Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–23

A LOGIC OF INJECTIVITY

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ABSTRACT: Injectivity of objects with respect to a set $\mathcal{H}$ of morphisms is an important concept of algebra and homotopy theory; here we study the logic of consequences of $\mathcal{H}$, by which we understand morphisms h such that injectivity with respect to $\mathcal{H}$ implies injectivity with respect to h. We formulate three simple deduction rules for the injectivity logic and for its finitary version (where morphisms between finitely ranked objects are considered only), and prove that they are sound (in all categories) and complete (in all "reasonable" categories).

1. Introduction

Recall that an object A is injective w.r.t. a morphism $h : P \to P'$ provided that every morphism from P to A factors through h. We address the following problem: given a set $\mathcal{H}$ of morphisms, which morphisms h are *consequences* of $\mathcal{H}$ in the sense of injectivity (i.e., every object injective w.r.t. all members of $\mathcal{H}$ is also injective w.r.t. h)? We denote the consequence relationship by $\mathcal{H} \models h$.

This is a classical topic in general algebra: the *equational logic* of Garrett Birkhoff [6] is a special case. In fact, an equation $s = t$ is a pair of elements of a free algebra F, and that pair generates a congruence $\sim$ on F. An algebra A satisfies $s = t$ iff it is injective w.r.t. the canonical epimorphism

$$h : F \to F/\sim .$$

Thus, if we restrict our sets $\mathcal{H}$ to regular epimorphisms with free domains, then the logic of injectivity becomes precisely the equational logic. However, there are other important cases in algebra: recall for example the concept of injective module, where $\mathcal{H}$ is the set of all monomorphisms (in the category of modules).

To mention an example from homotopy theory, recall that a *Kan complex* [9] is a simplicial set injective w.r.t. all the monomorphisms $\Delta_n^k \hookrightarrow \Delta_n$ (for

Received May 25, 2006.
The first author acknowledges financial support by the Czech Grant Agency, Project 201/06/0664.
The third author acknowledges financial support by the Center of Mathematics of the University of Coimbra and the School of Technology of Viseu.

n, $k \in \mathbb{N}$, $k \leq n$) where Δ_n is the complex generated by a single n-simplex (with 1-simplexes 0, 1, …, n) and Δ_n^k is the subcomplex obtained by deleting the 1-simplex k and all adjacent faces. We can ask for example whether Kan complexes can be specified by a simpler collection of monomorphisms, as a special case of our injectivity logic.

Injectivity is a Galois correspondence between objects and morphisms of a category. The closed families on the side of objects are called *injectivity classes*: for every set $\mathcal{H}$ of morphisms we obtain the injectivity class Inj$\mathcal{H}$, i.e., the class of all objects injective w.r.t. $\mathcal{H}$. In [4] small-injectivity classes in locally presentable categories were characterized as precisely the full accessible subcategories closed under products, and in [12] this was sharpened in the following sense. Let us call a morphism λ-*ary* if its domain and codomain are λ-presentable objects. Injectivity classes with respect to λ-ary morphisms are precisely the full subcategories closed under products, λ-filtered colimits, and λ-pure subobjects.

In the present paper we study closed sets in the side of morphisms, i.e., we develop a logic of the above consequence relationship $\models$. It has altogether three deduction rules, which are quite intuitive. Firstly, observe that every object injective w.r.t. a composite $h = h_2 \cdot h_1$ is injective w.r.t. h_1. This gives us the first deduction rule

$$\text{CANCELLATION}\ \frac{h_2 \cdot h_1}{h_1}$$

It is also easy to see that injectivity w.r.t. h implies injectivity w.r.t. any morphism h' opposite to h in a pushout (along an arbitrary morphism), which yields the rule

$$\text{PUSHOUT}\qquad \frac{h}{h'} \qquad \text{for every pushout}\quad \begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow{h'} & \cdot \end{array}$$

Finally, an object injective w.r.t. two composable morphisms is also injective w.r.t. their composite. The same holds for three, four, … morphisms – but also for a *transfinite composite* as used in homotopy theory. For example, given an ω-chain of morphisms

$$A_0 \xrightarrow{h_0} A_1 \xrightarrow{h_1} A_2 \xrightarrow{h_2} \cdots$$

then their ω-composite is the first morphism $c_0 : A_0 \to C$ of (any) colimit cocone $c_n : A_n \to C\,(n \in \mathbb{N})$ of the chain. Observe that c_0 is indeed a consequence of $\{h_i;\ i < \omega\}$. For every ordinal λ we have the concept of a λ-composite of morphisms (see 2.11 below) and the following deduction rule, expressing the fact that an object injective w.r.t. each h_i is injective w.r.t. the transfinite composite:

TRANSFINITE COMPOSITION $\dfrac{h_i\,(i < \lambda)}{h}$ for every λ-composite h of $(h_i)_{i<\lambda}$

We are going to prove that the Injectivity Logic based on the above three rules is sound and complete. That is, given a set $\mathcal{H}$ of morphisms, then $\mathcal{H} \models h$ holds for precisely those morphisms h which can be proved from assumptions in $\mathcal{H}$ using the three deduction rules above. This holds in a number of categories, e.g., in

(a) every variety of algebras,
(b) the category of topological spaces and many nice subcategories (e.g. Hausdorff spaces), and
(c) every locally presentable category of Gabriel and Ulmer.

We introduce the concept of a weakly locally presentable category encompassing (a)-(c) above, and prove the soundness and completeness of our Injectivity Logic in all such categories.

Observe that the above logic is infinitary, in fact, it has a proper class of deduction rules: one for every ordinal λ in the instance of TRANSFINITE COMPOSITION. We also present the corresponding Finitary Injectivity Logic: it is the restriction of the above logic to λ finite. Well, all we need to consider are the cases $\lambda = 2$, called COMPOSITION, and $\lambda = 0$, called IDENTITY:

COMPOSITION $\dfrac{h_1 \quad h_0}{h}$ for $h = h_1 \cdot h_0$

IDENTITY $\dfrac{}{\mathrm{id}_A}$

The resulting finitary deductive system has only four deduction rules; it is clearly sound, and the main result of our paper (Theorem 6.2) says that it is also complete with respect to finitary morphisms, i.e., morphisms with domain and codomain of finite rank. This implies the expected compactness theorem: every finitary consequence of a set $\mathcal{H}$ of finitary morphisms is a consequence of some finite subset of $\mathcal{H}$.

The completeness theorem for Finitary Injectivity Logic will then be extended to the k-ary Injectivity Logic, defined in the expected way. Then the full completeness theorem easily follows.

The fact that the full Injectivity Logic above is complete in weakly locally presentable categories can also be derived from Quillen's Small Object Argument [11], see Remark 3.9 below. However our sharpening to the k-ary logic for every cardinal k cannot be derived from that paper, and we consider this to be a major step.

Related work The finitary deductive system above was formulated in [5], where a (rather simple) proof of its completeness restricted to sets of finitary epimorphisms is presented. A related result was previously obtained by G. Roçu, see [13]. In the algebraic context, this restricted injectivity logic can be expressed by *quasi-equations*, that is, by sentences of the form

$$\forall \mathbf{x}(E(\mathbf{x}) \to F(\mathbf{x}))$$

where E and F are conjunctions of equations. More generally, the unrestricted injectivity logic for arbitrary finitary morphisms can be expressed by the *regular* sentences, i.e., of the form

$$\forall \mathbf{x}(E(\mathbf{x}) \to \exists \mathbf{y} F(\mathbf{x}, \mathbf{y})).$$

For a different approach to categorical regular logic (including a completeness theorem), see [10].

2. Logic of Injectivity

2.0. Assumption Throughout the paper we assume that we are working in a cocomplete category.

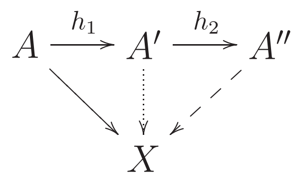
2.1. Definition A morphism h is called a *consequence* of a set of morphisms $\mathcal{H}$, notation

$$\mathcal{H} \models h$$

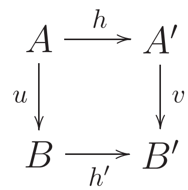
provided that every object injective w.r.t. all morphisms in $\mathcal{H}$ is also injective w.r.t. h.

2.2. Examples (1) A composite $h = h_2 \cdot h_1$ is a consequence of h_1 and h_2.

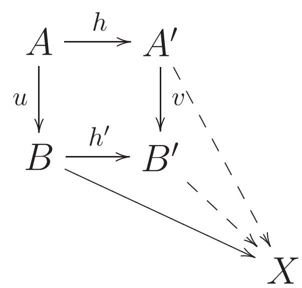
(2) Conversely, in every composite $h = h_2 \cdot h_1$ the morphism h_1 is a consequence of h:



(3) In every pushout



h' is a consequence of h:



2.3. Remark The above examples are exhaustive in the sense of the following logic, introduced in [5] (where, however, it was only applied to epimorphisms):

2.4. Definition The *Finitary Injectivity Deduction System* consists of one axiom

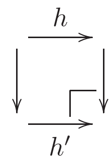
IDENTITY $\qquad \dfrac{}{\mathrm{id}_A}$

and three deduction rules

COMPOSITION $\qquad \dfrac{h \quad h'}{h' \cdot h}$ $\quad$ if $h' \cdot h$ is defined

CANCELLATION $\qquad \dfrac{h' \cdot h}{h}$

and

PUSHOUT $\dfrac{h}{h'}$ if 

We say that a morphism h is a *formal consequence* of a set $\mathcal{H}$ of morphisms (notation $\mathcal{H} \vdash h$) in the Finitary Injectivity Logic if there exists a proof of h from $\mathcal{H}$.

2.5. Remark In 2.4, a *proof of* h means, as usual, a finite sequence $h_1, \ldots, h_n = h$ of morphisms such that for every $i = 1, \ldots, n$ the morphism h_i lies in $\mathcal{H}$ or is a conclusion of one of the deduction rules whose premises lie in $\{h_1, \ldots, h_{i-1}\}$.

2.6. Lemma *The Finitary Injectivity Logic is sound, i.e., if a morphism* h *is a formal consequence of a set of morphisms* $\mathcal{H}$*, then* h *is a consequence of* $\mathcal{H}$*. Shortly:* $\mathcal{H} \vdash h$ *implies* $\mathcal{H} \models h$*.*

The proof follows from 2.2.

2.7. Remark Later we define finitary morphisms (as morphisms whose domains and codomains are finitely presentable (Section 3) or of finite rank (Section 5)), and in Section 6 we prove that the resulting Finitary Injectivity Logic is complete, i.e., that

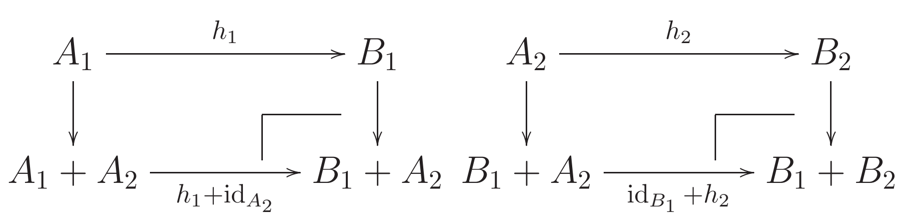
$$\mathcal{H} \models h \ \text{ implies } \ \mathcal{H} \vdash h$$

for every set $\mathcal{H}$ of finitary morphisms and every h finitary.

