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THE INVERSE EIGENVALUE PROBLEM FOR HERMITIAN MATRICES WHOSE GRAPH IS A CYCLE

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ABSTRACT: The inverse eigenvalue problem for Hermitian matrices whose graph is a cycle is discussed. Some results concerning the multiplicities of eigenvalues of a particular Hermitian matrix whose graph contains exactly one cycle are established.

KEYWORDS: Periodic Jacobi matrix, eigenvalues, multiplicities, graphs, cycle.
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1. Introduction

A periodic Jacobi matrix is a real symmetric matrix of the form

$$L = \begin{pmatrix} a_1 & b_1 & & & b_n \\ b_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ b_n & & & b_{n-1} & a_n \end{pmatrix}, \tag{1.1}$$

where $b_i > 0$, for $i = 1, \ldots, n$, and all the non-mentioned entries are zero.

An extensive attention has been paid in the literature to the theory of periodic Jacobi matrices (cf. [1, 2, 3, 8, 11]. Many problems on the spectra of periodic Jacobi matrices arise in a remarkable variety of applications, in pure and applied mathematics.

Ferguson [3] presented an algorithm for calculating L from some given spectral data, based on the Lanczos algorithm as treated by Boley and Golub [2], using a discrete version of Floquet theory. It is a typical inverse eigenvalue problem, a problem concerned the reconstruction of a matrix from prescribed spectral data. Recall that a Jacobi matrix is any real, symmetric tridiagonal matrix whose next to diagonal entries are positive, [5, 6, 7]. Let J denote the Jacobi matrix obtained by deleting from L the last row and column, with

$$\omega_J(\lambda) = \det(\lambda I - J) = (\lambda - \mu_1) \cdots (\lambda - \mu_{n-1}).$$

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The numbers $\rho_1, \ldots, \rho_{n-1}$ defined by $b_1 \cdots b_n = -\rho_j b_n^2 y_j^2 \omega'_J(\mu_j)$ are called the Floquet multipliers of L. Ferguson has shown that for given real numbers A, $B(>0)$, $\mu_1 > \cdots > \mu_{n-1}$ and $\rho_1, \ldots, \rho_{n-1}$ such that

$$\rho_j \, \omega'_J(\mu_j) < 0 \ , \ j = 1, \cdots, n-1$$

there exists an unique periodic Jacobi matrix L (1.1) such that

$$a_1 + a_2 + \cdots + a_n = A \quad \text{and} \quad b_1 \cdots b_n = B \ ,$$

where μ_i, for $i = 1, \cdots, n-1$, are the eigenvalues of J and the ρ_j the Floquet multipliers of L.

Ferguson also based his analysis on the partially characterization of periodic Jacobi matrices by van Moerbeke. In [11], van Moerbeke had considered a different periodic Jacobi matrix of order $2n$:

$$Q = \begin{pmatrix} a_1 & b_1 & & & & & b_n \\ b_1 & \ddots & \ddots & & & & \\ & \ddots & a_n & b_n & & & \\ & & b_n & a_1 & b_1 & & \\ & & & b_1 & \ddots & \ddots & \\ & & & & \ddots & \ddots & b_{n-1} \\ b_n & & & & & b_{n-1} & a_n \end{pmatrix} ,$$

where b_k's are real positive numbers and a_k's are any real numbers. The spectrum of Q was given in terms of the spectrum of the difference equation

$$Ly(k,\lambda) \equiv (a_{k-1}D^{-1} + b_k D^0 + a_k D^1) y(k,\lambda) = \lambda y(k,\lambda) \, , \quad k = 1, \cdots, 2n \ ,$$

with the boundary condition $y_{2n+i} = y_i$, giving an analogue of Floquet theory for the matrix Q.

Later, Andrea and Berry [1] presented some algorithms based on a continued fraction expansion for solving inverse eigenvalue problem for periodic Jacobi matrices.

Since the spectrum of a periodic Jacobi matrix consists of a finite union of closed intervals, so-called bands, recently E. Korotyaev and I.V. Krasovsky [8] showed new bounds, in terms of the matrix coefficients, on the width of the spectrum and on the total width of the gaps separating the bands. They considered the so-called q-periodic Jacobi matrix

$$(H\psi)_n = a_{n-1}\psi_{n-1} + b_n \psi_n + a_{n+1}\psi_{n+1}$$

on $\ell^2(\mathbb{Z})$, where $b_{n+q} = b_n \in \mathbb{R}$, $a_{n+q} = a_n > 0$, and $q > 1$ is the smallest period, and they estimated the spectra based on the analysis of the quasi-momentum. For that, they defined the Hermitian matrices

$$\begin{pmatrix} b_1 & a_1 - ia_2 \\ a_1 + ia_2 & b_2 \end{pmatrix},$$

if $q = 2$, and

$$\begin{pmatrix} b_1 & a_1 & & ia_q \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{q-1} \\ -ia_q & & a_{q-1} & b_q \end{pmatrix},$$

if $q > 2$.

