2.5}$$

This inequality is equivalent to

$$\xi\rho\Big(\frac{\eta}{\xi}\Big) \leq c\,\varphi(1,\tau)\xi\;\rho\left(\frac{\varphi(1/\tau,1)\eta}{\varphi(1,\tau)\xi}\right)$$

which follows from the submultiplicativity of ρ by

$$\rho\left(\frac{\eta}{\xi}\right) \leq c\;\rho(\tau)\rho\left(\frac{1}{\tau}\frac{\eta}{\xi}\right).$$

Now let operators $S_i \in \mathcal{A}(X,Z_i), R_i \in \mathcal{B}(X,W_i), i = 1,2,\dots,n$ be such that

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|).$$

For the proof of the second inequality in (2.4) let us define

$$\xi_i := \|S_i x\|, \quad \eta_i := \|R_i x\|, \quad \tau_i := \frac{\beta(R_i)}{\alpha(S_i)}.$$

Furthermore, let

$$Z = \Big(\bigoplus_{i=1}^{n} Z_i\Big)_1 \quad \text{and} \quad W = \Big(\bigoplus_{i=1}^{n} W_i\Big)_1.$$

and define $S \in \mathcal{A}(X,Z), R \in \mathcal{B}(X,W)$ by

$$S := \sum_{i=1}^{n} \rho(\tau_i) J_{Z_i}^{Z} S_i \quad \text{and} \quad R := \sum_{i=1}^{n} \frac{1}{\tau_i}\rho(\tau_i) J_{W_i}^{W} R_i.$$

Then we have

$$\|Sx\| = \sum_{i=1}^{n} \rho(\tau_i)\|S_i x\| \quad \text{and} \quad \|Rx\| = \sum_{i=1}^{n} \frac{1}{\tau_i}\rho(\tau_i)\|R_i x\|.$$

Using Lemma 2.1 and the inequality (2.5) yields

$$\begin{aligned}\|Tx\| \;&\leq\; \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) = \sum_{i=1}^{n} \varphi(\xi_i, \eta_i) \leq c\sum_{i=1}^{n} \varphi\Big(\rho(\tau_i)\xi_i, \frac{\rho(\tau_i)}{\tau_i}\eta_i\Big)\\ &\leq\; c\varphi\Big(\sum_{i=1}^{n} \rho(\tau_i)\xi_i, \sum_{i=1}^{n} \frac{\rho(\tau_i)}{\tau_i}\eta_i\Big) = c\varphi(\|Sx\|, \|Rx\|).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\varphi(\alpha(S), \beta(R)) &\leq \varphi\Big(\sum_{i=1}^{n} \rho(\tau_i)\alpha(S_i), \sum_{i=1}^{n} \frac{\rho(\tau_i)}{\tau_i}\beta(R_i)\Big)\\ &= \sum_{i=1}^{n} \alpha(S_i)\rho(\tau_i)\varphi(1,1) = \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)),\end{aligned}$$

which finishes the proof. ■

To show the completeness of $(\mathcal{A}, \mathcal{B})_\varphi$ we consider $(T_n) \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ such that $\sum_{n=1}^{\infty} \gamma(T_n) < \infty$. Our aim is to find an operator $T \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ such that $\gamma\left(T - \sum_{i=1}^{n} T_i\right)$ tends to zero as $n \to \infty$. By assumption there are Banach spaces Z_i, W_i and operators S_i, R_i such that $\|T_i x\| \leq \varphi(\|S_i x\|, \|R_i x\|)$ for all $x \in X$. Moreover, we obtain that $\sum_{i=1}^{\infty} \alpha(S_i)$ and $\sum_{i=1}^{\infty} \beta(R_i)$ are finite. Put $Z = \left(\bigoplus_{i=1}^{\infty} Z_i\right)_1$ and $W = \left(\bigoplus_{i=1}^{\infty} W_i\right)_1$. Now the completeness of $\mathcal{A}$ and $\mathcal{B}$ yields

$$\alpha\Big(S - \sum_{i=1}^{n} S_i\Big) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \beta\Big(R - \sum_{i=1}^{n} R_i\Big) \overset{n\to\infty}{\longrightarrow} 0$$

with $S = \Big(\bigoplus_{i=1}^{\infty} S_i\Big)_1$ and $R = \Big(\bigoplus_{i=1}^{\infty} S_i\Big)_1$. Consequently, we obtain for $T = \Big(\bigoplus_{i=1}^{\infty} T_i\Big)_1$ that

$$\gamma\Big(T - \sum_{i=1}^{n} T_i\Big) \le \varphi\Big(\alpha\Big(S - \sum_{i=1}^{n} S_i\Big), \beta\Big(R - \sum_{i=1}^{n} R_i\Big)\Big).$$