2.8. Example The following rule

FINITE COPRODUCT $\dfrac{h_1 \quad h_2}{h_1 + h_2}$

(where for $h_i : A_i \to B_i$ the morphism $h_1 + h_2 : A_1 + A_2 \to B_1 + B_2$ is the canonical coproduct morphism) is obviously sound. Here is a proof in the Finitary Injectivity Logic:
Using the pushouts



we can write

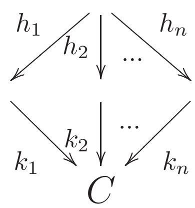
$$\cfrac{\cfrac{h_1 \qquad\qquad h_2}{h_1 + \mathrm{id}_{A_2} \qquad \mathrm{id}_{B_1} + h_2}}{h_1 + h_2} \quad \begin{array}{l} \text{via PUSHOUT} \\ \text{via COMPOSITION} \end{array}$$

since $h_1 + h_2 = (\mathrm{id}_{B_1} + h_2) \cdot (h_1 + \mathrm{id}_{A_2})$.

2.9. Example The following rule

$$\text{FINITE WIDE PUSHOUT} \qquad \frac{h_1 \ \ldots \ h_n}{h}$$

for every wide pushout



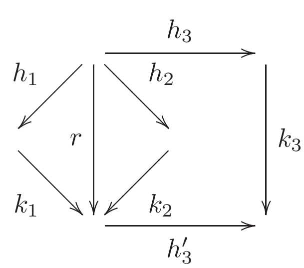
where $h = k_i \cdot h_i$

is sound. Here is a proof in the Finitary Injectivity Logic:

If $n = 2$ we have

$$\cfrac{\cfrac{h_1 \qquad h_2}{k_2}}{h = k_2 \cdot h_2} \quad \begin{array}{l} \text{via PUSHOUT} \\ \text{via COMPOSITION} \end{array}$$

If $n = 3$ denote by r a pushout of h_1, h_2, then a pushout, h_3',



of h_3 along r forms a wide pushout of h_1, h_2 and h_3:

$$\begin{array}{cl} h_1 \quad h_2 \quad h_3 & \\ \hline & \text{via PUSHOUT} \\ k_2 & \\ \hline & \text{via COMPOSITION} \\ r & \\ \hline & \text{via PUSHOUT} \\ k_3 & \\ \hline & \text{via COMPOSITION} \\ h = k_3 \cdot h_3 & \end{array}$$

Etc.

2.10. Remark We want to define a composition of a chain of λ morphisms for every ordinal λ (see the case $\lambda = \omega$ in the Introduction.). Recall that a λ-*chain* is a functor A from λ, the well-ordered category of all ordinals $i < \lambda$. We call A *smooth* if for every limit ordinal $i < \lambda$ we have

$$A_i = \operatorname*{colim}_{j<i} A_j$$

with the colimit cocone of all $a_{ji} = A(j \to i)$.

Recall that λ^+ denotes the successor ordinal, i.e., the set of all $i \leq \lambda$.

2.11. Definition A morphism h is called a λ-*composite* of morphisms $(h_i)_{i<\lambda}$, where λ is an ordinal, if there exists a smooth λ^+-chain A with connecting morphisms $a_{ij} : A_i \to A_j$ for $i \leq j \leq \lambda$ such that

$$h_i = a_{i,i+1} \quad \text{for all } i < \lambda$$

and

$$h = a_{0,\lambda}.$$

We call h a *transfinite composite* of morphisms if it is a λ-composite for some λ.

2.12. Examples $\lambda = 0$: No morphism h_i is given, just an object A_0; and $h = a_{0,0}$ is the identity morphism of A_0.

$\lambda = 1$: A morphism h_0 is given, and we have $h = a_{0,1} = h_0$. Thus, a 1-composite of h_0 is h_0.

$\lambda = 2$: This is the usual concept of composition: given morphisms h_0, h_1, their 2-composite exists iff they are composable. Then $h_1 \cdot h_0$ is the 2-composite.

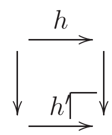
$\lambda = \omega$: This is the case mentioned in the Introduction. Observe that, unlike the previous cases, an ω-composite is only unique up to isomorphism.

2.13. Lemma *A transfinite composite of morphisms $(h_i)_{i<\lambda}$ is a consequence of these morphisms.*

Proof This is a trivial transfinite induction on λ. In case $\lambda = 0$ this states that id_A is a consequence of $\emptyset$, etc.

2.14. Definition The *Injectivity Deduction System* consists of the deduction rules

CANCELLATION $$\frac{h' \cdot h}{h}$$

PUSHOUT $$\frac{h}{h'}$$ for every pushout 

and the rule scheme (one rule for every ordinal λ)

TRANSFINITE COMPOSITION $$\frac{h_i \ (i < \lambda)}{h}$$ for every λ-composite h of $(h_i)_{i<\lambda}$

We say that a morphism h is a *formal consequence* of a set $\mathcal{H}$ of morphisms (notation $\mathcal{H} \vdash h$) in the Injectivity Logic if there exists a proof of h from $\mathcal{H}$.

2.15. Remark The Injectivity Logic is infinitary: here a proof of h is a chain $(h_i)_{i \leq n}$ of morphisms, where n is an ordinal, such that $h = h_n$, and each h_i either lies in $\mathcal{H}$, or is a conclusion of one of the deduction rules whose premises lie in $\{h_j\}_{j<i}$.

2.16. Lemma *The Injectivity Logic is sound, i.e., if a morphism h is a formal consequence of a set $\mathcal{H}$ of morphisms, then h is a consequence of $\mathcal{H}$. Shortly: $\mathcal{H} \vdash h$ implies $\mathcal{H} \models h$.*

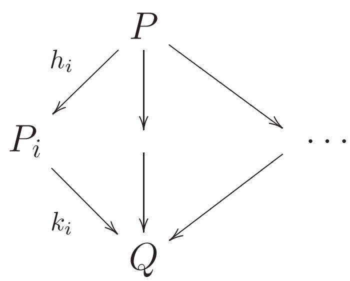
The proof (using 2.13) is elementary.

2.17. Remark In 2.14 we can replace TRANSFINITE COMPOSITION by the deduction rule WIDE PUSHOUT, see below, which makes use of the (obvious) fact that an object A injective w.r.t. a set $\{h_i\}_{i<\lambda}$ of morphisms having a common domain is also injective w.r.t. their wide pushout. Let us remark here that this rule does not replace PUSHOUT of 2.14 (because in the latter a pushout of h along an *arbitrary* morphism is considered).

2.18. Definition The deduction rule

$$\text{WIDE PUSHOUT} \qquad \frac{h_i \ (i < \lambda)}{h} \quad \text{for } h \text{ a wide pushout of } \{h_i\}_{i<\lambda}$$

applies, for every cardinal λ, to an arbitrary object P and an arbitrary set $\{h_i\}$ of λ morphisms with the common domain P and the following wide pushout



$h = k_i \cdot h_i$ (for any i)

Remark Again, this is a scheme of deduction rules: for every cardinal λ we have one rule λ-WIDE PUSHOUT. Observe that $\lambda = 0$ yields the rule IDENTITY.

2.19. Lemma *The Injectivity Deduction System 2.14 is equivalent to the deduction system*

COMPOSITION, CANCELLATION, PUSHOUT *and* WIDE PUSHOUT.

Proof (1) We can derive WIDE PUSHOUT from 2.14. For every ordinal number λ we derive the rule

$$\frac{h_i \ (i < \lambda)}{h} \quad \text{for } h \text{ a wide pushout of } \{h_i\}_{i<\lambda}$$

by transfinite induction on the ordinal λ. We are given an object P and morphisms $h_i : P \to P_i \, (i < \lambda)$. The case $\lambda = 0$ is trivial, from λ derive $\lambda + 1$ by using PUSHOUT, and for limit ordinals λ form the restricted multiple pushouts Q_j of morphisms h_i for $i < j$, and observe that they form a smooth chain whose composite is a multiple pushout of all h_i's.

(2) From the system in 2.19 we can derive the rule λ-COMPOSITION, where λ is an arbitrary ordinal: the case $\lambda = 0$ follows from 0-WIDE PUSHOUT. The isolated step uses COMPOSITION: a transfinite composite of $(h_i)_{i \leq \lambda}$ is simply $h_\lambda \cdot k$ where k is a transfinite composite of $(h_i)_{i<\lambda}$. In the limit case, use the

fact that a composite h of $(h_i)_{i<\lambda}$ is a wide pushout of $\{k_i\}_{i<\lambda}$, where k_i is a composite of $(h_j)_{j<i}$.

2.20. Remark For every infinite cardinal k the *k-ary Injectivity Deduction System* is the system 2.14 where λ ranges through ordinals smaller than k. A proof of a morphism h from a set $\mathcal{H}$ in the k-ary Injectivity Logic is, then, a proof of length $n < k$ using only the deduction rules with λ restricted as above. The last lemma can, obviously, be formulated under this restriction in case we use the scheme λ-WIDE PUSHOUT for all cardinals $\lambda < k$.

2.21. Definition The deduction rule

$$\text{COPRODUCT} \qquad \frac{h_i \ (i < \lambda)}{\coprod_{i<\lambda} h_i}$$

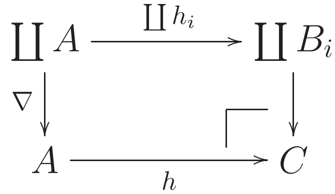
applies, for every cardinal λ, to an arbitrary collection of λ morphisms $h_i : A_i \to B_i$.

2.22. Lemma *The Injectivity Deduction System 2.14 is equivalent to the deduction system of 2.19 with* WIDE PUSHOUT *replaced by*

IDENTITY + COPRODUCT

Proof (1) COPRODUCT follows from 2.19. In fact, $\coprod_{i<\lambda} h_i : \coprod_{i<\lambda} A_i \to \coprod_{i<\lambda} B_i$ is a wide pushout of the morphisms $k_j : \coprod_{i<\lambda} A_i \to \coprod_{i<j} A_i + B_j + \coprod_{j<i<\lambda} A_i$, where j ranges through λ, with components $\mathrm{id}_{A_i}\,(i \neq j)$ and h_j, and k_j is a pushout of h_j along the j-th coproduct injection of $\coprod_{i<\lambda} A_i$.

(2) Conversely, WIDE PUSHOUT follows from IDENTITY+COPRODUCT. We obviously need to consider only $\lambda > 1$ and then we use the fact that given morphisms $h_i : A \to B_i\,(i < \lambda)$, their wide pushout $h : A \to C$ can be obtained from $\coprod_{i<\lambda} h_i$ by pushing out along the codiagonal $\nabla : \coprod_\lambda A \to A$:



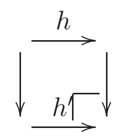
2.23. Remark The deduction system of the last lemma has five rules, but the advantage against the system 2.14 is that they are particularly simple to formulate.

IDENTITY $$\frac{}{\mathrm{id}_A}$$

CANCELLATION $$\frac{h_2 \cdot h_1}{h_1}$$

COMPOSITION $$\frac{h_2 \quad h_1}{h_2 \cdot h_1} \qquad \text{if } h_2 \cdot h_1 \text{ is defined}$$

PUSHOUT $$\frac{h}{h'}$$ given



COPRODUCT $$\frac{h_i \ (i < \lambda)}{\coprod_{i<\lambda} h_i}$$

We prove below that 2.14, and therefore the above equivalent deduction system, is not only sound but (in a number of categories) also complete.

3. Completeness in Locally Presentable Categories

3.1. Assumption In the present section we study injectivity in a *locally presentable* category $\mathcal{A}$ of Gabriel and Ulmer, see [7] or [3]. This means that:

(a) $\mathcal{A}$ is cocomplete,

and

(b) there exists a regular cardinal λ such that $\mathcal{A}$ has a set of λ-presentable objects whose closure under λ-filtered colimits is all in $\mathcal{A}$.

Recall that an object A is λ-*presentable* if its hom-functor $\hom(A, -) : \mathcal{A} \to$ **Set** preserves λ-filtered colimits. That is, given a λ-filtered diagram D with a colimit $c_i : D_i \to C$ $(i \in I)$ in $\mathcal{A}$, then for every morphism $f : A \to C$

(i) a factorization of f through c_i exists for some $i \in I$,

and

(ii) factorizations are essentially unique, i.e., given $i \in I$ and $c_i \cdot g' = c_i \cdot g''$ for some $g', g'' : A \to D_i$, there exists a connecting morphism $d_{ij} : D_i \to D_j$ of the diagram with $d_{ij} \cdot g' = d_{ij} \cdot g''$.

