Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–25

POINTWISE CONVERGENT EXPANSIONS IN q-FOURIER-BESSEL SERIES

L. D. ABREU, R. ALVAREZ-NODARSE AND J. L. CARDOSO

ABSTRACT: We define q-analogues of Fourier-Bessel series, by means of complete q-orthogonal systems constructed with the third Jackson q-Bessel function. Sufficient conditions for pointwise convergence of these series are obtained, in terms of a general convergence principle valid for other Fourier series on grids defined over numerable sets. The results are illustrated with specific examples of developments in q-Fourier-Bessel series.

KEYWORDS: q-Fourier series, third Jackson q-Bessel function.
AMS SUBJECT CLASSIFICATION (2000): 42C10, 33D15.

1. Introduction

The theory of Fourier-Bessel series, based on the orthogonality relation

$$\int_0^1 tJ_\nu(j_{\nu m}t)J_\nu(j_{\nu n}t)dt = 0, \text{ if } m \neq n\,,$$

where J_ν stands for the Bessel functions of order ν and $j_{\nu n}$ is their n-th zero, was developed by the classical analysis school of the early twenty century [19, XVIII], in a close parallelism to the classical theory of Fourier series and maintained its status as an active research topic till our days (see e.g. [12] and references therein).

Among the generalizations of the Bessel function, considerable attention has been given to the third Jackson (or Hahn-Exton) q-Bessel function $J_\nu^{(3)}(z;q)$ defined by

$$J_\nu^{(3)}(z;q) = z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q^{\nu+1};q)_k(q;q)_k}z^{2k},$$

where $\nu > -1$ is a real parameter. When $q \to 1^-$ we recover the Bessel function from $J_\nu^{(3)}(z;q)$, after a normalization. It is a well known fact (see

Received June 2, 2006.

e.g. [16, Prop. 3.5]) that these functions satisfy the orthogonality relation

$$\int_0^1 x\, J_\nu^{(3)}\big(j_{n\nu}qx;q^2\big)\, J_\nu^{(3)}\big(j_{m\nu}qx;q^2\big) d_qx = 0, \text{ if } m \neq n\,, \tag{1.1}$$

with $j_{1\nu} < j_{2\nu} < j_{3\nu} < \ldots$ where $j_{n\nu} := j_{n\nu}(q^2)$ are the zeros of $J_\nu(z;q^2)$ arranged in ascending order of magnitude and d_qx stands for the measure of the Jackson q-integral. The notation $J_\nu^{(k)}, k = 1, 2, 3\,,$ is due to Ismail [14] and is used to distinguish the three q-analogues of the Bessel function defined by Jackson. However, since the only q-Bessel function to appear on the text is $J_\nu^{(3)}$, we will drop the superscript for shortness of the notation and simply write

$$J_\nu(z;q^2) := J_\nu^{(3)}(z;q^2).$$

These functions appear as the matrix elements of irreducible unitary representations of the quantized universal enveloping algebra of the quantum group of plane motions [15]. They are, within some boundaries, the only functions q-orthogonal with respect to their own zeros [2] in the sense of (1.1).

It is the purpose of this note to develop convergence results for q-Fourier-Bessel series, based on the above orthogonality relation (1.1) and on results about the completeness of these system [4]. Associated with such a q-Fourier-Bessel theory there is a q-analogue of the Hankel transform with an inversion formula, introduced in [17], whose kernel is the third Jackson q-Bessel function. The relation between these q-Fourier-Bessel expansions and the q-Hankel transform was exploited in [1] to obtain a sampling theorem. However, since such sampling theorems sit in a reproducing kernel Hilbert space, convergence issues are of a different nature of those discussed here.

The q-Fourier theory to be present in this paper are different from the theory of Fourier expansions in terms of q-exponential functions [9, 18], since the latter is based on a continuous orthogonality property and the former is based on a discrete one. As we shall see, as a consequence of this fact, pointwise convergence issues are much simpler, and they follow from general properties of complete orthogonal systems in Hilbert spaces associated to general discrete orthogonality measures.