Moreover we have

$$\begin{aligned}\Big\|(T - \sum_{i=1}^{n} T_i)x\Big\| &\le \sum_{i=n+1}^{\infty} \varphi\left(\|S_i x\|, \|R_i x\|\right) \le \varphi\Big(\sum_{i=n+1}^{\infty} \|S_i x\|, \sum_{i=n+1}^{\infty} \|R_i x\|\Big)\\ &= \varphi\Big(\Big\|(S - \sum_{i=1}^{n} S_i)x\Big\|, \Big\|(R - \sum_{i=1}^{n} R_i)x\Big\|\Big).\end{aligned}$$

This shows our assertion.

We have thus proved that if ρ is submultiplicative then $[(\mathcal{A}, \mathcal{B})_\varphi, \overline{\gamma}]$ is a Banach ideal and γ is equivalent to the norm $\overline{\gamma}$. We collect the results obtained so far in the following theorem.

Theorem 2.5. *Let $\varphi \in \mathbf{AC}$ and $\rho : [0, \infty) \to \mathbb{R}_+$ be given by $\varphi(s, t) = s\rho(t/s)$. Let $\mathcal{A}$ and $\mathcal{B}$ be operators ideals. Then $(\mathcal{A}, \mathcal{B})_\varphi$ is an operator ideal. If, moreover, $(\mathcal{A}, \alpha)$, $(\mathcal{B}, \beta)$ are quasinormed Banach ideals and ρ is submultiplicative, then $[(\mathcal{A}, \mathcal{B})_\varphi, \overline{\gamma}]$ is a Banach ideal where $\overline{\gamma}$ is given by (ii)*

Straightforward computation yields the following reiteration property.

Proposition 2.6. *Let $\varphi, \varphi_0, \varphi_1 \in \mathbf{AC}$. Then*

$$\big((\mathcal{A}, \mathcal{B})_{\varphi_0}, (\mathcal{A}, \mathcal{B})_{\varphi_1}\big)_\varphi \subset (\mathcal{A}, \mathcal{B})_{\varphi(\varphi_0, \varphi_1)}.$$

Proof: Assume that $T \in \big((\mathcal{A}, \mathcal{B})_{\varphi_0}, (\mathcal{A}, \mathcal{B})_{\varphi_1}\big)_\varphi(X, Y)$. Then we find Banach spaces Z, W and operators $T_0 \in (\mathcal{A}, \mathcal{B})_{\varphi_0}(X, Z)$ and $T_1 \in (\mathcal{A}, \mathcal{B})_{\varphi_1}(X, W)$ such that

$$\|Tx\| \le \varphi(\|T_0 x\|, \|T_1 x\|) \quad \text{for all } \; x \in X.$$

By definition, for $T_i \in (\mathcal{A}, \mathcal{B})_{\varphi_i}(X, Y)$, $i = 0, 1$, we find Banach spaces Z_i, W_i and operators $S_i \in \mathcal{A}(X, Z_i)$ and $R_i \in \mathcal{B}(X, W_i)$ such that

$$\|T_i x\| \le \varphi_i(\|S_i x\|, \|R_i x\|) \quad \text{for all } \; x \in X, \quad i = 0, 1.$$

Define $S = S_0 \oplus_1 S_1$ and $S = R_0 \oplus_1 R_1$. By the monotonicity of φ, φ_0 and φ_1 we obtain

$$\begin{aligned}\|Tx\| \leq \varphi(\|T_0x\|, \|T_1x\|) &\leq \varphi\left(\varphi_0(\|S_0x\|, \|R_0x\|), \varphi_1(\|S_1x\|, \|R_1x\|)\right)\\ &\leq \varphi\left(\varphi_0(\|Sx\|, \|Rx\|), \varphi_1(\|Sx\|, \|Rx\|)\right).\end{aligned}$$

This shows that $T \in (\mathcal{A}, \mathcal{B})_{\varphi(\varphi_0,\varphi_1)}(X, Y)$, which completes the proof. ■