3.2. Examples Sets, presheaves, varieties of algebras and simplicial sets are examples of locally presentable categories. Categories such as **Top** (topological spaces) or **Haus** (Hausdorff spaces) are not locally presentable.

3.3. Remark (a) In the present section we prove that the Injectivity Logic is complete in every locally presentable category.

(b) The reader may decide to skip this section since we prove a more general result in Section 6. Both of our proofs are based on the fact that for every set $\mathcal{H}$ of morphisms the full subcategory Inj$\mathcal{H}$ (of all objects injective w.r.t. morphisms of $\mathcal{H}$) is *weakly reflective*. That is: every object $A \in \mathcal{A}$ has a morphism $r : A \to \overline{A}$, called a weak reflection, such that

(i) $\overline{A}$ lies in Inj$\mathcal{H}$

and

(ii) every morphism from A to an object of Inj$\mathcal{H}$ factors through r (not necessarily uniquely).

Here we will utilize the classical *Small Object Argument* of D. Quillen [11]: this tells us that every object A has a weak reflection $r : A \to \overline{A}$ in Inj$\mathcal{H}$ such that r is a transfinite composite of morphisms of the class

$$\widehat{\mathcal{H}} = \{k;\ k \text{ is a pushout of a member of } \mathcal{H} \text{ along some morphism}\}.$$

(c) The reason for proving the completeness based on the Small Object Argument in the present section is that the proof is short and elegant. However, by using a more refined construction of weak reflection in Inj$\mathcal{H}$, which we present in Section 5, we will be able to prove the completeness in the so-called weakly locally presentable categories, which include **Top** and **Haus**. However the proof will be technically much more involved.

The spirits of the two proofs are quite different. Given a consequence h of a set of morphisms, in this section we will show how to derive a formal proof of h from Quillen's construction of the weak reflection; this construction is "linear", forming a transfinite composite. In the next section, a weak reflection will be constructed as a colimit of a filtered diagram which somehow presents simultaneously all the possible formal proofs.

3.4. Definition A morphism is called λ-*ary* provided that its domain and codomain are λ-presentable objects. For $\lambda = \aleph_0$ we say *finitary.*

3.5. Remark (a) The λ-ary morphisms are precisely the λ-presentable objects of the arrow category $\mathcal{A}^{\to}$. In contrast, M. Hébert introduced in [8] λ-presentable morphisms; these are the morphisms $f : A \to B$ which are λ-presentable objects of the slice category $A \downarrow \mathcal{A}$. In the present paper we will not use the latter concept.

(b) We work now with the *Finitary Injectivity Logic*, i.e., the deduction system 2.4 applied to finitary morphisms. We generalize this to the k-ary logic below.

3.6. Theorem *The Finitary Injectivity Logic is complete. That is, given a set $\mathcal{H}$ of finitary morphisms, then every finitary morphism h which is a consequence of $\mathcal{H}$ is a formal consequence in the deduction system 2.4. Shortly:*

$$\mathcal{H} \models h \text{ implies } \mathcal{H} \vdash h.$$

Proof Without loss of generality $\mathcal{H}$ can be assumed to be closed under binary composition and binary coproduct (see Example 2.8). It then easily follows that the closure $\widehat{\mathcal{H}}$ of $\mathcal{H}$ under pushout is also closed under binary composition.

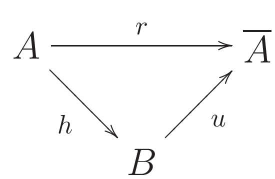
Given a finitary morphism $h : A \to B$ which is a consequence of $\mathcal{H}$, we prove that

$$\mathcal{H} \vdash h.$$

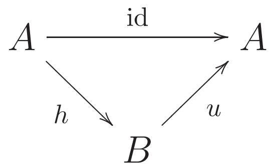
(a) The above object A has a weak reflection

$$r : A \to \overline{A}$$

such that r is a transfinite composition of morphisms in $\widehat{\mathcal{H}}$, see 3.3(b). Since $\mathcal{H} \vdash h$, it follows that $\overline{A}$ is injective w.r.t. h, which yields a morphism u forming a commutative triangle



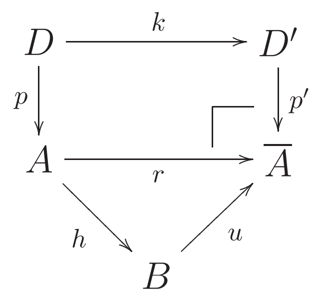
(b) Consider all commutative triangles as above where $r : A \to \overline{A}$ is any α-composite of morphisms in $\widehat{\mathcal{H}}$ for some ordinal α and u is arbitrary. We prove that the least possible α is 0 or 1. This finishes the proof of $\mathcal{H} \vdash h$: In case $\alpha = 0$, we have



and the formal proof of h is obvious:

$$\dfrac{\dfrac{}{u \cdot h}}{h} \quad \begin{array}{l}\text{via IDENTITY} \\ \text{via CANCELLATION}\end{array}$$

In case $\alpha = 1$, we have $r \in \widehat{\mathcal{H}}$ and thus there exists $k \in \mathcal{H}$ and a pushout

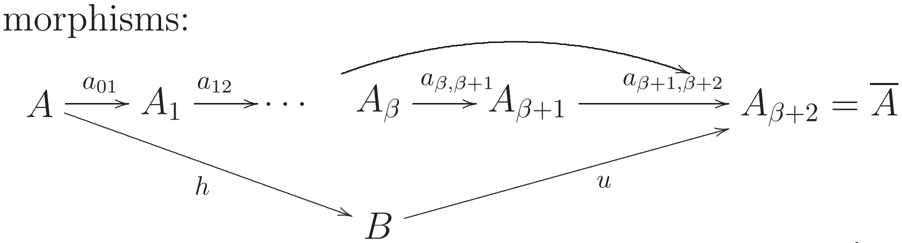


Then we have a proof of h as follows

$$\dfrac{\dfrac{k}{r}}{h} \quad \begin{array}{l}\text{via PUSHOUT} \\ \text{via CANCELLATION}\end{array}$$

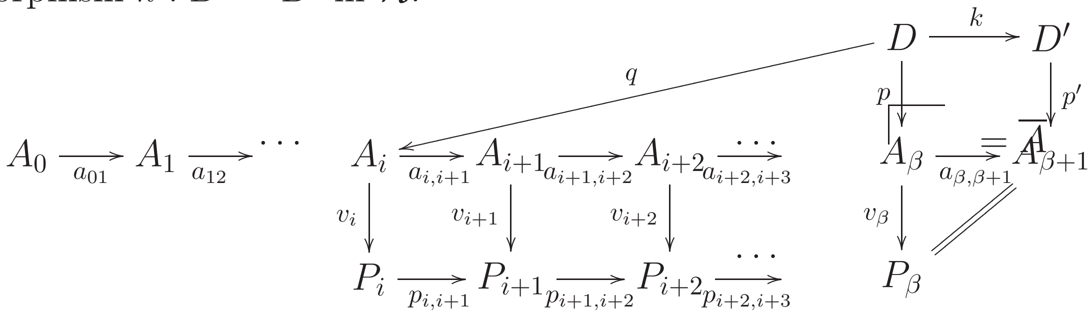
Assuming that the least possible α is larger than 1, we derive a contradiction:

A. Assume $\alpha = \beta + 2$ for some ordinal β. Then we can simply compose the two last morphisms:



We conclude that r is an $(\alpha - 1)$-composite of morphisms in $\widehat{\mathcal{H}}$, a contradiction.

B. Assume $\alpha = \beta + 1$ for some ordinal β. Due to A., β is a limit ordinal (since $\alpha \geq 2$). Since $a_{\beta,\beta+1}$ lies in $\widehat{\mathcal{H}}$, we can express it as a pushout of some morphism $k : D \to D'$ in $\mathcal{H}$:



We have a colimit $A_\beta = \operatorname{colim}_{i<\beta} A_i$ of a chain of morphisms. Hence, because D is finitely presentable, p factorizes as $p = a_{i\beta} \cdot q$ for some $i < \beta$ and some morphism $q : D \to A_i$. Let v_i be a pushout of k along q, and form a sequence v_j of pushouts of k along $a_{ij} \cdot q\,(j < \beta)$ as illustrated in the diagram above (taking colimits at the limit ordinals). Then it is easily seen, due to $p = a_{i\beta} \cdot q$, that $v_\beta = \operatorname{colim}_{j<\beta} v_j$ is a pushout of k along p. Thus, without loss of generality,

$$P_\beta = \overline{A} \text{ and } v_\beta = a_{\beta,\beta+1}.$$

Observe that, since $a_{j,j+1}$ lies in $\widehat{\mathcal{H}}$,

$$p_{j,j+1} \in \widehat{\mathcal{H}} \text{ for all } i \leq j < \beta.$$

Also $v_i \in \widehat{\mathcal{H}}$ since it is a pushout of k along q. Consequently, r is a composite of the β-chain of morphisms of $\widehat{\mathcal{H}}$ as follows:

$$\begin{array}{lll} & a_{j,j+1} & \text{for all } j < i, \\ & p_{i,i+1} \cdot v_i, & \\ \text{and} & & \\ & p_{j,j+1} & \text{for all } i < j < \beta \end{array}$$

This contradicts the minimality of α.

C. Assume α is a limit ordinal. The morphism

$$u : B \to \overline{A} = \operatorname*{colim}_{i<\alpha} A_i$$

factors, since B is finitely presentable, through some $a_{i\alpha},\ i < \alpha$:

$$u = a_{i\alpha} \cdot \overline{u} \text{ for some } \overline{u} : B \to A_i.$$

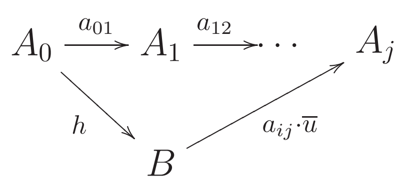
The parallel pair

$$A = A_0 \underset{a_{0i}}{\overset{\overline{u}\cdot h}{\rightrightarrows}} A_i$$

is clearly merged by the colimit morphism $a_{i\alpha}$ of $A_\alpha = \operatorname{colim}_{i<\alpha} A_i$. Since A is finitely presentable, $\hom(A, -)$ preserves that colimit, consequently (see (ii) in 3.1.b), the parallel pair is also merged by a connecting morphism $a_{ij} : A_i \to A_j$, for some $i < j < \alpha$:

$$a_{ij} \cdot \overline{u} \cdot h = a_{0j}.$$

This gives us a commutative triangle



in contradiction to the minimality of α.

3.7. Remark The above theorem immediatly generalizes to the k-ary Injectivity Logic, i.e., to the deduction system of 2.20 applied to k-ary morphisms. Recall that for every set of objects in a locally presentable category there exists a cardinal k such that all these objects are k-presentable. Consequently, for every set $\mathcal{H} \cup \{h\}$ of morphisms there exists k such that all members are k-ary. The proof that $\mathcal{H} \models h$ implies $\mathcal{H} \vdash h$ is completely analogously to 3.6: Cases A. and C. need no modification. Case B. is clear if the limit ordinal β is cofinal with some $\gamma < k$: by picking up a subchain of $(A_i)_{i<\beta}$ we derive that $a_{i,\beta} : A_0 \to A_\beta$ is a γ-composite of members of $\widehat{\mathcal{H}}$. It follows that r is a $(\gamma + 1)$-composite, and $\gamma + 1 < k$. In case β is not cofinal with any $\gamma < k$, every β-chain is a k-filtered category and the rest of the proof is as in 3.6.