We will organize our ideas in four sections. The second is devoted to the required background on q-analysis, the third gives the convergence results, first in a general set-up and then in the specialization to the q-Fourier-Bessel and other examples. We finish the paper with a section containing two examples of expansions in q-Fourier-Bessel series.

2. Definitions and preliminary results

Following the standard notations in [13], consider $0 < q < 1$ and define the q-shifted factorial for a nonnegative positive integer n

$$(a;q)_0 = 1, \quad (a;q)_n = (1-q)(1-aq)\cdots\left(1-aq^{n-1}\right), \quad n = 1, 2, \ldots$$

and the infinite case as

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

The $q-$Jackson integral in the interval $(0,a)$ is defined by

$$\int_0^a f(t)\, d_q t = (1-q)\sum_{k=0}^{\infty} f\left(aq^k\right) aq^k. \tag{2.1}$$

Using this definition we may consider an inner product by setting

$$\langle f, g\rangle = \int_0^1 f(t)\,\overline{g(t)} d_q t\,.$$

The resulting Hilbert space is commonly denoted by $L_q^2(0,1)$. The space $L_q^2(0,1)$ is a separable Hilbert space [5].

The third Jackson q-Bessel function has a countable infinite number of real and simple zeros $j_{k,\nu}$, $k = 0, 1, 2, \ldots$, as it was shown in [16]. Further properties of the zeros of the third Jackson q-Bessel function were considered in [3].

We will need the following theorem from [4]:

Theorem A *The orthonormal sequence* $(u_k^{(\nu)})_{k\geq 0}$ *defined by*

$$u_k^{(\nu)}(x) = \frac{x^{\frac{1}{2}} J_\nu(j_{k\nu} qx; q^2)}{\left\| x^{\frac{1}{2}} J_\nu(j_{k\nu} qx; q^2)\right\|_{L_q^2(0,1)}}$$

is complete in $L_q^2(0,1)$.

This means that, whenever a function f is in $L_q^2(0,1)$, if $\int_0^1 f(x) u_k^{(\nu)}(x) d_q x = 0$, $k = 1, 2, 3, \ldots$, then $f\left(q^k\right) = 0$, $k = 0, 1, 2, \ldots$.

3. q-Fourier-Bessel Series

Using the orthogonality relation (1.1), we may consider the $q-$*Fourier-Bessel series*, $S_q^\nu[f]$, associated with a function f,

$$S_q^\nu[f](x) = \sum_{k=1}^{\infty} a_k\,(f)\,J_\nu(qj_{k\nu}x;q^2), \tag{3.1}$$

with the coefficients a_k given by

$$a_k\,(f) = \frac{1}{\mu_k}\int_0^1 tf(t)J_\nu(qj_{k\nu}t;q^2)d_qt, \tag{3.2}$$

where [16, Prop. 3.5],

$$\begin{aligned}\mu_k &= \left\|x^{\frac{1}{2}}J_\nu(j_{k\nu}qx;q^2)\right\|^2_{L_q^2(0,1)} = \int_0^1 \left[x\,J_\nu^{(3)}\big(j_{n\nu}qx;q^2\big)\,\right]^2 d_qx\\ &=\frac{q-1}{2}q^{\nu-1}J_{\nu+1}(qj_{k\nu};q^2)J'_\nu(j_{k\nu};q^2).\end{aligned} \tag{3.3}$$

Throughout the paper we will use the set V_q defined by

$$V_q\ =\ \{q^n:\ n=0,1,2,\ldots\}\,, \tag{3.4}$$

which coincides with the support points of the q-integral (2.1) in $[0,1]$.

3.1. Pointwise convergence: A general set-up. With a view to study convergence of the series (3.1) when $x\in V_q$, we first establish a general result concerning the pointwise convergence of these series. The setting to be used in this section is a very general one, designed to cover not only the convergence of q-Fourier-Bessel series but also other Fourier systems based on discrete orthogonality relations, as in [8], [10] and [6].