These periodic Jacobi matrices suggest a generalization. In this work we will see a periodic Jacobi matrix as the adjacency matrix of weight cycle. We consider some Hermitian matrices whose graph has only one cycle and we will be concentrated on the multiplicities of the eigenvalues of such matrices.

2. The characteristic polynomial of a weighted graph

A graph $G = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \mathcal{V}(G)$ whose members are called vertices, and a set $\mathcal{E} = \mathcal{E}(G)$ of 2-subset of $\mathcal{V}$, whose members are called edges. By a digraph $D = (\mathcal{V}, \mathcal{A})$ we mean the same finite set $\mathcal{V}$, and a subset $\mathcal{A} = \mathcal{A}(D)$ of $\mathcal{V} \times \mathcal{V}$, whose members are called arcs. Note that an arc is an ordered pair (i, j), whereas an edge of a graph is an unordered pair $\{i, j\}$. We write in each context $i \sim j$. A forest is a graph without cycles and a tree is a connected forest.

Given an arc $e = (i, j)$ of D, $D \backslash e$ is obtained by deleting e but not the vertices i or j; the sub-digraph $D \backslash X$, where X is a subset of vertices of D, is obtained from D deleting the vertices X and all arcs incident in vertices of X.

Let $A = (a_{ij})$ be an $n \times n$ matrix. The graph of A, $G(A)$, is the pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\{i, j\}$, $i \neq j$, is an edge if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. Analogously the weighted digraph of $A = (a_{ij})$ is such (i, j) is an arc if and only if $a_{ij} \neq 0$. The matrix A can be viewed as a weighted adjacency matrix of digraph $D(A)$ on n vertices, with loops (arcs of the type (i, i)) allowed on the vertices.

We denote by $A(X)$, where X is a subset of vertices of the graph or digraph of A, the submatrix obtained by deleting from A the rows and columns of X.

A directed path from i_1 to i_r, P_{i_1,i_r}, in the digraph D is a sequence of distinct vertices $(i_1, i_2, \ldots, i_{r-1}, i_r)$ such that each arc $(i_1, i_2), \ldots, (i_{r-1}, i_r)$ is in $\mathcal{A}(D)$. We say that the length of P_{i_1,i_r}, $\ell(P_{i_1,i_r})$, is $r-1$. If to the path P_{i_1,i_r} we add the arc (i_r, i_1), then we have a directed cycle $(i_1, i_2, \ldots, i_r, i_1)$ (of length r). Analogously the path from i_1 to i_r in the graph G is a sequence of distinct vertices $(i_1, i_2, \ldots, i_{r-1}, i_r)$ such that each arc $\{i_1, i_2\}, \ldots, \{i_{r-1}, i_r\}$ is in $\mathcal{E}(G)$. If this path we add the arc $\{i_r, i_1\}$, then we have a cycle $(i_1, i_2, \ldots, i_r, i_1)$ of length r.

We have a general formula for the determinant:

Theorem 2.1 ([10]). *Given an $n \times n$ matrix $A = (a_{ij})$ and $r \in \{1, \ldots, n\}$, let us assume that $\{c_1, \ldots, c_m\}$ is the set of all directed cycles in $D(A) = D$ containing the vertex r, with $\ell_j = \ell(c_j)$. Then*

$$\det A = \sum_{k=1}^{m} (-1)^{\ell_k+1} \det A(\mathcal{V}(c_k)) \prod_{(i,j)\in\mathcal{A}(c_k)} a_{ij} \tag{2.1}$$

where $c_k = (\mathcal{V}(c_k), \mathcal{A}(c_k))$ and $\det A(\mathcal{V}(c_k)) = 1$ if c_k contain all vertices of D.

The set of cycles includes the cycles of one arc (a loop), the cycles with two arcs, (i, j, i), if $a_{ij} \neq 0$ and $a_{ji} \neq 0$, etc.

Suppose now A is Hermitian, with Theorem 2.1 we can give a general formula for the characteristic polynomial of A,

$$\det(\lambda I - A) \ ,$$

which we denote by $\varphi_A(\lambda)$:

Corollary 2.2. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ and $i \in \{1, \ldots, n\}$, let us assume that $\{c_1, \ldots, c_m\}$ is the set of all cycles in $G(A) = G$ containing the vertex i, with $c_k = (k_1, \ldots, k_{\ell_k}, k_1)$. Then*

$$\begin{aligned} \varphi_A(\lambda) \ &= \ (\lambda - a_{ii})\varphi_{A(i)}(\lambda) - \sum_{j\sim i} |a_{ij}|^2 \varphi_{A(i,j)}(\lambda) \\ &\quad -2\sum_{k=1}^{m} \mathrm{Re}\left(a_{k_1k_2} \cdots a_{k_{\ell_k-1}k_{\ell_k}} \bar{a}_{k_{\ell_k}k_1}\right) \varphi_{A(\mathcal{V}(c_k))}(\lambda) \ , \end{aligned} \tag{2.2}$$

If the graph of A has only one cycle, then we conclude the following:

Corollary 2.3. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ whose graph G has only one cycle, say $c = (1, \ldots, \ell, 1)$, if $i \in \{1, \ldots, \ell\}$ is a vertex of c, then*

$$\varphi_A(\lambda) = (\lambda - a_{ii})\varphi_{A(i)}(\lambda) - \sum_{j \sim i} |a_{ij}|^2 \varphi_{A(i,j)}(\lambda) - 2\,\mathrm{Re}\,(a_{12} \cdots a_{\ell-1,\ell}\bar{a}_{\ell,1})\, \varphi_{A(\mathcal{V}(c))}(\lambda)\ .$$

If $G(A)$ has only one cycle, then we say the graph is unicycle.