3.8. Corollary *The Injectivity Logic is sound and complete in every locally presentable category.*

In fact, given

$$\mathcal{H} \models h$$

find a cardinal k such that all members of $\mathcal{H} \cup \{h\}$ are k-ary morphisms. Then h is a formal consequence of $\mathcal{H}$ by 3.7.

3.9. Remark The above corollary also follows from the Small Object Argument (see 3.3(b)): if $h : A \to B$ is a consequence of $\mathcal{H}$ and if $r : A \to \overline{A}$ is the corresponding weak reflection, then r is clearly a formal consequence of $\mathcal{H}$. Since $\overline{A}$ is injective w.r.t. h, it follows that r factors through h, thus, h is a formal consequence of r (via CANCELLATION).

4. Weakly Locally Presentable Categories

4.1. Remark Recall that a *factorization system* in a category is a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms containing all isomorphisms and closed under composition such that

(a) every morphism $f : A \to B$ has a factorization $f = m \cdot e$ with $e : A \to C$ in $\mathcal{E}$ and $m : C \to B$ in $\mathcal{M}$

and

(b) given another such factorization $f = m' \cdot e'$ there exists a unique "diagonal fill-in" morphism d making the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & C \\ {\scriptstyle e'}\downarrow & \swarrow{\scriptstyle d} & \downarrow{\scriptstyle m} \\ C' & \xrightarrow[m']{} & B \end{array}$$

commutative.

The factorization system is called *left-proper* if every morphism of $\mathcal{E}$ is an epimorphism. In that case the $\mathcal{E}$-quotients of an object A are the quotient objects of A represented by morphisms of $\mathcal{E}$ with domain A.

4.2. Definition Let $(\mathcal{E}, \mathcal{M})$ be a factorization system. We say that an object A has $\mathcal{M}$-*rank* λ, where λ is a regular cardinal, provided, that

(a) $\hom(A, -)$ preserves λ-filtered diagrams of $\mathcal{M}$-morphisms (i.e., given a λ-filtered diagram D whose connecting morphisms lie in $\mathcal{M}$, then every morphism $f : A \to \mathrm{colim} D$ factors, essentially uniquely, through a colimit map of D)

and

(b) A has less than λ $\mathcal{E}$-quotients.

If $\lambda = \aleph_0$ we say that the object A has *finite* $\mathcal{M}$-*rank*.

4.3. Examples (1) For the factorization system (Iso, All), rank λ is equivalent to λ-presentability.

(2) In the category **Top** of topological spaces, choose $(\mathcal{E}, \mathcal{M}) =$ (Epi, Strong Mono). Here the $\mathcal{M}$-subobjects are precisely the embeddings of subspaces. Every topological space A of cardinality α has $\mathcal{M}$-rank λ whenever $\lambda > 2^{2^\alpha}$. In fact, $\hom(A, -)$ preserves λ-directed unions of subspaces since $\alpha < \lambda$. And the amount of quotient objects of A (carried by epimorphisms) is at most $\sum_{\beta \leq \alpha} E_\beta T_\beta$ where E_β is the number of equivalence relations on A of order β and T_β is the number of topologies on a set of cardinality β. Since E_β and T_β are both $\leq 2^{2^\beta}$, we have $\sum_{\beta\leq\alpha} E_\beta T_\beta \leq \alpha \cdot 2^{2^\alpha} \cdot 2^{2^\alpha} < \lambda$, thus we conclude that A has less than λ quotients.

4.4. Remark Every $\mathcal{E}$-quotient of an object of $\mathcal{M}$-rank λ also has $\mathcal{M}$-rank λ. In fact (a) in 4.2 follows easily by diagonal fill-in, and (b) is obvious.

4.5. Definition A category $\mathcal{A}$ is called *weakly locally presentable* provided that it has a left-proper factorization system $(\mathcal{E}, \mathcal{M})$ such that

(i) $\mathcal{A}$ is cocomplete;
(ii) every object has an $\mathcal{M}$-rank, and all objects of the same $\mathcal{M}$-rank form a set up to isomorphism;
(iii) for every cardinal μ the collection of all objects of $\mathcal{M}$-rank μ is closed under $\mathcal{E}$-quotients and under μ-small colimits, i.e., colimits of diagrams with less than μ morphisms;

and

(iv) the subcategory of all objects of $\mathcal{A}$ and all morphisms of $\mathcal{M}$ is closed under filtered colimits in $\mathcal{A}$.

Remark The statement (iv) means that, given a filtered colimit with connecting morphisms in $\mathcal{M}$, then

(a) the colimit cocone is formed by morphisms of $\mathcal{M}$

and

(b) every other cocone of $\mathcal{M}$-morphisms has the unique factorizing morphism in $\mathcal{M}$.

4.6. Examples (1) Every locally presentable category is weakly locally presentable: choose

$$\mathcal{E} \equiv \text{isomorphisms}, \ \mathcal{M} \equiv \text{all morphisms}.$$

In fact, see [3], 1.9 for the proof of (ii), whereas (iii) and (iv) hold trivially.

(2) Choose

$$\mathcal{E} \equiv \text{epimorphisms}, \ \mathcal{M} \equiv \text{strong monomorphisms}.$$

Here categories such as **Top** (which are not locally presentable) are included. In fact, for a space A of cardinality α we have that $\hom(A, -)$ preserves λ-filtered colimits (=unions) of subspaces whenever $\alpha < \lambda$. Thus, by choosing a cardinal $\lambda > \alpha$ bigger than the number of quotients of A we get an $\mathcal{M}$-rank of A. It is easy to verify (iii) and (iv) in **Top**.

(3) Let $\mathcal{B}$ be a full, isomorphism closed, $\mathcal{E}$-reflective subcategory of a weakly locally presentable category $\mathcal{A}$. If $\mathcal{B}$ is closed under filtered colimits of $\mathcal{M}$-morphisms in $\mathcal{A}$, then $\mathcal{B}$ is weakly locally presentable. In fact, $\mathcal{B}$ is closed under $\mathcal{M}$ in the sense that given $m : A \to B$ in $\mathcal{M}$ with $B \in \mathcal{B}$, then $A \in \mathcal{B}$. (Indeed, we have a reflection $r_A : A \to A'$ in $\mathcal{E}$ and $m = m' \cdot r_A$ for a unique m'; this implies that $r_A \in \mathcal{E}$ is an isomorphism, thus, $A \in \mathcal{B}$.) Therefore the restriction of $(\mathcal{E}, \mathcal{M})$ to $\mathcal{B}$ yields a factorization system. It fulfils (ii)-(iv) of 4.5 because $\mathcal{B}$ is closed under filtered colimits of $\mathcal{M}$-morphisms.

(4) The category **Haus** of Hausdorff spaces is weakly locally presentable: it is an epireflective subcategory of **Top** closed under filtered unions of subspaces.

4.7. Observation In a weakly locally presentable category the class $\mathcal{M}$ is closed under transfinite composition. This follows from (iv).

4.8. Definition A morphism is called *k-ary* if its domain and codomain have $\mathcal{M}$-rank k. In case $k = \aleph_0$ we speak of *finitary morphisms.*

5. A construction of weak reflections

5.1. Assumption In the present section $\mathcal{A}$ denotes a weakly locally presentable category, and, for every infinite cardinal k, $\mathcal{A}_k$ a set of objects of $\mathcal{M}$-rank k closed under $\mathcal{E}$-quotients and k-small colimits. In particular, one may of course choose $\mathcal{A}_k$ to be a set of representatives of all the objects of $\mathcal{M}$-rank k up to isomorphism.

Given a set $\mathcal{H} \subseteq \mathcal{M}$ of k-ary morphisms of $\mathcal{A}_k$ (considered as a full subcategory of $\mathcal{A}$), [1] provides a construction of a weak reflection in $\operatorname{Inj} \mathcal{H}$, which generalizes the Small Object Argument (see 3.3). However, this does not appear to be sufficient to prove our Completeness Theorem for the finitary case. The aim of this section is to present a different, more appropriate construction.

We begin with the case $k = \omega$ and come back to the general case at the end of this section.

5.2. Convention (a) Morphisms with domain and codomain in $\mathcal{A}_\omega$ are called *petty.*

(b) Given a set $\mathcal{H}$ of petty morphisms,

$$\overline{\mathcal{H}}$$

denotes the closure of $\mathcal{H}$ under finite composition and pushout in $\mathcal{A}_\omega$. (That is, $\overline{\mathcal{H}}$ is the closure of $\mathcal{H} \cup \{\mathrm{id}_A;\ A \in \mathcal{A}_\omega\}$ under binary composition and pushout along petty morphisms.)

(c) Since $\overline{\mathcal{H}} \subseteq \mathrm{mor}\mathcal{A}_\omega$ is a set, we can, for every object B of $\mathcal{A}_\omega$, index all morphisms of $\overline{\mathcal{H}}$ with domain B by a set – and that indexing set can be chosen to be independent of B. That is, we assume that a set T is given and that for every object $B \in \mathcal{A}_\omega$,

$$\{h_B(t) : B \to B(t)\ ;\ t \in T\} \tag{5.1}$$

is the set of all morphisms of $\overline{\mathcal{H}}$ with domain B.

5.3. Diagram $\mathbf{D_A}$ For every object $A \in \mathcal{A}_\omega$ we define a diagram D_A in $\mathcal{A}$ and later prove that a weak reflection of A in $\mathrm{Inj}\,\mathcal{H}$ is obtained as a colimit of D_A. The domain $\mathcal{D}$ of D_A, independent of A, is the poset of all finite words

$$\varepsilon,\ M_1,\ M_1M_2,\ \dots,\ M_1 \dots M_k\ \ (k < \omega)$$

where ε denotes the empty word and each M_i is a finite subset of T. The ordering is as follows:

$$M_1 \dots M_k \le N_1 \dots N_l \quad \text{iff} \quad k \le l \quad \text{and} \quad M_1 \subseteq N_1,\ \dots,\ M_k \subseteq N_k.$$

Observe that ε is the least element.

We denote the objects $D_A(M_1 \dots M_k)$ of the diagram D_A by

$$A_M \text{ where } M = M_1 \dots M_k,$$

and if $M_1 \dots M_k \le N_1 \dots N_l = N$, we denote by

$$a_{M,N} : A_M \to A_N$$

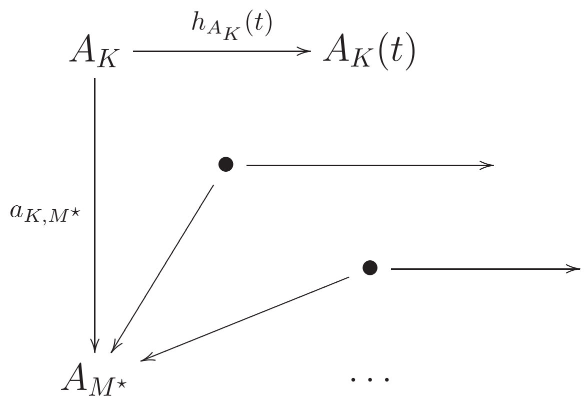
the corresponding connecting morphism of D_A. We define these objects and connecting morphisms by induction on the length k of the word $M = M_1 \dots M_k$ considered.

Case $k = 0$: $A_\varepsilon = A$.