Let $\mathcal{N}=\{a_n\,|\,n\in\mathbb{N}\}$ be a numerable space and let μ a positive measure on $\mathcal{N}$ such that $\mu_n=\mu(\{a_n\})>0$. We will denote by $\mathcal{L}^2_\mu$, the space of all functions $f:\mathcal{N}\mapsto\mathbb{C}$, such that

$$\|f\|_2^2=\sum_{n=1}^{\infty}|f(a_n)|^2\mu_n<+\infty.$$

In such a space, the scalar product $\langle f,g\rangle$ of two functions is defined by

$$\langle f,g\rangle=\sum_{n=1}^{\infty}\overline{f(a_n)}g(a_n)\mu_n.$$

An orthonormal basis in $\mathcal{L}^2_\mu$ is the sequence of functions $(e_n)_n$ such that

$$e_n(a_k) = \begin{cases} \mu_n^{-1/2}, & k = n, \\ 0, & k \neq n. \end{cases} \tag{3.5}$$

To check that the above system is a complete orthonormal system in $\mathcal{L}^2_\mu$, notice that the function g_N, $N \in \mathbb{N}$, defined as

$$g_N = f - \sum_{n=1}^{N} \langle f, e_n \rangle e_n, \qquad f \in \mathcal{L}^2_\mu$$

is such that $g_N(a_k) = 0$ for all $k \leq N$ and $g_N(a_k) = f(a_k)$ for all $k > N$. Therefore,

$$\|g_N\|_2^2 = \sum_{n=N+1}^{\infty} |f(a_n)|^2 \mu_n \to 0, \text{ as } N \to \infty.$$

Thus, for an arbitrary $f \in \mathcal{L}^2_\mu$, we have

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n,$$

with convergence in norm $\| \cdot \|_2^2$. This is also true for any other complete orthonormal system $(u_n)_n$, i.e., for an arbitrary $f \in \mathcal{L}^2_\mu$ one has

$$f = \sum_{n=1}^{\infty} \langle f, u_n \rangle u_n,$$

with convergence in norm $\|\cdot\|_2^2$. It remains only to check when the convergence of the above series is pointwise. The answer to this question is in the following lemma.

Lemma 1. *Let $(u_n)_n$ a complete orthonormal system of functions in $\mathcal{L}^2_\mu$. Then for any arbitrary $f \in \mathcal{L}^2_\mu$*

$$f(a_k) = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, u_n \rangle u_n(k), \quad \forall a_k \in \mathcal{N}.$$

Proof: Let a_k be an arbitrary element of $\mathcal{N}$. Then, the function $d_k := \mu_k^{-1/2} e_k$, where e_k is the function given in (3.5), satisfies the property

$$\langle f, d_k \rangle = \langle f, \mu_k^{-1/2} e_k \rangle = f(a_k) \mu_k^{-1/2} e_k(a_k) \mu_k = f(a_k).$$

In particular, $\langle u_n, d_k\rangle = u_n(a_k)$. Then,

$$f(a_k) = \langle f, d_k\rangle = \left\langle \lim_{N\to\infty}\sum_{n=1}^{N}\langle f, u_n\rangle u_n, d_k\right\rangle = \lim_{N\to\infty}\sum_{n=1}^{N}\langle f, u_n\rangle\langle u_n, d_k\rangle,$$

and, therefore,

$$f(a_k) = \lim_{N\to\infty}\sum_{n=1}^{N}\langle f, u_n\rangle u_n(a_k).$$

■

3.2. Application to q-Fourier-Bessel series. Let be $\mathcal{N} = V_q$ and in V_q define the measure μ associated to the Jackson q-integral (2.1). Let $L_q^2(0,1)$ be the corresponding $\mathcal{L}_\mu^2$ space. Since the set of functions