Corollary 2.4. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ whose graph G is a cycle, say $(1, \ldots, n, 1)$, and $i \in \{1, \ldots, n\}$, the characteristic polynomial of A is*

$$\begin{aligned} \varphi_A(\lambda) &= (\lambda - a_{ii})\varphi_{A(i)}(\lambda) - |a_{i-1,i}|^2 \varphi_{A(i-1,i)}(\lambda) \\ &\qquad - |a_{i,i+1}|^2 \varphi_{A(i,i+1)}(\lambda) - 2\,\mathrm{Re}\,(a_{12} \cdots a_{n-1,n}\bar{a}_{n,1})\ . \end{aligned} \tag{2.3}$$

Corollary 2.5. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ whose graph G is a path, say $(1, \ldots, n)$, and $i \in \{1, \ldots, n\}$, the characteristic polynomial of A is*

$$\varphi_A(\lambda) = (\lambda - a_{ii})\varphi_{A(i)}(\lambda) - \sum_{j \sim i} |a_{ij}|^2 \varphi_{A(i,j)}(\lambda)\ .$$

Let $A = (a_{ij})$ be a Hermitian matrix. We denote by $A^+ = (b_{ij})$ the symmetric matrix such that $b_{ij} = |a_{ij}|$.

Corollary 2.6. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ whose graph G is a path, say $(1, \ldots, n)$, and $i \in \{1, \ldots, n\}$, then*

$$\varphi_A(\lambda) = \varphi_{A^+}(\lambda)\ .$$

Actually, the above result is still true for any Hermitian matrix whose graph is a tree.

3. Prior Results

The main aim of an inverse eigenvalue problem is to construct a matrix that maintains a certain specific structure as well as some given spectral property. For example, given distinct real numbers $\mu_1, \ldots, \mu_{n-1}$ and nonzero real numbers, $u_1, \ldots, u_{n-1}$, whose squares sum to one, Ferguson [3] used

Lanzcos algorithm to get a Jacobi matrix

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-2} \\ & & b_{n-2} & a_{n-1} \end{pmatrix}, \tag{3.1}$$

such that $u_1, \ldots, u_{n-1}$ are the first components of a set $Y_1, \ldots, Y_{n-1}$ of real orthonormal eigenvectors of J associated with eigenvalues $\mu_1, \ldots, \mu_{n-1}$:

1: Set:
- **1.1:** $b_0 = 1$;
- **1.2:** $u_{0,j} = 0$, for $j = 1, 2, \ldots, n-1$;
- **1.3:** $u_{1,j} = u_j$, for $j = 1, 2, \ldots, n-1$.

2: Iteration $i = 1, 2, \ldots, n-2$:
- **2.1:** $a_i = \sum_{\ell=1}^{n-1} \mu_\ell u_{i,\ell}^2$;
- **2.2:** $b_i = \sqrt{\sum_{\ell=1}^{n-1} ((\mu_\ell - a_i)u_{i,\ell} - b_{i-1}u_{i-1,\ell})^2}$;
- **2.3:** $u_{i+1,j} = ((\mu_j - a_i)u_{i,j} - b_{i-1}u_{i-1,j})/b_i$, for $j = 1, 2, \ldots, n-1$.

3: $a_{n-1} = \sum_{\ell=1}^{n-1} \mu_\ell u_{n-1,\ell}^2$,

where $u_{i,j}$, for $j = 1, 2, \ldots, n-1$, is the i-th component of Y_j.

Leal Duarte [9] generalized this result to any Hermitian matrix whose graph is a tree.

To prove this algorithm, Ferguson consider important relationships between the eigenvalues and eigenvectors of J.

Since J is a real, symmetric matrix, J has a set of real distinct eigenvalues $\mu_1, \ldots, \mu_{n-1}$ with associated orthonormal eigenvectors $Y_1, \ldots, Y_{n-1}$. If Y is a matrix whose j column is Y_j, then

$$JY = YD\,, \quad \text{where} \quad D = \operatorname{diag}(\mu_1, \ldots, \mu_{n-1})\,.$$