Induction step: Assume that all objects A_M with M of length less than or equal to k and all connecting morphisms between them are defined. For every word M of length $k+1$ denote by

$$M^\star \leq M$$

the prefix of M of length k, and define the object A_M as a colimit of the following finite diagram



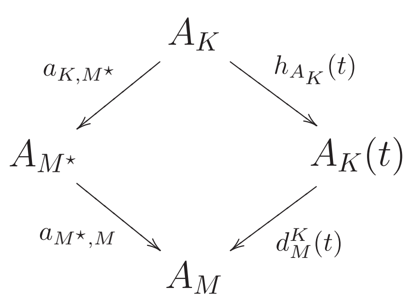
where K ranges over all words $K \in \mathcal{D}$ with $K \leq M^\star$ and t ranges over the set M_{k+1}. Thus, A_M is equipped with (universal) morphisms

$$a_{M^\star,M} : A_{M^\star} \to A_M \quad \text{(connecting morphism of } D_A)$$

and

$$d_M^K(t) : A_K(t) \to A_M \quad \text{for all } K \leq M^\star,\ t \in M_{k+1},$$

forming commutative squares

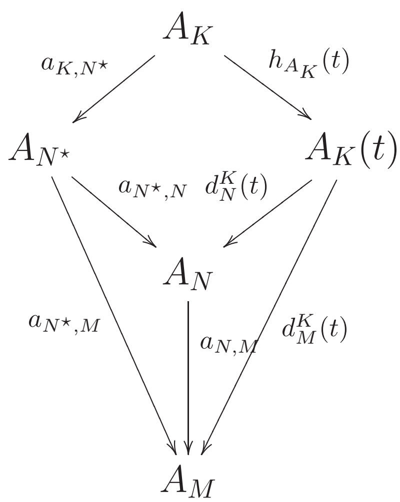


(5.2)

This defines the objects A_M for all words of length $k+1$. Next we define connecting morphisms

$$a_{N,M} : A_N \to A_M$$

for all words $N \leq M$. If the length of N is at most k, then $N \leq M^\star$ and we define $a_{N,M}$ through the (already defined) connecting morphism $a_{N,M^\star}$ by composing it with the above $a_{M^\star,M}$. If N has length $k+1$, we define $a_{N,M}$ as the unique morphism for which the diagrams



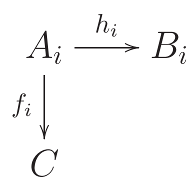
$(K \leq N^\star,\ t \in N_{k+1})$ (5.3)

commute.

It is easy to verify that the morphisms $a_{N,M}$ are well-defined and that $D_A : \mathcal{D} \to \mathcal{A}$ preserves composition and identity morphisms.

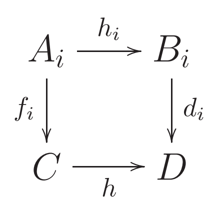
5.4. Lemma *All connecting morphisms of the diagram D_A lie in $\overline{\mathcal{H}}$.*

Proof We first observe that, given a finite diagram



$(i \in I)$

with all h_i in $\overline{\mathcal{H}}$, a colimit



$(i \in I)$ (5.4)

is obtained by first considering pushouts h_i' of h_i along f_i and then forming a wide pushout h of all $h_i'\,(i \in I)$. Consequently, the connecting morphisms of D_A are formed by repeating one of the following steps: a finite wide pushout of morphisms in $\overline{\mathcal{H}}$, a composition of morphisms in $\overline{\mathcal{H}}$, and a pushout of a morphism in $\overline{\mathcal{H}}$ along a petty morphism. Since $\overline{\mathcal{H}}$ is closed, by 5.2, under the latter, then it is closed under the first one in the obvious sense, see the construction of a finite wide pushout described in Example 2.9.

5.5. Lemma *For every object A_M of the diagram D_A and every morphism $h : A_M \to B$ of $\overline{\mathcal{H}}$ there exists a connecting morphism $a_{M,N} : A_M \to A_N$ of D_A which factors through h.*

Proof We have $M = M_1 \dots M_k$ and $h = h_{A_M}(t)$ for some $t \in T$. Put

$$N = M_1 \dots M_k\{t\}.$$

Then for $K = M$ the definition of $d_N^K(t)$ (see (5.2)) gives the following commutative diagram:

$$\begin{array}{ccc} A_M & \xrightarrow{h_{A_M}(t)} & A_M(t) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle d_N^K(t)} \\ A_M & \xrightarrow[a_{M,N}]{} & A_N \end{array}$$

Consequently,

$$a_{M,N} = d_N^K(t) \cdot h_{A_M}(t)$$

as required.

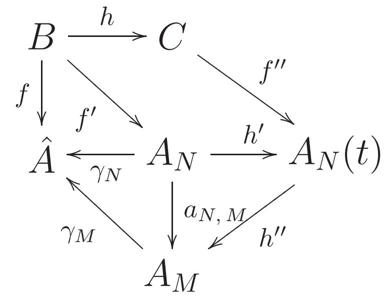
5.6. Proposition *Let $\mathcal{H}$ be a set of petty morphisms with $\overline{\mathcal{H}} \subseteq \mathcal{M}$. Then for every object $A \in \mathcal{A}_\omega$ a colimit $\gamma_M : A_M \to \hat{A}$ $(M \in \mathcal{D})$ of the diagram D_A yields a weak reflection of A in $Inj\mathcal{H}$ via*

$$r_A = \gamma_\varepsilon : A \to \hat{A}.$$

Proof (1) $\hat{A}$ is injective w.r.t. $\mathcal{H}$: We want to prove that given $h \in \mathcal{H}$ and f as follows

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \\ \hat{A} & & \end{array}$$

then f factors through h. Firstly, since $\hat{A} = \mathrm{colim} D_A$ is a directed colimit of $\overline{\mathcal{H}}$-morphisms (see 5.4) with $\overline{\mathcal{H}} \subseteq \mathcal{M}$, and B has finite $\mathcal{M}$-rank (because $B \in \mathcal{A}_\omega$), it follows that $\hom(B, -)$ preserves the colimit of D_A. Thus, there exists a colimit morphism $\gamma_N : A_N \to \hat{A}$ through which f factors, $f = \gamma_N \cdot f'$.



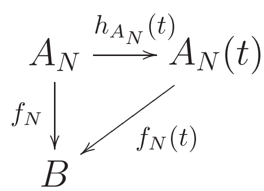
By pushing $h \in \mathcal{H}$ out along f' we obtain a morphism $h' \in \overline{\mathcal{H}}$. Then by 5.5 there exists $M \geq N$ such that $a_{N,M} = h'' \cdot h'$ for some $h'' : A_N(t) \to A_M$. The above commutative diagram proves that f factors through h.

(2) Let B be injective w.r.t. $\mathcal{H}$. For every morphism $f : A \to B$ we define a compatible cocone $f_M : A_M \to B$ of the diagram D_A by induction on

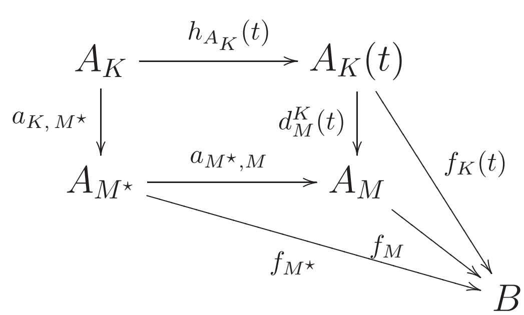
$$k = \text{the length of the word } M$$

such that $f_\varepsilon = f$. Then the desired factorization of f is obtained via the (unique) factorization $g : \hat{A} \to B$ with $g \cdot \gamma_M = f_M$: in fact, $g \cdot r_A = f$.

For $k \mapsto k+1$, choose for every word N of length k and every $t \in T$ a morphism $f_N(t)$ forming a commutative triangle



(recalling that B is $\overline{\mathcal{H}}$-injective because it is $\mathcal{H}$-injective). Then for every word M of length $k+1$ we have a unique factorization $f_M : A_M \to B$ making the following diagrams



(5.5)

commutative for all $K \leq M^\star$ and $t \in M_{k+1}$.

Let us verify the compatibility

$$f_M = f_N \cdot a_{M,N} \qquad \text{for all } M \leq N \text{ in } \mathcal{D}. \tag{5.6}$$

The last diagram yields $f_{M^\star} = f_M \cdot a_{M^\star, M}$. Therefore, it is sufficient to prove (5.6) for words M and N of the same length $k+1$. In order to do that, we will show that

$$f_M \cdot d_M^K(t) = f_N \cdot a_{M,N} \cdot d_M^K(t), \text{ for all } K \leq M^\star \text{ and } t \in M_{k+1}, \tag{5.7}$$

and

$$f_M \cdot a_{M^\star,M} = f_N \cdot a_{M,N} \cdot a_{M^\star,M}. \tag{5.8}$$

Concerning (5.7), we have

$$\begin{aligned} f_M \cdot d_M^K(t) &= f_K(t) \\ &= f_N \cdot d_N^K(t), \text{ by replacing } M \text{ by } N \text{ in (5.5)} \\ &= f_N \cdot a_{M,N} \cdot d_M^K(t), \text{ by (5.3).} \end{aligned}$$

As for (5.8), we have

$$\begin{aligned} f_M \cdot a_{M^\star,M} &= f_{M^\star} \\ &= f_{N^\star} \cdot a_{M^\star,N^\star} \\ &= f_N \cdot a_{N^\star,N} \cdot a_{M^\star,N^\star}, \text{ by replacing } M \text{ by } N \text{ in (5.5)} \\ &= f_N \cdot a_{M,N} \cdot a_{M^\star,M}. \end{aligned}$$

5.7. Convention Generalizing the above construction from ω to any infinite cardinal k, we call the morphisms of $\mathcal{A}_k$ *k-petty*. Let us now denote by

$$\overline{\mathcal{H}}_k$$

the closure of $\mathcal{H}$ under k-composition (2.11) and pushout in A_k. Following 2.20, $\overline{\mathcal{H}}_k$ is closed under k-wide pushout. We again assume that a set T is given such that, for every object $B \in \mathcal{A}_k$ we have an indexing $h_B(t) : B \to B(t)$, $t \in T$ of all morphisms of $\overline{\mathcal{H}}_k$ with domain B.

5.8. Diagram $\mathbf{D_A}$ The poset $\mathcal{D}$ of 5.3 is generalized to a poset $\mathcal{D}_k$: Let $\mathcal{P}_kT$ be the poset of all subsets of T of power $< k$. The elements of $\mathcal{D}_k$ are all functions

$$M : \lambda \to \mathcal{P}_kT$$

where $\lambda < k$ is an ordinal, including the case $\varepsilon : 0 \to \mathcal{P}_kT$. The ordering is as follows: for $N : \lambda' \to \mathcal{P}_kT$ put

$$M \leq N \quad \text{iff} \quad \lambda \leq \lambda' \quad \text{and} \quad M_i \subseteq N_i \text{ for all } i < \lambda.$$

We define, for every $A \in \mathcal{A}_k$, the diagram $D_A : \mathcal{D}_k \to \mathcal{A}$. The objects $D_A(M) = A_M$ and the connecting morphisms $a_{M,N} : A_M \to A_N$ $(M \leq N)$ are defined by transfinite induction on $\lambda < k$. For $\lambda = 0$ we have $A_\varepsilon = A$. The isolated step is precisely as in 5.3, where for $M : \lambda + 1 \to \mathcal{P}_kT$ we denote by $M^\star : \lambda \to \mathcal{P}_kT$ the domain-restriction. The limit steps are defined via colimits of smooth chains, see 2.10: if $\lambda < k$ is a limit ordinal and $M : \lambda \to \mathcal{P}_kT$ is given, then A_M is a colimit of the chain $A_{M/i}$ $(i < \lambda)$, where M/i is the domain restriction of M to i, with the connecting morphisms $a_{M/i, M/j} : A_{M/i} \to A_{M/j}$ for all $i \leq j < \lambda$. The proof that these chains are smooth is an easy transfinite induction.