$$u_n^\nu(x) := \frac{x^{\frac{1}{2}}J_\nu(j_{n\nu}qx;q^2)}{\left\|x^{\frac{1}{2}}J_\nu(j_{n\nu}qx;q^2)\right\|},$$

is a complete orthogonal system in $L_q^2(0,1)$, then, for an arbitrary $f \in L_q^2(0,1)$, i.e., f such that

$$\int_0^1 |f(x)|^2 d_qx < +\infty,$$

we have the equality

$$f(q^k) = \lim_{N\to\infty}\sum_{n=1}^{N}\langle f, u_n^\nu\rangle u_n^\nu(a_k), \quad \forall k = 0,1,2,\ldots,$$

where

$$a_k\,(f) = \frac{1}{\mu_k}\int_0^1 tf(t)J_\nu(qj_{k\nu}t;q^2)d_qt\,,$$

being μ_k the norm of the q-Bessel functions given in (3.3). This summarizes in the following theorem.

Theorem 1. *If $f \in L_q^2(0,1)$, then the q-Fourier-Bessel series (3.1) converges to the function f at every point $x = q^{k-1}$, $k = 1,2,3,\ldots$ of the set V_q.*

Remark 1. *Let us mention that in the case of the standard trigonometric series the equivalent result of Lemma 1 ($\mathcal{L}^2$ convergence implies pointwise convergence) is not true. In fact this problem leads to the celebrated Carleson Theorem (see e.g. [7]). The main difference between these two cases is that,* *contrary to the case of the discrete space $\mathcal{L}^2_\mu$ (see the function d_k used in the proof of Lemma 1), for the $\mathcal{L}^2_{[0,2\pi]}$ space for an arbitrary $a \in [0, 2\pi]$ there not exists a function f_a such that $\langle f, f_a \rangle = f(a)$.*

Remark 2. *In the paper* [10] *were established some convergence theorems of the q-Fourier series associated with the q-trigonometric orthogonal system $\left\{1, C_q\left(q^{\frac{1}{2}}\omega_k x\right), S_q\left(q\omega_k x\right)\right\}$, where the q-cosines C_q and q-sinus S_q are given in terms of the third q-Bessel functions by*

$$C_q(z) = q^{-3/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(3)}_{-1/2}\left(q^{-3/4}z;q^2\right),$$

$$S_q(z) = q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(3)}_{1/2}\left(q^{-1/4}z;q^2\right),$$

being ω_k is the positive zeros of the function S_q. Notice that, since this orthogonal system is also a complete system in $L^2_q(0,1)$, then we have that the q-trigonometric Fourier series converge to f at every point of V_q, i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k C_q\left(q^{\frac{1}{2}}\omega_k x\right) + b_k S_q\left(q\omega_k x\right)\right], \quad x = q^m,\ m = 0, 1, 2, \ldots,$$

with $a_0 = \int_{-1}^{1} f(t) d_q t$ and, for $k = 1, 2, 3, \ldots$,

$$a_k = \frac{1}{\mu_k}\int_{-1}^{1} f(t) C_q\left(q^{\frac{1}{2}}\omega_k t\right) d_q t, \quad b_k = \frac{q^{\frac{1}{2}}}{\mu_k}\int_{-1}^{1} f(t) S_q\left(q\omega_k t\right) d_q t,$$

where,

$$\mu_k = (1-q) C_q(q^{1/2}\omega_k) S_q'(\omega_k).$$

Then, for any $f \in L^2_q(0,1)$, the q-trigonometric Fourier series defined above (see [10]*) converges to the function f at every point $x = q^{k-1}$, $k = 1, 2, 3, \ldots$ of the set V_q (3.4). Thus, the corresponding open problem posed in the concluding remarks section of* [10] *is completely solved.*

Remark 3. *In* [6] *a rigorous theory of q-Sturm-Liouville systems was developed. In particular it was shown that the set of all normalized eigenfunctions forms an orthonormal basis for $L^2_q(0,a)$. Therefore Lemma 1 can be used to show that the Fourier expansions in terms of the eigenfunctions of q-Sturm-Liouville systems are pointwise convergent.*