3. Factoring through interpolation spaces

Let us start this section by recalling some basic notation and results from interpolation theory. A pair of Banach spaces (X_0, X_1) is said to be an *interpolation couple* (or compatible couple) if both spaces are continuously embedded into a certain Hausdorff topological vector space V. For an interpolation couple (X_0, X_1) we put

$$\Delta(X_0, X_1) = \{x \in X_0 \cap X_1 : \ \|x\|_\Delta = \max(\|x_0\|_{X_0}, \|x_1\|_{X_1}) < \infty\},$$
$$\Sigma(X_0, X_1) = \{x \in X_0 + X_1 : \ \|x\|_\Sigma = \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \|x_1\|_{X_1}\} < \infty\}.$$

The linear spaces $\Delta(X_0, X_1)$ and $\Sigma(X_0, X_1)$ equipped with the norms $\|\cdot\|_\Delta$ and $\|\cdot\|_\Sigma$, respectively are Banach spaces. A Banach space X is said to be an *intermediate space* with respect to (X_0, X_1) if $\Delta(X_0, X_1) \hookrightarrow X \hookrightarrow \Sigma(X_0, X_1)$ (continuous inclusion). If additionally for every linear operator $T : X_0+X_1 \to X_0 + X_1$ such that the restrictions $T_0 : X_0 \to X_0$ and $T_1 : X_1 \to X_1$ are bounded we have that $T : X \to X$ is bounded, then we refer X to as *interpolation space* with respect to (X_0, X_1). Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples. From now on, the notation $T : (X_0, X_1) \to (Y_0, Y_1)$ means that $T : \Sigma(X_0, X_1) \to \Sigma(Y_0, Y_1)$ is a linear operator such that the restrictions of T given by $T_0 : X_0 \to X_1$ and $T_1 : Y_0 \to Y_1$ are bounded. A functor $\mathcal{F}$ from the category of all compatible couples into the category of all Banach spaces is called *interpolation functor* (or *interpolation method*) if for any couple (X_0, X_1) the Banach space $\mathcal{F}(X_0, X_1)$ is an intermediate space and $T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)$ is bounded for all couples (X_0, X_1), (Y_0, Y_1) and any $T : (X_0, X_1) \to (Y_0, Y_1)$. The closed graph theorem implies that for any interpolation functor $\mathcal{F}$ there exists a constant $C > 0$ such that we have

$$\|T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)\| \leq C \max\Big(\|T : X_0 \to Y_0\|, \|T : X_1 \to Y_1\|\Big).$$

If C can be chosen equal to one then we say that $\mathcal{F}$ is an *exact* interpolation functor. Fundamental examples of the exact interpolation methods are the *real method of interpolation* $(\cdot,\cdot)_{\theta,p}$ with $0<\theta<1$ and $1\le p\le\infty$ which takes its origins from the classical Marcinkiewicz theorem and the *complex method of interpolation* $[\cdot,\cdot]_\theta$. The idea of this method goes back to the Riesz-Thorin theorem.

In the sequel for $t>0$ let $t\mathbb{R}$ denote the real line $\mathbb{R}$ equipped with the norm $\|x\|_{t\mathbb{R}}=t|x|$. If $\mathcal{F}$ is an exact interpolation functor, its *characteristic function* φ is defined by

$$\varphi(s,t)\mathbb{R}=\mathcal{F}(s\mathbb{R},t\mathbb{R}).$$

In particular we may work with a function from the class **AC**. For a compatible pair (X_0,X_1) of Banach spaces, we define $(X_0,X_1)_{\varphi,1}$ as the space of all $x\in\Sigma(X_0,X_1)$ for which there exists a sequence $(x_n)\subseteq\Delta(X_0,X_1)$ such that $x=\sum_{n=1}^\infty x_n$ in $\Sigma(X_0,X_1)$ and $\sum_{n\ge1}\varphi(\|x_n\|_0,\|x_n\|_1)<\infty$. Equipped with the norm

$$\|x\|_{\varphi,1}=\inf\Big\{\sum_{n\ge1}\varphi(\|x_n\|_0,\|x_n\|_1):\quad x=\sum_{n=1}^\infty x_n\Big\}$$

$(X_0,X_1)_{\varphi,1}$ becomes a Banach space. It can be shown, that the space $(X_0,X_1)_{\varphi,1}$ is an interpolation space. Moreover the interpolation functor $(\cdot,\cdot)_{\varphi,1}$ turns out to be exact and φ is its characteristic function. In addition, the interpolation functor $(\cdot,\cdot)_{\varphi,1}$ possesses a certain minimal property in the following sense. If $\mathcal{F}$ is an arbitrary exact interpolation functor and φ is its characteristic function, then for any Banach couple (X_0,X_1) the following inclusion holds

$$(X_0,X_1)_{\varphi,1}\subseteq\mathcal{F}(X_0,X_1).$$

In addition, the embedding constant is less than one, i.e the above inclusion is a contraction. For more information and proofs we refer the reader to [2] and [3].