It is also easy to see that all the above results hold: $\hat{A} = \mathrm{colim} D_A$ is an $\mathcal{H}$-injective weak reflection of A, and all connecting morphisms of D_A are members of $\overline{\mathcal{H}}$. Consequently, the proof of the following proposition is analogous to that of 5.6:

5.9. Proposition *Let $\mathcal{H}$ be a set of k-petty morphisms with $\overline{\mathcal{H}_k} \subseteq \mathcal{M}$. Then for every object $A \in \mathcal{A}_k$ a colimit $\gamma_M : A_M \to \hat{A}$ of D_A yields a weak reflection of A in Inj$\mathcal{H}$ via $r_A = \gamma_\varepsilon : A \to \hat{A}$.*

6. Completeness in Weakly Locally Presentable Categories

6.1. Assumption Throughout this section $\mathcal{A}$ denotes a weakly locally presentable category. We first prove the completeness of the finitary logic. Recall that the finitary morphisms are those where the domain and codomain are of finite $\mathcal{M}$-rank. Let us remark that whenever the class $\mathcal{M}$ is closed under pushout, then the method of proof of Theorem 3.6 applies again. However, this excludes examples such as **Haus** (where strong monomorphisms are not closed under pushout).

6.2. Theorem *The Finitary Injectivity Logic is complete. That is, given a set $\mathcal{H}$ of finitary morphisms, then every finitary morphism h which is a consequence of $\mathcal{H}$ is a formal consequence (in the deduction system of 2.4). Shortly: $\mathcal{H} \models h$ implies $\mathcal{H} \vdash h$.*

6.3. Remark We do not need the full strength of weak local presentation for this result. We are going to prove the completeness under the following milder assumptions on $\mathcal{A}$:

(i) $\mathcal{A}$ is cocomplete and has a left-proper factorization system $(\mathcal{E}, \mathcal{M})$;
(ii) $\mathcal{A}_\omega$ is a set of objects of finite $\mathcal{M}$-rank, closed under finite colimits and $\mathcal{E}$-quotients;
(iiii) $\mathcal{M}$ is closed under filtered colimits in $\mathcal{A}$ (see 4.5 (iv)).

The statement we prove is, then, concerned with petty morphisms (see 5.2). We show that for every set $\mathcal{H}$ of petty morphisms we have

$$\mathcal{H} \models h \text{ implies } \mathcal{H} \vdash h \text{ (for all } h \text{ petty).}$$

The choice of $\mathcal{A}_\omega$ as a set of representatives of all objects of finite $\mathcal{M}$-rank yields the statement of the theorem.

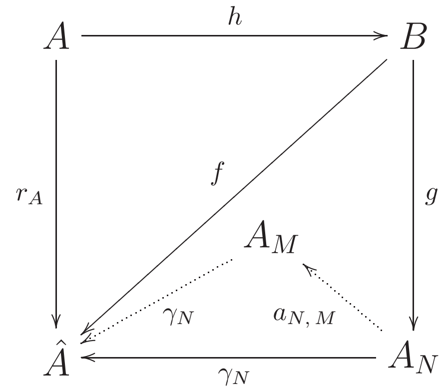
Proof of 6.2 and 6.3 Let then $\mathcal{H}$ be a set of petty morphisms, and let

$$\overline{\mathcal{H}}$$

denote the closure of $\mathcal{H}$ as in 5.2.

(1) We first prove that the theorem holds whenever $\overline{\mathcal{H}} \subseteq \mathcal{M}$. Moreover, for every petty consequence $\mathcal{H} \models h$ we have a formal proof of h from assumptions in $\mathcal{H}$ such that the use of PUSHOUT is always restricted to pushing out along petty morphisms.

To prove this, consider, for the given petty consequence $h : A \to B$ of $\mathcal{H}$, the weak reflection $r_A : A \to \hat{A}$ in $\operatorname{Inj} \mathcal{H}$ of 5.6. The object $\hat{A}$ is injective w.r.t. h, thus r_A factors through h via some $f : B \to \hat{A}$:



Since $B \in \mathcal{A}_\omega$, it has finite $\mathcal{M}$-rank, and 5.4 implies that $\hom(B, -)$ preserves the colimit $\hat{A} = \operatorname{colim} D_A$. Then f factors through one of the colimit morphisms $\gamma_N : A_N \to \hat{A}$:

$$f = \gamma_N \cdot g \quad \text{for some } g : B \to A_N.$$

We know that $r_A = \gamma_\varepsilon$ is the composite of the connecting morphism $a_{\varepsilon, N} : A \to A_N$ of D_A and γ_N, therefore,

$$\gamma_N \cdot a_{\varepsilon, N} = r_A = \gamma_N \cdot g \cdot h.$$

This implies that the colimit morphism γ_N merges the parallel pair $a_{\varepsilon, N}$, $g \cdot h : A \to A_N$. Now the domain A has finite $\mathcal{M}$-rank, thus $\hom(A, -)$ also preserves $\hat{A} = \operatorname{colim} D_A$. Consequently, by (ii) in 3.1(b) the parallel pair is also merged by some connecting morphism $a_{N,M} : A_N \to A_M$ of D_A:

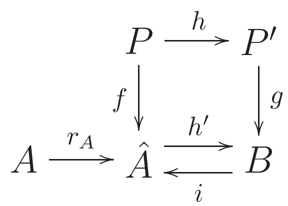
$$a_{N,M} \cdot a_{\varepsilon, N} = a_{N,M} \cdot g \cdot h : A \to A_M.$$

The left-hand side is simply $a_{\varepsilon, M}$, and this is a morphism of $\overline{\mathcal{H}}$, see Lemma 5.4. Recall that the definition of $\overline{\mathcal{H}}$ implies that every morphism in $\overline{\mathcal{H}}$ can be proved from $\mathcal{H}$ using Finitary Injectivity Logic in which PUSHOUT is only applied to pushing out along petty morphisms. Thus, we have a proof of the right-hand side $a_{N,M} \cdot g \cdot h$. The last step is deriving h from this by CANCELLATION.

(2) Assuming $\mathcal{H} \subseteq \mathcal{E}$, then $\operatorname{Inj} \mathcal{H}$ is a reflective subcategory of $\mathcal{A}$, and for every object $A \in \mathcal{A}_\omega$ the reflection map $r_A : A \to \hat{A}$ is a formal consequence of $\mathcal{H}$ lying in $\mathcal{E}$:

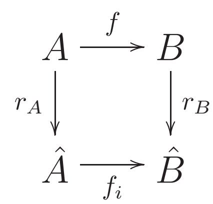
$$\mathcal{H} \vdash r_A \text{ and } r_A \in \mathcal{E}.$$

In fact, from $\mathcal{H} \subseteq \mathcal{E}$ it follows that $\overline{\mathcal{H}} \subseteq \mathcal{E}$ (since $\mathcal{E}$ is closed under composition and pushout). Since A has only finitely many $\mathcal{E}$-quotients, see 4.2, we can form a finite wide pushout, $r_A : A \to \hat{A}$, of all $\mathcal{E}$-quotients of A lying in $\overline{\mathcal{H}}$. Clearly, $\mathcal{H} \vdash r_A$, in fact, $r_A \in \overline{\mathcal{H}}$.
The object $\hat{A}$ is injective w.r.t. $\mathcal{H}$: given $h : P \to P'$ in $\mathcal{H}$ and $f : P \to \hat{A}$, form a pushout h' of h along f. This is an $\mathcal{E}$-quotient in $\overline{\mathcal{H}}$, then the same is true for $h' \cdot r_A$. Consequently, r_A factors through $h' \cdot r_A$, and the factorization, $i : B \to \hat{A}$, is an epimorphism split by h', thus, $f = i \cdot g \cdot h$:



The morphism r_A is a weak reflection: given a morphism u from A to an object C of $\mathrm{Inj}\,\mathcal{H}$, then u factors through r_A because C is injective w.r.t. $\overline{\mathcal{H}}$ and $r_A \in \overline{\mathcal{H}}$.

(3) Let $\mathcal{H}$ be arbitrary. We begin our proof by defining an increasing sequence of sets $\mathcal{E}_i \subseteq \mathcal{E}$ of petty morphisms ($i \in Ord$). For every member $f : A \to B$ of $\overline{\mathcal{H}}$ we denote by f_i a reflection of f in $\mathrm{Inj}\,\mathcal{E}_i$:



First step: $\mathcal{E}_0 = \{\mathrm{id}_A;\ A \in \mathcal{A}_\omega\}$. Here $\mathrm{Inj}\,\mathcal{E}_0 = \mathcal{A}$, thus $f_0 = f$.

Isolated step: For each $f \in \overline{\mathcal{H}}$, let $f_i = f_i'' \cdot f_i'$ be the $(\mathcal{E}, \mathcal{M})$-factorization of the reflection f_i of f in $\mathrm{Inj}\,\mathcal{E}_i$, and put

$$\mathcal{E}_{i+1} = \mathcal{E}_i \cup \{f_i';\ f \in \overline{\mathcal{H}}\}.$$

Limit step: $\mathcal{E}_j = \cup_{i<j} \mathcal{E}_i$ for limit ordinals j.

We prove that for every ordinal i we have

$$\mathcal{H} \vdash f_i' \quad \text{for every } f \in \overline{\mathcal{H}} \tag{6.1}$$

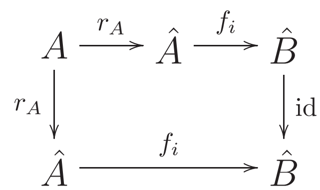
and

$$\mathrm{Inj}\,\mathcal{H} = \mathrm{Inj}\,\mathcal{E}_i \cap \mathrm{Inj}\{f_i\}_{f \in \overline{\mathcal{H}}}. \tag{6.2}$$

For $i = 0$, (6.1) and (6.2) are trivial (use CANCELLATION for (6.1) and IDENTITY for (6.2)). Given $i > 0$, assuming that $\mathcal{H} \vdash f'_j$ for all $j < i$, with $f : A \to B$ in $\overline{\mathcal{H}}$, that is, $\mathcal{H} \vdash \mathcal{E}_i$, we have, by (2), that

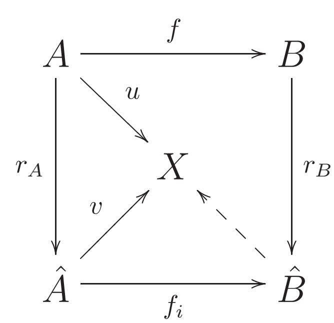
$$\mathcal{H} \vdash r_B \tag{6.3}$$

where r_B is the reflection of B in $\operatorname{Inj}\mathcal{E}_i$. Thus, $\mathcal{H} \vdash f_i \cdot r_A$. Moreover, r_A is an epimorphism, therefore the following square



is a pushout, which proves $\mathcal{H} \vdash f_i$ (via PUSHOUT). $\mathcal{H} \vdash f'_i$ then follows by CANCELLATION.

To prove (6.2), observe that (6.1) implies $\operatorname{Inj}\mathcal{H} \subseteq \operatorname{Inj}\mathcal{E}_i$, and our previous argument yields $\operatorname{Inj}\mathcal{H} \subseteq \operatorname{Inj}\{f_i\}_{f\in\overline{\mathcal{H}}}$. Thus, it remains to prove the reverse inclusion: every object X injective w.r.t. $\mathcal{E}_i \cup \{f_i\}_{f\in\overline{\mathcal{H}}}$ is injective w.r.t. $\mathcal{H}$. In fact, given $f : A \to B$ in $\mathcal{H}$ and a morphism $u : A \to X$, then since $X \in \operatorname{Inj}\mathcal{E}_i$ we have a factorization $u = v \cdot r_A$, and then the injectivity of X w.r.t. f_i yields the desired factorization of u through f.