It is also known

$$(\mu I - J)^{-1} = Y(\mu I - D)^{-1}Y^T\,. \tag{3.2}$$

Since the characteristic polynomial of J is

$$\omega_J(\lambda) = \det(\lambda I - J) = (\lambda - \mu_1)\cdots(\lambda - \mu_{n-1})$$

then

$$\omega_J'(\mu) = \sum_{\ell=1}^{n-1} \prod_{i(\neq \ell)=1}^{n-1} (\mu - \mu_\ell) = \sum_{\ell=1}^{n-1} \frac{\omega_J(\mu)}{\mu - \mu_\ell}\,,$$

and we have

$$\omega_J'(\mu_j) = \prod_{i(\neq j)=1}^{n-1} (\mu_j - \mu_i) \ .$$

Comparing the entries in row 1, column n-1 of (3.2) we obtain the identity

$$b_1 \cdots b_{n-2} = \omega_J'(\mu_j) u_{1,j} u_{n-1,j} \quad , j = 1, \ldots, n-1 \ . \tag{3.3}$$

Ferguson also treated an inverse eigenvalue problem for periodic Jacobi matrices. Let L be the periodic matrix as in (1.1) and J be the Jacobi matrix obtained by deleting from L the last row and column. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of J, $u_1, \ldots, u_{n-1}$ be the first components of a set $Y_1, \ldots, Y_{n-1}$ of real orthonormal eigenvectors of J associated with eigenvalues $\mu_1, \ldots, \mu_{n-1}$ and $\omega_J(\mu)$ be the characteristic polynomial of J.

Definition 3.1. The **Floquet multipliers** $\rho_1, \ldots, \rho_{n-1}$ of L corresponding to $\mu_1, \ldots, \mu_{n-1}$ are the numbers defined by the relation

$$b_1 \cdots b_n = -\rho_j \omega_J'(\mu_j) b_n^2 u_j^2 \ , \quad j = 1, \ldots, n-1 \ .$$

With this facts in mind, Ferguson introduced the following matrices:

$$L_\rho = \begin{pmatrix} a_1 & b_1 & & & \frac{1}{\rho} b_n \\ b_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ \rho b_n & & & b_{n-1} & a_n \end{pmatrix} ,$$

where $\rho \neq 0$, and he proved some interesting spectral properties of L_ρ.

Theorem 3.1 ([3]). *The characteristic polynomial of L_ρ admits the representation*

$$\det(\lambda I - L_\rho) = b_1 \cdots b_n \left\{ \Delta(\lambda) - (\rho + \frac{1}{\rho}) \right\} \ ,$$

where $\Delta(\lambda)$, called the discriminant of L_ρ, is independent of ρ. The Floquet multipliers $\rho_1, \cdots, \rho_{n-1}$ of L corresponding to the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ of J satisfy the relation

$$(-1)^j \Delta(\mu_j) = (-1)^j (\rho_j + \frac{1}{\rho_j}) \geq 2 \ , \quad j = 1, \ldots, n-1 \ .$$

Furthermore, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of L, which are the roots of $\Delta(\lambda) = 2$, are real and can be ordered so that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots .$$

The main important result of [3] is the following:

Theorem 3.2 ([3])**.** *There exists a periodic Jacobi matrix (1.1) with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if the real numbers $\lambda_1, \ldots, \lambda_n$ can be rearranged such that*

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \cdots .$$

To construct a Periodic Jacobi matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \cdots$$

Ferguson used the following procedure: let $A = \lambda_1 + \cdots + \lambda_n$, $\mu_1 > \cdots > \mu_{n-1}$ be real numbers and B be a positive real number such that

$$(-1)^j \Delta(\mu_j) \geq 2 \, , \quad j = 1, \ldots, n-1,$$

and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

where $\Delta(\lambda) = 2 + \frac{1}{B}(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$. Consider $\rho_1, \ldots, \rho_{n-1}$ such that

$$\Delta(\mu_j) = \rho_j + \frac{1}{\rho_j} \, , \quad j = 1, \ldots, n-1.$$

With $\omega_J(\mu) = (\mu - \mu_1) \cdots (\mu - \mu_{n-1})$,

- **1:** Set:
  - **1.1:** $b_n = \sqrt{-\sum_{k=1}^{n-1} \frac{B}{\rho_k \omega'_J(\mu_k)}}$
  - **1.2:** $u_j = \frac{1}{b_n} \sqrt{-\frac{B}{\rho_j \omega'_J(\mu_j)}}$, for $j = 1, \ldots, n-1$.
- **2:** Recover the Jacobi matrix J (3.1) from μ_i's and u_i's.
- **3:** Set:
  - **3.1:** $b_{n-1} = \frac{B}{b_1 \cdots b_{n-2} b_n}$;
  - **3.2:** $a_n = A - (a_1 + \cdots + a_{n-1})$.

4. Inverse eigenvalue problem

Suppose now that A is a Hermitian matrix whose graph is exactly the cycle $C = (1, \dots, n, 1)$, i.e.,

$$A = \begin{pmatrix} a_1 & b_1 & & b_n \\ \bar{b}_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ \bar{b}_n & & \bar{b}_{n-1} & a_n \end{pmatrix} , \tag{4.1}$$

where a_ℓ's are real numbers and b_ℓ's are nonzero complex numbers.