In what follows, let X_0,X_1 be Banach spaces such that $X_0\hookrightarrow X_1$. Let us deal with sequences given by

$$a_k:=\rho(2^{-k})\quad\text{and}\quad b_k:=2^k\rho(2^{-k})\quad\text{for } k\in\mathbb{N}.$$

Next, observe the following basic properties

(i) The sequence (a_k) is decreasing.
(ii) The sequence (b_k) is increasing.
(iii) $\rho(\tau)\le\max\{1,\tau\}$ for any $\tau\in\mathbb{R}_+$.

For $x \in X_0$, define the following expressions.

$$u(x) = \inf\Big\{ \sum_{k=1}^{n} \varphi(\|x_k\|_0, \|x_k\|_1) : \quad n = 1, 2, \ldots;\ x_k \in X_0;\ x = \sum_{k=1}^{n} x_k \Big\}.$$

$$v(x) = \inf\Big\{ \sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1) : \quad x_k \in X_0;\ x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k \Big\}.$$

$$w(x) = \inf\Big\{ \max\Big(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1 \Big) : \quad x_k \in X_0;\ x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k \Big\}.$$

Observe that any representation $x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k$ with $x_k \in X_0$ such that either $\sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1) < \infty$ or $\sum_{k=1}^{\infty} \|a_k x_k\|_0 < \infty$ is absolutely convergent. All above expressions are norms on X_0. The following proposition tells us that these norms are equivalent provided that the function ρ is submultiplicative.

Proposition 3.1. *If $\rho(st) \le c\rho(s)\rho(t)$ for every $s, t \in \mathbb{R}_+$ then*

$$v(x) \le u(x) \le c\, w(x) \le 2c\, v(x) \textit{ for all } x \in X_0.$$

Proof: The inequality $v(x) \le u(x)$ is trivial.

Let us show that $u(x) \le c\ w(x)$. For that, assume $x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k$ with $x_k \in X_0$ satisfies $\sum_{k=1}^{\infty} \|a_k x_k\|_0 < \infty$ and $\sum_{k=1}^{\infty} \|b_k x_k\|_1 < \infty$. Given $n \in \mathbb{N}$, define $\widetilde{x}_n = x - \sum_{k=1}^{n} x_k \in X_0$. Since

$$\left\| a_n \left(x - \sum_{k=1}^{n} x_k \right) \right\|_0 \le a_n \|x\|_0 + \sum_{k=1}^{n} a_k \|x_k\|_0 \le a_n \|x\|_0 + \sum_{k=1}^{n} \|a_k x_k\|_0,$$

the sequence $(\|a_n \widetilde{x}_n\|_0)$ is bounded. Since

$$\|b_n \widetilde{x}_n\|_1 \le b_n \sum_{k=n+1}^{\infty} \|x_k\|_1 \le \sum_{k=n+1}^{\infty} \|b_k x_k\|_1$$

is a null sequence, we can, for given $\varepsilon > 0$, choose n large enough that

$$\varphi(\|a_n \widetilde{x}_n\|_0, \|b_n \widetilde{x}_n\|_1) < \varepsilon.$$

The submultiplicativity of ρ implies

$$\varphi(s,t) = s\rho\left(\frac{t}{s}\right) \le c\ s\rho(2^{-k})\rho\left(2^k \frac{t}{s}\right) = c\ s\ a_k \rho\left(\frac{b_k t}{a_k s}\right) = c\ \varphi(a_k s, b_k t)$$

for all $s, t \geq 0$ and $k = 1, 2, \ldots$.
Hence

$$\begin{aligned}
u(x) &\leq \sum_{k=1}^{n} \varphi(\|x_k\|_0, \|x_k\|_1) + \varphi(\|\widetilde{x}_n\|_0, \|\widetilde{x}_n\|_1) \\
&\leq c \sum_{k=1}^{n} \varphi(\|a_k x_k\|_0, \|b_k x_k\|_1) + c\varphi(\|a_n \widetilde{x}_n\|_0, \|b_n \widetilde{x}_n\|_1) \\
&\leq c \sum_{k=1}^{\infty} \varphi(\|a_k x_k\|_0, \|b_k x_k\|_1) + c\varepsilon \leq c\varphi\left(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\right) + c\varepsilon \\
&\leq c \max\left(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\right) + c\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$u(x) \leq c \max\left(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\right).$$