(4) Since $\mathcal{A}_\omega$ is a small category, there exists an ordinal j with

$$\mathcal{E}_j = \mathcal{E}_{j+1}.$$

We want to apply (1) to the category

$$\mathcal{A}' = \operatorname{Inj}\mathcal{E}_j,$$

and the set

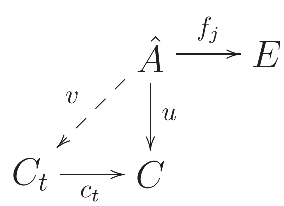
$$\mathcal{A}'_\omega = \mathcal{A}_\omega \cap \operatorname{obj}\mathcal{A}'.$$

Let us verify that $\mathcal{A}'$ satisfies the assumptions (i) – (iii) of Remark 6.3 w.r.t.

$$\mathcal{E}' = \mathcal{E} \cap \mathrm{mor}\mathcal{A}' \quad \text{and} \quad \mathcal{M}' = \mathcal{M} \cap \mathrm{mor}\mathcal{A}'.$$

Ad(i): $\mathcal{A}'$ is cocomplete because it is reflective in $\mathcal{A}$. Moreover, since the reflection maps lie in $\mathcal{E}$, it follows that $(\mathcal{E}', \mathcal{M}')$ is a factorization system: in fact, $\mathcal{A}'$ is closed under factorization in $\mathcal{A}$. Since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{A})$, we have $\mathcal{E}' \subseteq \mathrm{Epi}(\mathcal{A}')$.

Ad(iii): It is sufficient to prove that $\mathcal{A}'$ is closed under filtered colimits of $\mathcal{M}'$-morphisms in $\mathcal{A}$. In fact, let D be a filtered diagram in $\mathcal{A}'$ with connecting morphisms in $\mathcal{M}$, and let $c_t : C_t \to C$ $(t \in T)$ be a colimit of D in $\mathcal{A}$. Then $C \in \mathcal{A}'$, i.e., C is injective w.r.t. $f_j : \hat{A} \to E$ for every $f \in \overline{\mathcal{H}}$. This follows from $\hat{A}$ having finite $\mathcal{M}$-rank (because $A \in \mathcal{A}_\omega$ implies $\hat{A} \in \mathcal{A}_\omega$ due to the fact that $r_A : A \to \hat{A}$ is an $\mathcal{E}$-quotient): since $\hom(\hat{A}, -)$ preserves the colimit of D, every morphism $u : \hat{A} \to C$ factors through some of the colimit morphisms:



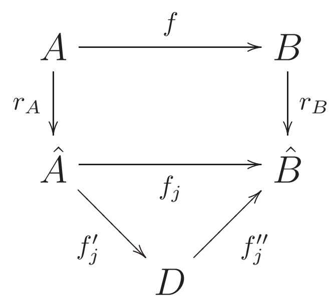
Since $C_t \in \mathcal{A}'$ is injective w.r.t. f_j, we have a factorization of v through f_j, and therefore, u also factors through f_j. This proves $C \in \mathcal{A}'$.

Ad(ii): Due to the above, every object of $\mathcal{A}'$ having a finite $\mathcal{M}$-rank in $\mathcal{A}$ has a finite $\mathcal{M}'$-rank in $\mathcal{A}'$. Also, a finite colimit of objects of $\mathcal{A}'$ in $\mathcal{A}'$ is a reflection (thus, an $\mathcal{E}$-quotient) of the corresponding finite colimit in $\mathcal{A}$. Thus, it lies in $\mathcal{A}'_\omega$.

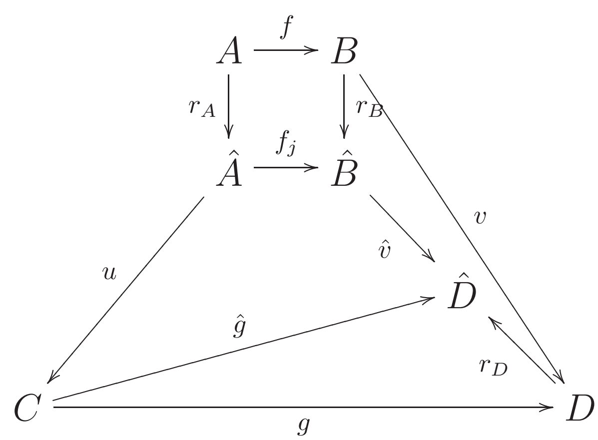
Next we claim that the set $\mathcal{H}' = \{f_j;\ f \in \overline{\mathcal{H}}\}$ fulfils

$$\mathcal{H}' \subseteq \mathcal{M}'$$

and $\mathcal{H}'$ is closed under petty identities, composition, and pushouts along petty morphisms. In fact, in the above $(\mathcal{E}, \mathcal{M})$-factorization of f_j:



we know that f'_j lies in $\mathcal{E}_{j+1} = \mathcal{E}_j$ and $\hat{A}$ is injective w.r.t. $\mathcal{E}_j$, thus, f'_j is a split monomorphism (as well as an epimorphism, since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{A})$). Thus, f'_j is an isomorphism, which implies $f_j \in \mathcal{M}$. $\mathcal{H}'$ contains id_A for every $A \in \mathcal{A}'_\omega$ because $\overline{\mathcal{H}}$ contains it; $\mathcal{H}'$ is closed under composition because $\overline{\mathcal{H}}$ is (and $f \mapsto f_j$ is the action of the reflector functor from $\mathcal{A}$ to $\mathrm{Inj}\,\mathcal{E}_j$). Finally, $\mathcal{H}'$ is closed under pushout along petty morphisms. In fact, to form a pushout of $f_j : \hat{A} \to \hat{B}$ along $u : \hat{A} \to C$ in $\mathcal{A}' = \mathrm{Inj}\,\mathcal{E}_j$, we form a pushout, g, of f along $u \cdot r_A$ in $\mathcal{A}$, and compose it with the reflection map r_D of the codomain D:



Since C lies in $\mathcal{A}'$, we can assume $r_C = \mathrm{id}_C$, and the reflection $\hat{g} = r_D \cdot g$ of g in $\mathcal{A}'$ is then a pushout of f_j along u. Now $f \in \overline{\mathcal{H}}$ implies $g \in \overline{\mathcal{H}}$, and we have $\hat{g} = g_j \in \mathcal{H}'$.

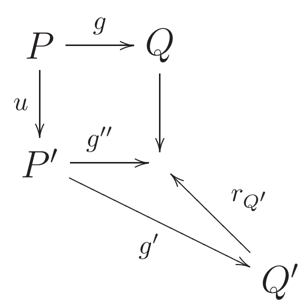
(5) We are ready to prove that if a petty morphism $h : A \to B$ is a consequence of $\mathcal{H}$, then $\mathcal{H} \vdash h$ in $\mathcal{A}$. We write $\mathcal{H} \vdash_{\mathcal{A}} h$ for the latter since we work within two categories: when we apply (1) to $\mathcal{A}'$ we use $\vdash_{\mathcal{A}'}$ for formal consequence in $\mathcal{A}'$. Analogously with $\models_{\mathcal{A}}$ and $\models_{\mathcal{A}'}$. Let $\hat{h} : \hat{A} \to \hat{B}$ be a reflection of h in $\mathcal{A}'$, then

$$\mathcal{H}' \models_{\mathcal{A}'} \hat{h}$$

because every object $C \in \mathcal{A}' = \operatorname{Inj} \mathcal{E}_j$ which is injective w.r.t. $\mathcal{H}' = \{f_j\}_{f \in \overline{\mathcal{H}}}$ is, due to (6.2), injective w.r.t. $\mathcal{H}$ in $\mathcal{A}$. Then C is injective w.r.t. h, and from $C \in \mathcal{A}'$ it follows easily that C is injective w.r.t. $\hat{h}$. Due to (4) we can apply (1). Therefore,

$$\mathcal{H}' \vdash_{\mathcal{A}'} \hat{h}.$$

We thus have a proof of $\hat{h}$ from $\mathcal{H}'$ in $\mathcal{A}'$. We modify it to obtain a proof of h from $\mathcal{H}$ in $\mathcal{A}$. We have no problems with a line of the given proof that uses one of the assumptions $f_j \in \mathcal{H}'$: we know from (6.1) that $\mathcal{H} \vdash_{\mathcal{A}} f_j$, and we substitute that line with a formal proof of f_j in $\mathcal{A}$. No problem is, of course, caused by the lines using COMPOSITION or CANCELLATION. But we need to modify the lines using PUSHOUT because $\mathcal{A}'$ is not closed under pushout in $\mathcal{A}$. However, a pushout, g'', of a morphism g along a petty morphism u in $\mathcal{A}'$



is obtained from a pushout, g', of g along u in $\mathcal{A}$ by composing it with a reflection map $r_{Q'}$ of the pushout codomain. Recall that P, P', $Q \in \mathcal{A}_\omega$ imply $Q' \in \mathcal{A}_\omega$. Thus, we can replace the line g'' of the given proof by using PUSHOUT in $\mathcal{A}$ (deriving g'), followed by a proof of $r_{Q'}$ (recall from (6.3) that $\mathcal{H} \vdash_{\mathcal{A}} r_{Q'}$) and an application of COMPOSITION. We thus proved that

$$\mathcal{H} \vdash_{\mathcal{A}} \hat{h}.$$

Since $r_B \cdot h = \hat{h} \cdot r_A$ and $\mathcal{H} \vdash_{\mathcal{A}} r_A$ (see (6.3)), we conclude $\mathcal{H} \vdash_{\mathcal{A}} \hat{h} \cdot r_A$; by CANCELLATION then $\mathcal{H} \vdash_{\mathcal{A}} h$.

6.4. Corollary (Compactness Theorem) *Let $\mathcal{H}$ be a set of finitary morphisms. Every finitary morphism which is a consequence of $\mathcal{H}$ is a consequence of a finite subset of $\mathcal{H}$.*

6.5. Remark We proceed by generalizing the completeness result from finitary to k-ary, where k is an arbitrary infinite cardinal. The *k-ary logic*, then, deals with k-ary morphisms (i.e., those having both domain and codomain of $\mathcal{M}$-rank k) and the k-ary Injectivity Deduction System of 2.20.

6.6. Theorem *The k-ary Injectivity Logic is complete. That is, given a set $\mathcal{H}$ of k-ary morphisms, then every k-ary morphism which is a consequence of $\mathcal{H}$ is a formal consequence (in the k-ary Injectivity Deduction System).*

Proof The whole proof is completely analogous to that of Theorem 6.2. As described in Remark 6.3 we work under the following milder assumptions on the category $\mathcal{A}$:

(i) $\mathcal{A}$ is cocomplete and has a left-proper factorization system $(\mathcal{E}, \mathcal{M})$;
(ii) $\mathcal{A}_k$ is a set of objects of $\mathcal{M}$-rank k, closed under colimits of less than k morphisms and under $\mathcal{E}$-quotients;
(iii) $\mathcal{M}$ is closed under k-filtered colimits in $\mathcal{A}$.

The statement we prove is concerned with k-petty morphisms (see 5.7). We denote by $\overline{\mathcal{H}}_k$ the closure of $\mathcal{H}$ as in 5.7. We write $\mathcal{H} \vdash h$ for the k-ary Injectivity Logic.

(1) The theorem holds whenever $\overline{\mathcal{H}}_k \subseteq \mathcal{M}$. The proof, based on the construction of a weak reflection $\hat{A} = \mathrm{colim} D_A$ of 5.8, is completely analogous to that of (1) in 6.2.

(2) Assuming $\mathcal{H} \subseteq \mathcal{E}$, then $\mathrm{Inj}\,\mathcal{H}$ is a reflective subcategory, and the reflection maps r_A fulfil $\mathcal{H} \vdash r_A$ and $r_A \in \mathcal{E}$. This is analogous to the proof of (2) of 6.2.

(3) The definition of $\mathcal{E}_i$ is precisely as in the proof of 6.2.

(4) For the first ordinal j with $\mathcal{E}_j = \mathcal{E}_{j+1}$ the category $\mathcal{A}' = \mathrm{Inj}\,\mathcal{E}_j$ fulfils the assumptions (i)-(iii) above, and the set $\mathcal{H}' = \{f_j;\ f \in \overline{\mathcal{H}}\}$ fulfils $\mathcal{H}' = \overline{\mathcal{H}'} \subseteq \mathcal{M}$.