Proposition 4.1. *The eigenvalues of A defined in (4.1), $\lambda_1, \dots, \lambda_n$, can be ordered as*

$$\lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \cdots , \quad \textit{if}\ \mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n) < 0$$

or

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \cdots , \quad \textit{if}\ \mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n) > 0 .$$

Proof: By Corollary 2.4, the characteristic polynomial of A is

$$\varphi_A(\lambda) = (\lambda - a_1)\varphi_{A(1)}(\lambda) - |b_1|^2 \varphi_{A(1,2)}(\lambda) - |b_n|^2 \varphi_{A(1,n)}(\lambda) - 2\,\mathrm{Re}\left(b_1 \cdots b_{n-1}\bar{b}_n\right) .$$

If one considers the symmetric matrix

$$A^+ = \begin{pmatrix} a_1 & |b_1| & & & |b_n| \\ |b_1| & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & |b_{n-1}| \\ |b_n| & & & |b_{n-1}| & a_n \end{pmatrix} ,$$

then, again by Corollary 2.4, the characteristic polynomial of A^+ is

$$\varphi_{A^+}(\lambda) = (\lambda - a_1)\varphi_{A^+(1)}(\lambda) - |b_1|^2 \varphi_{A^+(1,2)}(\lambda) - |b_n|^2 \varphi_{A^+(1,n)}(\lambda) - 2\,|b_1 \cdots b_n| .$$

Hence by Corollary 2.6, $\varphi_{A^+(1)}(\lambda) = \varphi_{A(1)}(\lambda)$, $\varphi_{A^+(1,2)}(\lambda) = \varphi_{A(1,2)}(\lambda)$, $\varphi_{A^+(1,n)}(\lambda) = \varphi_{A(1,n)}(\lambda)$, then

$$\varphi_A(\lambda) = \varphi_{A^+}(\lambda) + 2|b_1 \cdots b_n| - 2\,\mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n) .$$

If

$$\Delta_{A^+}(\lambda) = |b_1 \cdots b_n|^{-1} \varphi_{A^+}(\lambda) + 2$$

(the so-called discriminant of A^+), then

$$\varphi_A(\lambda) = |b_1 \cdots b_n| \left(\Delta_{A^+}(\lambda) - 2\,\frac{\mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n)}{|b_1 \cdots b_n|} \right) .$$

Since $|\mathrm{Re}(b_1 \cdots b_{n-1}| \leq |b_1 \cdots b_{n-1}|$, then the eigenvalues of A, which are the roots of $\Delta_{A^+}(\lambda) = 2\,\frac{\mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n)}{|b_1 \cdots b_n|}$, verify

$$-2 \leq 2\,\frac{\mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n)}{|b_1 \cdots b_n|} \leq 2 .$$

If $\mu_1 > \cdots > \mu_{n-1}$ are the eigenvalues of the Jacobi matrix obtained by deleting from A^+ the last row and column, using Theorem 3.1,

$$(-1)^j \Delta_{A^+}(\mu_j) \geq 2, \quad j = 1, \ldots, n-1.$$

Consequently, the eigenvalues of A are real and can be ordered so that

$$\lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \cdots , \quad \text{if } \mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n) < 0$$

or

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \cdots , \quad \text{if } \mathrm{Re}(b_1 \cdots b_{n-1}\bar{b}_n) > 0 ,$$

because the coefficient $|b_1 \cdots b_n|^{-1}$ of λ^n in $\Delta_{A^+}(\lambda)$ is positive. ■

We state now the main result of this section:

Theorem 4.2. *There exists a Hermitian matrix A as in (4.1), with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if the (real) numbers $\lambda_1, \ldots, \lambda_n$ can be rearranged such that*

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \cdots \tag{4.2}$$

or

$$\lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \cdots . \tag{4.3}$$

Proof: The necessity follows from Proposition 4.1.

¿From Theorem 3.2, if the real numbers $\lambda_1, \ldots, \lambda_n$ verify (4.2), then we can find such matrix. Suppose that the real numbers are under the conditions (4.3). Set

$$A = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Let us consider the real numbers $\mu_1 > \mu_2 > \cdots > \mu_{n-1}$ and $B(> 0)$ such that

$$(-1)^j \Delta(\mu_j) \geq 2, \quad \text{for } j = 1, 2, \ldots, n-1,$$

and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots ,$$

where

$$\Delta(\lambda) = -2 + \frac{1}{B}(\lambda - \lambda_1)\cdots(\lambda - \lambda_n).$$

Consider $\rho_1, \dots, \rho_{n-1}$ such that

$$\Delta(\mu_j) = -\rho_j - \frac{1}{\rho_j}, \quad \text{for } j = 1, 2, \dots, n-1.$$

With $\omega_J(\lambda) = (\mu - \mu_1)\cdots(\mu - \mu_{n-1})$ setting

1: $b_n = \sqrt{\sum_{\ell=1}^{n-1} \frac{B}{\rho_\ell \omega'_J(\mu_\ell)}}$;