Taking the infimum on the right side yields

$$u(x) \leq c\, w(x).$$

Finally, we have to prove that

$$w(x) \leq 2\, v(x) \quad \text{for all} \quad x \in X_0.$$

So assume $x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k$ with $x_k \in X_0$ and $\sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1) < \infty$. As already observed, this implies the convergence of $\sum_{k=1}^{\infty} \|x_k\|_1$. Let

$$I_n = \{k \in \mathbb{N} : \; 2^n \|x_k\|_1 \leq \|x_k\|_0 \leq 2^{n+1} \|x_k\|_1\}.$$

Since $\|x_k\|_1 \leq \|x_k\|_0$, we have $\bigcup_{n \geq 0} I_n = \mathbb{N}$. Set $y_n = \sum_{k \in I_n} x_k$. Then $x \overset{X_1}{=} \sum_{k=1}^{\infty} y_k$. Now, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \|a_n y_n\|_0 &\leq \sum_{n=0}^{\infty} \sum_{k \in I_n} \|\rho(2^{-n}) x_k\|_0 = \sum_{n=0}^{\infty} \sum_{k \in I_n} \varphi(\|x_k\|_0, 2^{-n} \|x_k\|_0) \\
&\leq \sum_{n=0}^{\infty} \sum_{k \in I_n} \varphi(\|x_k\|_0, 2\|x_k\|_1) \leq 2 \sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1).
\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\|b_n y_n\|_1 &\le \sum_{n=0}^{\infty}\sum_{k\in I_n}\|2^n\rho(2^{-n})x_k\|_1 = \sum_{n=0}^{\infty}\sum_{k\in I_n}\varphi(2^n\|x_k\|_1,\|x_k\|_1)\\ &\le \sum_{n=0}^{\infty}\sum_{k\in I_n}\varphi(\|x_k\|_0,\|x_k\|_1) = \sum_{k=1}^{\infty}\varphi(\|x_k\|_0,\|x_k\|_1).\end{aligned}$$

Altogether

$$w(x) \le \max\left(\sum_{n=0}^{\infty}\|a_n y_n\|_0, \sum_{n=0}^{\infty}\|b_n y_n\|_1\right) \le 2\sum_{k=1}^{\infty}\varphi(\|x_k\|_0,\|x_k\|_1).$$

Turning to the infimum on the right hand side yields the claim. ■

We generalize now a result obtained by U. Matter, see [10, Theorem A].

Theorem 3.2. *Let $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(X,Z)$ be such that*

$$\|Tx\| \le \varphi(\|Sx\|,\|x\|) \quad \textit{holds for all } x \in X.$$

Let $\ker(S)$ denote the kernel of S. Moreover, let ρ be submultiplicative. Then there exists an operator $D : (X/\ker(S), Z)_{\varphi,1} \to Y$ such that the operator T factors as follows: $T = DJ_\varphi Q$, where Q and J_φ denote the canonical quotient map $X \to X/\ker(S)$ and the continuous embedding $X/\ker(S) \hookrightarrow (X/\ker(S), Z)_{\varphi,1}$, respectively. In other words, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \Big\downarrow Q & & \Big\uparrow D \\ X/\ker(S) & \xrightarrow{J_\varphi} & (X/\ker(S), Z)_{\varphi,1} \end{array}$$

Proof: Consider the canonical factorization of S

$$S : X \xrightarrow{Q} X/\ker(S) \xrightarrow{\widetilde{S}} Z.$$

By (2.2) we have $\ker(S) \subset \ker(T)$. This implies that T factors as follows

$$X \xrightarrow{Q} X/\ker(S) \xrightarrow{P} X/\ker(T) \xrightarrow{T_1} Y.$$

We may write $T = \widetilde{T}Q$, where $\widetilde{T} = T_1 P$. Furthermore, from (2.2) we obtain

$$\|\widetilde{T}\widetilde{x}\| \le \varphi(\|\widetilde{S}\widetilde{x}\|,\|\widetilde{x}\|) \text{ for all } \widetilde{x} \in X/\ker(S).$$