4. Examples of q-Fourier-Bessel expansions

Consider

$$g(x;q) = x^{\nu}\frac{(x^2q^2;q^2)_\infty}{(x^2q^{2\mu-2\nu};q^2)_\infty}.$$

Using the q-binomial theorem [13, (1.3.2)] we have

$$g(x;q) = \frac{x^{\nu}}{(1-x^2)}\left[\sum_{n=0}^{\infty}\frac{(q^{2\mu-2\nu};q^2)_n}{(q^2;q^2)_n}\,x^{2n}\right]^{-1}.$$

Since

$$\lim_{q\to 1^-}\sum_{n=0}^{\infty}\frac{(q^{2\mu-2\nu};q^2)_n}{(q^2;q^2)_n}\,x^{2n} = \sum_{n=0}^{\infty}\frac{(2\mu-2\nu)_n}{n!}\,x^{2n} = (1-x^2)^{-2\mu+2\nu},$$

it becomes clear that $g(x;q)$ is a q-analogue of $g(x) = x^{\nu}(1-x^2)^{2\mu-2\nu-1}$.

Let consider the case when $\nu > 0$, $\mu > 1/2$. Then $g(x;q) \in L_q^2(0,1)$ and by Theorem 1 the q-Fourier-Bessel series is pointwise convergent in V_q. Setting $x = qj_{k\nu}$ in the formula [1, Eq. (4.11)] (where a misprint is corrected; see also [11, (54)] for a similar formula with a different normalization)

$$\frac{(q^2;q^2)_\infty}{(q^{2u-2\nu};q^2)_\infty}x^{\nu-u}J_u^{(3)}(x;q^2) = \frac{1}{1-q}\int_0^1 t^{\nu+1}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2u-2\nu};q^2)_\infty}J_\nu^{(3)}(xt;q^2)d_qt,$$

follows that

$$\int_0^1 t\,g(t;q)J_\nu(qj_{k\nu}t;q^2)d_qt = (1-q)(qj_{k\nu})^{\nu-\mu}\frac{(q^2;q^2)_\infty}{(q^{2\mu-2\nu};q^2)_\infty}J_\mu(qj_{k\nu};q^2)\,. \quad (4.1)$$

Therefore, from (3.2)-(3.3), (4.1) follows that

$$\frac{x^{\nu}\;(x^2q^2;q^2)_\infty}{(x^2q^{2\mu-2\nu};q^2)_\infty} = -\frac{2\,q^{1-\mu}\big(q^2;q^2\big)_\infty}{\big(q^{2(\mu-\nu)};q^2\big)_\infty}\sum_{k=1}^{\infty}\frac{(j_{k\nu})^{\nu-\mu}J_\mu\big(j_{qk\nu};q^2\big)}{J_{\nu+1}\big(j_{k\nu}q;q^2\big)J'_\nu(j_{k\nu};q^2)}J_\nu\big(j_{qk\nu}x;q^2\big)\,,$$

for all $x = q^n$, $n = 0,1,2,\dots$.

Notice that putting $\mu = \nu + 1$ in the above formula it gives the following expansion of $f(x) = x^{\nu}$:

$$x^{\nu} = -2\,q^{-\nu}\sum_{k=1}^{\infty}\frac{J_\nu(j_{k\nu}qx;q^2)}{j_{k\nu}J'_\nu(j_{k\nu};q^2)}$$

for all $x = q^n$, $n = 0,1,2,\dots$.

The convergence of the expansion of x^{ν} in the classical Fourier-Bessel series is studied in [19, 18.22] using contour integral methods.