(5) The theorem is then proved by applying (1) to $\mathcal{A}'$ and $\mathcal{H}'$: we get $\mathcal{H}' \vdash \hat{h}$ in $\mathcal{A}'$ and we derive $\mathcal{H} \vdash h$ in $\mathcal{A}$ precisely as in the proof of 6.2.

6.7. Corollary *The Injectivity Logic is sound and complete. That is, given a set $\mathcal{H}$ of morphisms of a weakly locally presentable category, then the consequences of $\mathcal{H}$ are precisely the formal consequences of $\mathcal{H}$ (in the Injectivity Deduction System). Shortly:*

$$\mathcal{H} \models h \quad \textit{iff} \quad \mathcal{H} \vdash h \quad \textit{(for all morphisms } h\textit{)}$$

In fact, soundness was proved in Section 2. Completeness follows from Theorem 6.6: since $\mathcal{H}$ is a set, and since every object of $\mathcal{A}$ has an $\mathcal{M}$-rank, see 4.5(ii), there exists k such that all domains and codomains of morphisms of $\mathcal{H} \cup \{h\}$ have $\mathcal{M}$-rank k.

7. Counterexamples

7.1. Example In "nice" categories which are not weakly locally presentable the completeness theorem can fail. Here we refer to $\vdash$ of the Deduction System 2.14 (and the logic concerning arbitrary morphisms). We denote by

$$\mathbf{CPO(1)}$$

the category of unary algebras defined on CPO's. Recall that a CPO is a poset with directed joins, and the corresponding category, **CPO**, has as morphisms the *continuous functions* (i.e., those preserving directed joins). The category $\mathbf{CPO(1)}$ has as objects the triples $(A, \sqsubseteq, \alpha)$ where $(A, \sqsubseteq)$ is a CPO and $\alpha : A \to A$ is a unary operation. Morphisms are the continuous algebra homomorphisms.

First let us observe that the assumption of cocompleteness is fulfilled.

Lemma $\mathbf{CPO(1)}$ *is cocomplete.*

Proof The category **CPO** is easily seen to be cocomplete. The category $\mathbf{CPO(1)}^*$ of partial unary algebras on CPO's (defined as above except that we allow $\alpha : A' \to A$ for any $A' \subseteq A$) is monotopological over **CPO**, see [2], since for every monosource
$f_i : (A, \sqsubseteq) \to (A_i, \sqsubseteq_i, \alpha_i)$ $(i \in I)$ we define a partial operation α on A at an element $x \in A$ iff α_i is defined at $f_i(x)$ for every i, and then

$$\alpha x = y \ \text{ iff } \ f_i(y) = \alpha_i(f_i(x)) \ \text{ for all } i \in I.$$

Consequently, $\mathbf{CPO(1)}^*$ is cocomplete by [2], 21.42 and 21.15. Further, $\mathbf{CPO(1)}$ is a full reflective subcategory of $\mathbf{CPO(1)}^*$: form a free unary algebra on the given partial unary algebra, ignoring the ordering, and then extend the ordering trivially (i.e., the new elements are pairwise incomparable, and incomparable with any of the original elements). Thus, $\mathbf{CPO(1)}$ is cocomplete.

We will find morphisms h_1, h_2 and k of $\mathbf{CPO(1)}$ with

$$\{h_1, h_2\} \models k \quad \text{but} \quad \{h_1, h_2\} \not\vdash k.$$

(i) We define a morphism h_1 that expresses, by injectivity, the condition

(h1) $$x \sqsubseteq \alpha x \quad \text{for all } x \in A.$$

Let $=$ denote the discrete order on the set $\mathbf{N}$ of natural numbers, and $\sqsubseteq$ that order enlarged by $0 \sqsubseteq 1$. Let $s : \mathbf{N} \to \mathbf{N}$ be the successor operation. Then

$$h_1 = \text{id} : (\mathbf{N}, =, s) \to (\mathbf{N}, \sqsubseteq, s)$$

is a morphism such that an algebra is injective w.r.t. h_1 iff it fulfils (h1) above.

(ii) The condition

(h2) $$A \neq \emptyset$$

is expressed by the injectivity w.r.t.

$$h_2 : \emptyset \to (\mathbf{N}, =, s)$$

where $\emptyset$ is the empty (initial) algebra. The following morphism k expresses the existence of a fixed point of α:

$$k : \emptyset \to 1$$

where 1 is a one-element (terminal) algebra.

Proposition *$\{h_1, h_2\} \models k$ but $\{h_1, h_2\} \not\vdash k$.*

Proof To prove $\{h_1, h_2\} \models k$, let $(A, \sqsubseteq, \alpha)$ be injective w.r.t. h_1 and h_2, i.e., fulfill $x \sqsubseteq \alpha(x)$ and be nonempty. Define a smooth chain $(a_i)_{i \in Ord}$ in $(A, \sqsubseteq)$ by transfinite induction: $a_0 \in A$ is any chosen element. Given a_i put $a_{i+1} = \alpha(a_i)$; we know that $a_i \sqsubseteq a_{i+1}$. Limit steps are given by (directed) joins, $a_j = \bigsqcup_{i<j} a_i$. Since A is small, there exist i with $a_i = a_{i+1}$, that is, a_i is a fixed point of α. Thus, A is injective w.r.t. k.

To prove $\{h_1, h_2\} \not\vdash k$, it is sufficient to find an extension $\mathcal{K}$ of the category $\mathbf{CPO(1)}$ in which $\mathbf{CPO(1)}$ is closed under colimits (therefore $\vdash$ has the same meaning in $\mathbf{CPO(1)}$ and in $\mathcal{K}$) and in which there exists an object which is injective w.r.t. h_1 and h_2 but not w.r.t. k. Thus k cannot be proved in $\mathcal{K}$ from h_1, h_2; consequently it cannot be proved in $\mathbf{CPO(1)}$ either.

We define $\mathcal{K}$ by adding a single new object K to $\mathbf{CPO(1)}$. The only morphism with domain K is id_K. For every algebra $(A, \sqsubseteq, \alpha)$ of $\mathbf{CPO(1)}$ we call a function $f : A \to Ord$ a *coloring* of A provided that it is continuous and fulfils $f(\alpha(x)) = f(x) + 1$ for all $x \in A$.
The hom-object of A and K in $\mathcal{K}$ is defined to be the class of all colorings of A. The composition in $\mathcal{K}$ is defined "naturally": given a continuous homomorphism
$h : (A, \sqsubseteq, \alpha) \to (B, \leq, \beta)$, then for every coloring $f : B \to Ord$ of B we have a coloring $f \cdot h : A \to Ord$ of A. The category $\mathbf{CPO(1)}$ is a full subcategory of $\mathcal{K}$ closed under (small) colimits. In fact, given a colimit cocone $a_i : A_i \to A$ $(i \in I)$ in $\mathbf{CPO(1)}$, then for every compatible cocone of colorings $f_i : A_i \to Ord$ $(i \in I)$ there exists an ordinal j such that all ordinals in $\cup_{i \in I} f_i[A_i]$ are smaller than j. Let $B = (j^+, \leq, \overline{s})$ be the object of $\mathbf{CPO(1)}$

where $\leq$ is the usual linear ordering of j^+ (the poset of all ordinals smaller or equal to j), and $\overline{s}$ is the successor map except $\overline{s}(j) = j$. Then the codomain restriction f_i' of each f_i defines a continuous homomorphism $f_i' : A_i \to B$, and we obtain a compatible cocone $(f_i')_{i \in I}$ for our diagram. The unique continuous homomorphism $g : A \to B$ with $g \cdot a_i = f_i'$ yields, by composing it with the inclusion $j^+ \hookrightarrow Ord$, a coloring $f : A \to Ord$ with $f \cdot a_i = f_i$ $(i \in I)$.

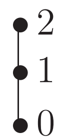
It is obvious that K is injective w.r.t. h_1: every coloring of $(\mathbf{N}, =, s)$ is also a coloring of $(\mathbf{N}, \sqsubseteq, s)$. And K is injective w.r.t. h_2 (because the inclusion $\mathbf{N} \hookrightarrow Ord$ is a coloring of $(\mathbf{N}, =, s)$). But K is not injective w.r.t. k, since 1 has no coloring.

7.2. Example None of the deduction rules of the Finitary Injectivity Deduction System can be left out. For each of them we present an example of a finite complete lattice $\mathcal{A}$ in which the reduced deduction system is not complete (for finitary morphisms).

(1) IDENTITY The deduction system CANCELLATION, COMPOSITION and PUSHOUT is not complete because nothing can be derived from the empty set of assumptions, although $\emptyset \models \mathrm{id}_A$.

(2) CANCELLATION In the poset

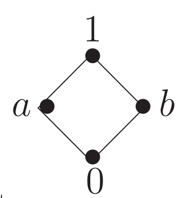
$\mathcal{A}$:



the only object injective w.r.t. $\{0 \to 2\}$ is 2, thus, we see that $\{0 \to 2\} \models 0 \to 1$. However, $0 \to 1$ cannot be derived from $0 \to 2$ by means of IDENTITY, COMPOSITION and PUSHOUT because the set of all morphisms of $\mathcal{A}$ except $0 \to 1$ is closed under composition and pushout.

(3) COMPOSITION In $\mathcal{A}$ above we clearly have $\{0 \to 1, 1 \to 2\} \models 0 \to 2$. However, the set of all morphisms except $0 \to 2$ is closed under left cancellation and pushout.

(4) PUSHOUT In the poset



we have $\{0 \to a\} \models b \to 1$, but we cannot derive $b \to 1$ from $0 \to a$ using IDENTITY, COMPOSITION and CANCELLATION because the set of all morphisms except $b \to 1$ is closed under composition and cancellation.

7.3. Example Here we demonstrate that in the Finitary Injectivity Logic we cannot restrict the statement of the completeness theorem from the given weakly locally presentable category $\mathcal{A}$ to its full subcategory $\mathcal{A}_\omega$ on all objects of finite rank: although the relation $\vdash$ works entirely in $\mathcal{A}_\omega$, the relation $\models$ does not.

More precisely, let $\mathcal{H} \models_\omega h$ mean that every $\mathcal{H}$-injective object of finite $\mathcal{M}$-rank is also h-injective. And let $\vdash_\omega$ be the formal consequence w.r.t. Deduction System 2.4. Then the implication

$$\mathcal{H} \models_\omega h \text{ implies } \mathcal{H} \vdash_\omega h$$

does NOT hold in general for sets of finitary morphisms.

Indeed, let $\mathcal{A} = \mathcal{G}ra$ be the category of graphs, i.e., binary relational structures (A, R), $R \subseteq A \times A$, and the usual graph homomorphisms. Recall that $\mathcal{G}ra$ is locally finitely presentable, and the finitely presentable objects are precisely the finite graphs. Let us call a graph a *clique* if $R = A \times A - \Delta_A$. Denote by C_n a clique of cardinality n, and let $\mathbf{0}$ be the initial object (empty graph).

For the set

$$\mathcal{H} = \{\mathbf{0} \to C_n\}_{n \in \mathbb{N}}$$

we have the following property:

every finite $\mathcal{H}$-injective graph G has a loop (i.e., a morphism from 1 to G).

In fact, if G has cardinality less than n and is injective w.r.t. $\mathbf{0} \to C_n$, then we have a homomorphism $f : C_n \to G$. Since f cannot be one-to-one, there exist $x \neq y$ in C_n with $f(x) = f(y)$ – and the last element defines a loop of G because (x, y) is an edge of C_n. Hence

$$\mathcal{H} \models_\omega (\mathbf{0} \to \mathbf{1}).$$

However, $\mathbf{0} \to \mathbf{1}$ cannot be proved in the Finitary Injectivity Logic. In fact, the graph

$$G = \coprod_{n \in \mathbb{N}} C_n$$

demonstrates that $\mathcal{H} \not\models (\mathbf{0} \to \mathbf{1})$.

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