For $\widetilde{x} \in X/\ker(S)$ with $\widetilde{x} = \sum_{k=1}^{n} \widetilde{x}_k$ we have

$$\|\widetilde{T}\widetilde{x}\| \leq \sum_{k=1}^{n} \|\widetilde{T}\widetilde{x}_k\| \leq \sum_{k=1}^{n} \varphi(\|\widetilde{S}\widetilde{x}_k\|, \|\widetilde{x}_k\|).$$

Proposition 3.1 ensures that there exists a constant $C > 0$ such that

$$\|\widetilde{T}\widetilde{x}\| \leq C\|\widetilde{x}\|_{\varphi,1}$$

Define $D_0 : X/\ker(S) \to Y,\ D_0(\widetilde{x}) := \widetilde{T}\widetilde{x}$. By density we may extend the operator D_0 to a continuous map $D : (X/\ker(S), Z)_{\varphi,1} \to Y$. Of course $T = DJ_\varphi Q$. ■

4. (p, φ)-absolutely continuous operators

Our goal in this section is to investigate some important examples of the interpolative construction introduced in Section 2. We begin by reviewing some of the needed results on absolutely p-summing operators. Suppose that $1 \leq p < \infty$. Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be *absolutely p-summing* or just p-summing, if it takes weak ℓ_p-sequences (x_n) of X (i.e. $(\langle x_n, x'\rangle) \in \ell_p$ for all $x' \in X'$) to strong ℓ_p sequences (Tx_n) of Y (i.e. $(\|Tx_n\|) \in \ell_p$). In fact T is p-summing if and only if there exists a constant C such that for any choice of $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leq C \sup_{x' \in B_{X'}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|\Big)^{1/p}.$$

The least constant C for which the above inequality holds is denoted by $\pi_p(T)$. The class of p-summing operators endowed with the norm π_p constitutes an injective maximal Banach ideal denoted by $[\Pi_p, \pi_p]$. The fundamental characterization of p-summing operators developed by A. Pietsch (see [13]) may be formulated as follows. An operator $T \in \mathcal{L}(X, Y)$ is p-summing if and only if there exist a constant $C \geq 0$ and a regular probability measure μ on $B_{X'}$ such that for each $x \in X$

$$\|Tx\| \leq C\Big(\int_{B_{X'}} |\langle x, x'\rangle|^p d\mu(x')\Big)^{1/p}.$$

By applying the definition of $(\mathcal{A}, \mathcal{B})_\varphi$ to ideals $[\mathcal{A}, \alpha] = [\Pi_p, \pi_p]$ and $[\mathcal{B}, \beta] = [\mathcal{L}, \|\cdot\|]$ we may consider the Banach ideal $[\Pi_{p,\varphi}, \pi_{p,\varphi}] := [(\Pi_p, \mathcal{L})_\varphi, \overline{\gamma}]$ of all

(p,φ)-absolutely continuous operators. By Theorem 2.4, the ideal norm $\pi_{p,\varphi}$ is equivalent to

$$\widetilde{\pi}_{p,\varphi}(T) = \inf_{\lambda}\{\varphi(\pi_p(S),\lambda) : \quad \|Tx\| \le \varphi(\|Sx\|, \lambda\|x\|)\},$$

provided that ρ is submultiplicative. The subsequent result characterizes (p,φ)-absolutely continuous operators by a special factorization property through a suitable space given by the interpolation method $(\cdot,\cdot)_{\varphi,1}$.

Theorem 4.1. *Let ρ be submultiplicative. For every operator $T \in \mathcal{L}(X,Y)$, the following statements are equivalent*

(i) *T is (p,φ)-absolutely continuous.*

(ii) *There exist a probability measure μ on $B_{X'}$ and a constant $C > 0$ such that*

$$\|Tx\| \le \varphi(C\|J_\mu x\|, \|x\|), \quad \textit{for every } x \in X,$$

where $J_\mu : X \to L_p(\mu)$ is the restriction of the canonical map $J_p : C(B_{X'}) \to L_p(\mu)$ and is given by $x \mapsto \langle x, \cdot\rangle$.