Acknowledgements. The authors are very grateful to Prof. Juan Arias for stimulating discussions and for pointing out the Lemma 1. This work was partially supported by CMUC from University of Coimbra (LDA, JLC) and by the Ministerio de Ciencias y Tecnología of Spain under the grant BFM-2003-6335-C03-01 and the Junta de Andalucía under grant FQM-262 (RAN). LDA was also supported by a post-doctoral grant from Fundação Ciência e Tecnologia from Portugal during the completion of this work.

References

[1] L. D. Abreu, *A q-Sampling theorem related to the q-Hankel transform,* Proc. Amer. Math. Soc. **133**, 1197-1203, (2005).

[2] L. D. Abreu, *Functions q-orthogonal with respect to their own zeros,* Proc. Amer. Math. Soc. **134**, 2695-2701, (2006).

[3] L. D. Abreu, J. Bustoz, J. L. Cardoso, *The roots of the third Jackson q-Bessel function*, Internat. J. Math. Math. Sci. **67**, 4241-4248, (2003).

[4] L. D. Abreu, J. Bustoz *On the completeness of sets of q-Bessel functions* $J_{\nu}^{(3)}(x;q)$, in *Theory and Applications of Special Functions. A volume dedicated to Mizan Rahman*, 29–38, Dev. Math., **13**, Springer, New York, 2005 (eds. M. E. H. Ismail and H. T. Koelink).

[5] M. H. Annaby, *q-type sampling theorems,* Result. Math., **44**(3-4), 214-225, (2003).

[6] M. H. Annaby, Z. S. Mansour, *Basic Sturm-Liouville problems.* J. Phys. A, Math Gen. **38**(17), 3775-3797, (2005).

[7] J. Arias de Reyna, *Pointwise Convergence of Fourier Series*, Springer. Lect. Notes Math. 1785, (2000).

[8] J. Bustoz, J. L. Cardoso, *Basic Analog of Fourier Series on a q-Linear Grid*, J. of Approx. Theory, **112**, 134-157, (2001).

[9] J. Bustoz, S. Suslov, *Basic analogues of Fourier series on a q-quadratic grid*, Meth. Appl. Anal. **5**, 1-38, (1998).

[10] J. L. Cardoso, *Basic Fourier series on a q-linear grid: Convergence theorems,* J. Math. Anal. Appl., in press (published online 17 November 2005).

[11] A. Fitouhi, M. M. Hamza, F. Bouzeffour, *The q-*j_{α} *Bessel function.* J. Approx. Theory **115**(1), 144-166, (2002).

[12] O. Ciaurri, L. Roncal, *The Bochner-Riesz means for Fourier-Bessel expansions.* J. Funct. Anal. **228**(1), 89-113, (2005).

[13] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, UK, 1990.

[14] M. E. H. Ismail, *The Zeros of Basic Bessel functions, the functions* $J_{v+ax}(x)$ *and associated orthogonal polynomials*, J. Math. Anal. Appl. **86**, 1-19, (1982).

[15] H. T. Koelink, *The quantum group of plane motions and the Hahn-Exton q-Bessel function.* Duke Math. J. **76** (2), 483-508, (1994).

[16] H. T. Koelink, R. F. Swartouw, *On the zeros of the Hahn-Exton q-Bessel Function and associated q-Lommel polynomials*, J. Math. Anal. Appl. **186**, 690-710, (1994).

[17] T. H. Koornwinder, R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. **333** (1), 445-461, (1992).
[18] S. K. Suslov, *An introduction to basic Fourier series.* Dev. Math., **9**, Dordrecht: Kluwer Academic Publishers, 2003.
[19] G. N. Watson, *A treatise on the Theory of Bessel Functions,* second edition, Cambridge University Press, Cambridge, 1966.

L. D. Abreu
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade de Coimbra, Apdo. 3008, 3001-454 Coimbra, PORTUGAL
E-mail address: abreu@mat.uc.pt

R. Alvarez-Nodarse
Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, SPAIN
E-mail address: ran@us.es

J. L. Cardoso
Departamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, Apdo. 1013, 5001-801 Vila Real, PORTUGAL
E-mail address: jluis@utad.pt