2: $u_\ell = \frac{1}{b_n}\sqrt{\frac{B}{\rho_\ell \omega'_J(\ell)}}$, for $\ell = 1, 2, \dots, n-1$,

and using the Lanzcos algorithm already described, with μ_ℓ's and u_ℓ's we can get the Jacobi matrix (3.1) J. Finally, setting

$$b_{n-1} = \frac{B}{b_1 b_2 \dots b_{n-2} b_n}$$

and

$$a_n = A - (a_1 + a_2 + \cdots + a_{n-1}),$$

then

$$T_1 = \begin{pmatrix} a_1 & b_1 & & & & b_n \\ b_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & b_{n-2} & \\ & & & b_{n-2} & a_{n-1} & -b_{n-1} \\ b_n & & & & -b_{n-1} & a_n \end{pmatrix}$$

is the desired matrix. In fact, let

$$T_\rho = \begin{pmatrix} a_1 & b_1 & & & & \frac{1}{\rho} b_n \\ b_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & b_{n-2} & \\ & & & b_{n-2} & a_{n-1} & -b_{n-1} \\ \rho b_n & & & & -b_{n-1} & a_n \end{pmatrix}$$

Since $u_\ell = \frac{1}{b_n}\sqrt{\frac{B}{\rho_\ell \omega'_J(\ell)}}$ and $B = b_1 \cdots b_n$, then

$$b_1 \cdots b_n = \rho_\ell \omega'_J(\mu_\ell) b_n^2 u_\ell^2,$$

for $\ell = 1, 2, \ldots, n-1$. If $Y_1, \ldots, Y_{n-1}$ are the set of orthonormal eigenvectors of J, corresponding to its eigenvalues $\mu_1, \ldots, \mu_{n-1}$, obtained using the Lanzcos algorithm already described, then $u_{i,\ell}$ denote the i-th component of Y_ℓ. From the last equality and (3.3) we obtain

$$\rho_\ell b_n u_{1,\ell} - b_{n-1} u_{n-1,\ell} = 0 \,, \quad \ell = 1, \ldots, n-1.$$

So,

$$T_{\rho_\ell} \begin{pmatrix} Y_\ell \\ 0 \end{pmatrix} = \mu_\ell \begin{pmatrix} Y_\ell \\ 0 \end{pmatrix}, \qquad \text{for } \ell = 1, \ldots, n-1.$$

Therefore μ_ℓ is an eigenvalue of T_{ρ_ℓ}, for $\ell = 1, 2, \ldots, n-1$. Using elementary properties of determinants, it is easy to see that $\frac{d}{d\rho} \det(\lambda I - t_\rho) = b_1 \cdots b_n (1 - \frac{1}{\rho^2})$. When both sides are integrated with respect to ρ, we obtain

$$\det(\lambda I - T_\rho) = b_1 \cdots b_n \left\{ \Delta_T(\lambda) + (\rho + \frac{1}{\rho}) \right\}.$$

Then

$$\Delta_T(\mu_\ell) = -\rho_\ell - \frac{1}{\rho_\ell} \,, \quad \text{for } \ell = 1, 2, \ldots, n-1.$$

Notice that

$$\Delta_T(\lambda) = \frac{1}{b_1 \cdots b_n} \left(\lambda^n - A\lambda^{n-1} + \cdots \right).$$

and thus the coefficients of λ^n and λ^{n-1} in $\Delta_T(\lambda)$ and $\Delta(\lambda)$, respectively, are the same. Henceforth, $\Delta_T(\lambda) - \Delta(\lambda)$ is a polynomial of degree less than or equal to $n-2$. But $\Delta_T(\mu_\ell) - \Delta(\mu_\ell) = 0$, for $\ell = 1, 2, \ldots, n-1$, which means that $\Delta_T(\lambda) - \Delta(\lambda)$ has $n-1$ distinct roots. Therefore, $\Delta_T = \Delta$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T_1. ∎

Example 4.1. Given the numbers $6, 3, 1$, we want to find a Hermitian matrix (4.1) whose eigenvalues are

$$\lambda_1 = 6 = \lambda_2 > \lambda_3 = 3 = \lambda_4 > \lambda_5 = 1.$$

We can get with $A = 19$, $B = 1$, $\mu_1 = 6 > \mu_2 = 5 > \mu_3 = 3, \mu_4 = 2$ and the above procedure described in Theorem 4.2, the matrix

$$\begin{pmatrix} 4 & \sqrt{2} & 0 & 0 & \sqrt{2-\sqrt{3}} \\ \sqrt{2} & 4+\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 4-\frac{\sqrt{3}}{2} & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 4 & -\sqrt{2+\sqrt{3}} \\ \sqrt{2-\sqrt{3}} & 0 & 0 & -\sqrt{2+\sqrt{3}} & 3 \end{pmatrix}$$

which eigenvalues are

$$\lambda_1 = \lambda_2 = 6 > \lambda_3 = \lambda_4 = 3 > \lambda_5 = 1.$$

Corollary 4.3. *Any eigenvalue of a Hermitian matrix of the form (4.1) has at most multiplicity* 2.