(iii) *There exist a probability measure μ on $B_{X'}$, a constant $C > 0$ and an operator $R : (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \to Y$ such that $\|R\| \le C$ and $T = RJ_{\mu,\varphi}Q$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \Big\downarrow{\scriptstyle Q} & & {\scriptstyle R}\Big\uparrow \\ X/\ker(J_\mu) & \xrightarrow{J_{\mu,\varphi}} & (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \end{array}$$

Proof: The equivalence of (i) and (ii) follows immediately from the definition of (p,φ)-absolutely continuous operators and the Pietsch factorization theorem for p-summing operators. For a proof of the implication (iii) to (i) observe that the continuous embedding $\widetilde{J}_\mu : X/\ker(J_\mu) \to L_p(\mu)$ is p-summing. This fact together with the properties of interpolation norms shows that the embedding $J_{\mu,\varphi} : X/\ker(J_\mu) \to (X/\ker(J_\mu), L_p(\mu))_{\varphi,1}$ is (p,φ)-absolutely continuous. We finally show $(ii) \Rightarrow (iii)$. By assumption there is a probability measure μ on $B_{X'}$ such that

$$\|C^{-1}Tx\| \le \varphi(\|J_\mu x\|, \|x\|) \quad \text{for all } x \in X.$$

Thus by setting $Z = L_p(\mu)$ and $S = J_\mu$ and applying Theorem 3.2, we obtain that $C^{-1}T = DJ_{\mu,\varphi}Q$ for a suitable operator $D : (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \to Y$. Finally the operator $R = CD$ possesses the desired properties. ∎

5. Applications to approximation quantities and entropy numbers

We start with the introduction of some basic notation for approximation quantities and entropy numbers of linear operators between Banach spaces. Given a closed linear subspace M of X, the inclusion mapping of M into X will be denoted by J_M^X. The k-th *Gelfand number* of $T \in \mathcal{L}(X,Y)$ is given by

$$c_k(T) = \inf\{\|TJ_M^X\| : \quad M \subset X \text{ and } \operatorname{codim} M < k\}.$$

The k-th *entropy number* of a bounded set $M \subset X$ is defined as

$$\varepsilon_k(M) = \inf\Big\{\varepsilon > 0 \mid \ \exists x_1, \dots, x_k \in X \text{ such that } M \subset \bigcup_{i=1}^{k}(x_i + \varepsilon B_X)\Big\}.$$

Furthermore the k-th inner entropy number of a bounded set $M \subset X$ is given by

$$\varphi_k(M) = \sup\Big\{\rho > 0 \mid \ \exists x_1, \dots, x_k \in M \text{ such that } \|x_i - x_k\| \geq 2\rho \text{ for } i \neq k\Big\}.$$

For an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces we put

$$\varepsilon_n(T) = \varepsilon_n(T(B_X)) \quad \text{and} \quad \varphi_n(T) = \varphi_n(T(B_X)).$$

Moreover, we study the quantities

$$e_n(T) = \varepsilon_{2^{n-1}}(T) \quad \text{and} \quad f_n(T) = \varphi_{2^{n-1}}(T),$$

called *dyadic entropy numbers* and *inner dyadic entropy numbers*, respectively. For any operator $T \in \mathcal{L}(X,Y)$ we have

$$f_n(T) \leq e_n(t) \leq 2f_n(t). \tag{5.1}$$

The speed of convergence to zero of a sequence of entropy numbers measures "quality" of compactness of the operator under consideration.

Throughout this section we use symmetric quasi Banach sequence spaces. By this term we mean a quasi Banach space E consisting of scalar sequences such that $\|(x_n)\|_E = \|(x_n^*)\|_E$. To avoid trivial cases we assume that E contains a sequence with full support. For arbitrary Banach spaces X, Y we define

$$\mathcal{L}_E^{(s)}(X,Y) = \{T \in \mathcal{L}(X,Y) : \quad (s_n(T)) \in E\}$$

with $s = c$ or $s = e$. The above linear space equipped with the quasi-norm defined by

$$\|T|\mathcal{L}_E^{(s)}\| = \|(s_n(T))\|_E,$$

becomes a quasi Banach operator ideal.

More information on entropy numbers and approximation quantities may be found in [8] and [4].

Lemma 5.1. *If $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(X, Z)$ and $R \in \mathcal{L}(X, W)$ are such that (2.2) holds then*

$$c_{n+m-1}(T) \le \varphi(c_n(S), c_m(R)). \tag{5.2}$$

Proof: For given $\varepsilon > 0$ we may choose subspaces M and N of X such that

$$\|SJ_M^X\| \le (1+\varepsilon)c_n(S) \quad \text{and} \quad \operatorname{codim}(M) < n,$$

$$\|RJ_N^X\| \le (1+\varepsilon)c_m(R) \quad \text{and} \quad \operatorname{codim}(N) < m.$$

Define $L := M \cap N$. It is easy to verify that

$$\operatorname{codim}(L) \le \operatorname{codim}(M) + \operatorname{codim}(N) < m + n - 1.$$

Using the above inequality we obtain

$$c_{n+m-1}(T) \le \|TJ_L^X\| = \sup_{\|x\|\le 1} \|(TJ_J^X)x\| \le \sup_{\|x\|\le 1} \varphi(\|(SJ_M^X)x\|, \|(RJ_N^X)x\|)$$

$$\le (1+\varepsilon)\varphi(c_n(S), c_m(R)).$$

Letting $\varepsilon \to 0$ we conclude that (5.2) holds. ■

Using a similar argument as in [13] we obtain the following inequality for dyadic entropy numbers.

Lemma 5.2. *If $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(X, Z)$ and $R \in \mathcal{L}(X, W)$ are such that (2.2) holds then*

$$e_{n+m-1}(T) \le 2\varphi(e_n(S), e_m(R)). \tag{5.3}$$

Proof: Suppose that $\sigma_0 > e_n(S)$ and $\sigma_1 > e_m(R)$. Then we find $z_1, \dots z_{q_0} \in Z$ and $w_1, \dots w_{q_1} \in W$ with

$$S(B_X) \subseteq \bigcup_{h=1}^{q_0} \{z_h + \sigma_0 B_Z\} \quad \text{and} \quad R(B_X) \subseteq \bigcup_{h=1}^{q_1} \{w_h + \sigma_1 B_W\}$$

respectively, and $q_0 \leq 2^{n-1}, q_1 \leq 2^{m-1}$. For given $x_1, \ldots, x_p \in B_X$ with $p > 2^{(n+m-1)-1}$ we define

$$I_h := \{i : \; Sx_i \in z_h + \sigma_0 B_Z\}.$$

Since $\sum_{h=1}^{q_0} \operatorname{card}(I_h) \geq p > q_0 q_1$, we have $\operatorname{card}(I_{h_0}) > q_1$ for some h_0. Hence there exist $i, j \in I_{h_0}$ such that Rx_i and Rx_j belong to the same $w_{h_1} + \sigma_1 B_W$. This means that

$$\|Sx_i - Sx_j\| \leq 2\sigma_0 \;\text{ and }\; \|Rx_i - Rx_j\| \leq 2\sigma_1.$$

Thus we obtain

$$\|Tx_i - Tx_j\| \leq 2\varphi(\sigma_0, \sigma_1).$$

By (5.1) we obtain

$$e_{n+m-1}(T) \leq 2f_{n+m-1}(T) \leq 2\varphi(\sigma_0, \sigma_1).$$

Since this is true for arbitrary $\sigma_0 > e_n(S)$ and $\sigma_1 > e_m(R)$, this completes the proof. ■

Let E, E_0, E_1 be quasi normed sequence spaces. The function φ is said to be (E, E_0, E_1)-regular, if for every non-decreasing, positive sequences (s_n) and (t_n) the following inequality holds

$$\Big\|(\varphi(s_n, t_n))\Big\|_E \leq \varphi\Big(\|(s_n)\|_{E_0}, \|(t_n)\|_{E_1}\Big). \tag{5.4}$$

Example. In case when $E = \ell_p, E_0 = \ell_{p_0}$ and $E_1 = \ell_{p_1}$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\varphi(s,t) = s^{1-\theta} t^{\theta}$ for $0 < \theta < 1$ the above condition becomes the Hlder inequality.

Im summary we can state the following result which generalizes results obtained by H. Jarchow and U. Matter in [7].

Proposition 5.3. *Let φ be a (E, E_0, E_1)-regular function. Then*

$$\left(\mathcal{L}_{E_0}^{(c)}, \mathcal{L}_{E_1}^{(c)}\right)_{\varphi} \subseteq \mathcal{L}_E^{(c)}$$

and

$$\left(\mathcal{L}_{E_0}^{(e)}, \mathcal{L}_{E_1}^{(e)}\right)_{\varphi} \subseteq \mathcal{L}_E^{(e)}.$$

Proof: We give the proof only for the case of Gelfand numbers. The entropy number case is similar. Let us assume that φ is a (E, E_0, E_1)-regular function. Then using Lemma 5.1 and Inequality (5.4) we obtain

$$\|(c_n(T))\|_E \leq \| (\varphi(c_n(S), c_n(R))) \|_E \leq \varphi\big(\|(c_n(S))\|_{E_0}, \|(c_n(R))\|_{E_1}\big).$$

This completes the proof. ∎

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