5. An unicycle case

Let us go back to real matrices and let us consider now the symmetric matrix

$$P = \begin{pmatrix} a_1 & b_1 & & & & b_n \\ b_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & a_{n-2} & b_{n-2} & b_{n-1} \\ & & & b_{n-2} & a_{n-1} & 0 \\ b_n & & & b_{n-1} & 0 & a_n \end{pmatrix},$$

where $b_i > 0$, for $i = 1, \ldots, n$, whose graph is a cycle to which we add an edge to no one of its vertices.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of L, then we know that they are all real and the maximum multiplicity of each is two (cf. [4]). With $b_i > 0$, for $i = 1, \ldots, n-2$, in J defined in (3.1), as a set of real distinct eigenvalues $\mu_1, \ldots, \mu_{n-1}$ with associated orthonormal eigenvectors $Y_1, \ldots, Y_{n-1}$, if Y is a matrix whose j column is Y_j, then

$$JY = YD \quad \text{where} \quad D = \operatorname{diag}(\mu_1, \ldots, \mu_{n-1}) .$$

It is also known

$$(\mu I - J)^{-1} = Y(\mu I - D)^{-1} Y^T .$$

By comparing the entries in row 1 column $n-2$ we obtain the identity

$$b_1 \cdots b_{n-3}(\mu_j - a_{n-1}) = \omega'_J(\mu_j) u_{1,j} u_{n-2,j} , \tag{5.1}$$

for $j = 1, \ldots, n-1$.

Set now

$$P_\rho = \begin{pmatrix} a_1 & b_1 & & & & \frac{1}{\rho} b_n \\ b_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & a_{n-2} & b_{n-2} & b_{n-1} \\ & & & b_{n-2} & a_{n-1} & 0 \\ \rho b_n & & & b_{n-1} & 0 & a_n \end{pmatrix},$$

where $\rho \neq 0$.

Definition 5.1. Let J be a matrix characterized by the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ and by the set $Y_1, \cdots, Y_{n-1}$ of real orthonormal eigenvectors associated with $\mu_1, \ldots, \mu_{n-1}$, whose first components are $u_1, \cdots, u_{n-1}$. Let $\omega_J(\mu)$ be the characteristic polynomial of J. We define $\rho_1, \ldots, \rho_{n-1}$, the **unicycle Floquet multipliers** of P, corresponding to $\mu_1, \ldots, \mu_{n-1}$ as

$$b_1 b_2 \cdots b_{n-3}(\mu_j - a_{n-1}) b_{n-1} b_n = -\rho_j \omega'_J(\mu_j) b_n^2 u_j^2 \,, \tag{5.2}$$

for $j = 1, \ldots, n-1$.

Remark 5.1. Notice that if $\mu_j \neq a_{n-1}$, since $\omega'_J(\mu_j) \neq 0$ and $u_j \neq 0$, then

$$\rho_j = -\frac{b_1 b_2 \cdots b_{n-3}(\mu_j - a_{n-1}) b_{n-1}}{\omega'_J(\mu_j) b_n u_j^2} \,.$$

Otherwise, $\rho_j = 0$.

Proposition 5.1. *If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of P, then*

$$\lambda_1 > a_{n-1} > \lambda_n \,.$$

Proof: We know that $J = P(n)$ has eigenvalues $\mu_1 > \cdots > \mu_{n-1}$ which verify the interlacing relation

$$\lambda_1 \geq \mu_1 > \cdots > \mu_{n-1} \geq \lambda_n \,.$$

It is also known that $J(1)$ (whose graph is a tree obtained from another tree deleting and end vertex) has eigenvalues $\theta_1 > \cdots > \theta_{n-2}$ which verify

$$\lambda_1 \geq \mu_1 > \theta_1 > \cdots > \theta_{n-2} > \mu_{n-1} \geq \lambda_n \,.$$

Repeating this procedure and since $n \geq 4$, then $J(1, \ldots, n-2) = (a_{n-1})$ is itself an eigenvalue, which verifies the relation

$$\lambda_1 \geq \mu_1 > \theta_1 > \cdots > a_{n-1} > \cdots > \theta_{n-2} > \mu_{n-1} \geq \lambda_n \,.$$

Therefore

$$\lambda_1 > a_{n-1} > \lambda_n \ .$$

■

Proposition 5.2. *If a_{n-1} is an eigenvalue of P, then its (algebraic) multiplicity is* 1*.*

Proof: Let us calculate the characteristic of the matrix $a_{n-1}I - P$. From

$$a_{n-1}I - P = \begin{pmatrix} a_{n-1}-a_1 & -b_1 & & & & -b_n \\ -b_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & a_{n-1}-a_{n-2} & -b_{n-2} & -b_{n-1} \\ & & & -b_{n-2} & 0 & 0 \\ -b_n & & & -b_{n-1} & 0 & a_{n-1}-a_n \end{pmatrix} ,$$

using elementary transformations we get

$$\begin{pmatrix} a_{n-1}-a_1 & -b_1 & & & & & & -b_n \\ -b_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & -b_{n-4} & & & \\ & & & -b_{n-4} & a_{n-1}-a_{n-3} & 0 & & \\ & & & & 0 & 0 & -b_{n-2} & -b_{n-1} \\ & & & & & -b_{n-2} & 0 & 0 \\ -b_n & & & & & -b_{n-1} & 0 & a_{n-1}-a_n \end{pmatrix} = Q \ .$$

If R is the submatrix of Q resulting from deleting row n and column $n-3$, since $b_i > 0$, $i = 1, \ldots, n$, then

$$\operatorname{rank}(R) = n - 1.$$

Thus $\operatorname{rank}(a_{n-1}I - P) \geq n-1$. But the multiplicity of a_{n-1} is greater or equal than 1, and therefore $\operatorname{rank}(a_{n-1}I - P) = n - 1$ and (algebraic) multiplicity of a_{n-1} is equal to 1. ■

The main result of this section states:

Theorem 5.3. *The characteristic polynomial of P_ρ may be represented by*

$$\det(\lambda I - P_\rho) = b_1 b_2 \cdots b_{n-3}(\lambda - a_{n-1}) b_{n-1} b_n \left\{ \xi(\lambda) - \left(\rho + \frac{1}{\rho} \right) \right\} , \qquad (5.3)$$

where $\xi(\lambda)$ is independent of ρ. The numbers $\rho_1, \ldots, \rho_{n-1}$ of P corresponding to the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ of J satisfy the relation

$$\xi(\mu_j) = \rho_j + \frac{1}{\rho_j} \;, \quad \textit{if } \mu_j \neq a_{n-1} \;. \tag{5.4}$$

Moreover, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of P and k is the integer satisfying $\lambda_k \geq a_{n-1} > \lambda_{k+1}$, then the eigenvalues of P can be ordered as

$$\begin{array}{ll} \lambda_1 > \lambda_2 \geq \lambda_3 > \cdots \geq \lambda_{k-1} > \lambda_k > \lambda_{k+1} \geq \lambda_{k+2} > \cdots & \textit{if } k \textit{ is even} \\ \lambda_1 > \lambda_2 \geq \lambda_3 > \cdots > \lambda_{k-1} \geq \lambda_k > \lambda_{k+1} > \lambda_{k+2} \geq \cdots & \textit{if } k \textit{ is odd.} \end{array}$$

Proof: It is straightforward to show that

$$\frac{d}{d\rho} \det(\lambda I - P_\rho) = -b_1 b_2 \cdots b_{n-3}(\lambda - a_{n-1}) b_{n-1} b_n \left(1 - \frac{1}{\rho^2}\right) \;,$$

and integrating both sides this equality in order to ρ we get (5.3) where $b_1 b_2 \cdots b_{n-3}(\lambda - a_{n-1}) b_{n-1} b_n$ and $\xi(\lambda)$ do not depend on ρ.

Let J be the matrix obtained by deleting from P the last row and column and let $u_1, \ldots, u_{n-1}$ be the first entries of the unitary orthogonal eigenvectors $Y_1, \ldots, Y_{n-1}$ of J corresponding to the eigenvalues $\mu_1, \ldots, \mu_{n-1}$. From (5.1) and (5.2) we get

$$\rho_j = -\frac{b_{n-1} u_{n-2,j}}{b_n u_{1,j}}, \qquad \text{for } j = 1, \ldots, n-1,$$

and therefore

$$P_{\rho_j} \begin{pmatrix} Y_j \\ 0 \end{pmatrix} = \mu_j \begin{pmatrix} Y_j \\ 0 \end{pmatrix}, \qquad \text{for } j = 1, \ldots, n-1.$$

Consequently, μ_j is eigenvalue of P_{ρ_j}, $j = 1, \ldots, n-1$, and from (5.3) we get (5.4) provided $\mu_j \neq a_{n-1}$.

¿From the definition of ρ_j, (5.2), since $\omega'_J(\mu_j) \neq 0$, for $j = 1, \ldots, n-1$, we have

$$\begin{array}{ll} \omega'_J(\mu_j)\rho_j < 0 & \text{if } \mu_j > a_{n-1}, \\ \omega'_J(\mu_j)\rho_j > 0 & \text{if } \mu_j < a_{n-1}, \\ \omega'_J(\mu_j)\rho_j = 0 & \text{if } \mu_j = a_{n-1}. \end{array}$$

¿From (5.4), since $(-1)^j \omega'_J(\mu_j) < 0$, we obtain

$$\begin{array}{ll} (-1)^j \xi(\mu_j) \geq 2 & \text{if } \mu_j > a_{n-1}, \\ (-1)^{j+1} \xi(\mu_j) \geq 2 & \text{if } \mu_j < a_{n-1}, \end{array}$$

provided $\left|\rho_j + \frac{1}{\rho_j}\right| \geq 2$.

Proposition 5.1 gives the inequalities

$$\lambda_1 > a_{n-1} > \lambda_n \; ;$$

let k be the integer such that

$$\lambda_k \geq a_{n-1} > \lambda_{k+1}.$$

By Proposition 5.2 (algebraic) multiplicity of a_{n-1} is least or equal to 1. Let us split now the proof on two cases:

1st case: a_{n-1} is not an eigenvalue of P.

In this case, let us assume $\lambda_k > a_{n-1} > \lambda_{k+1}$.

If k is even, then since the eigenvalues of P are the roots of $\xi(\lambda) = 2$ and $(b_1 b_2 \cdots b_{n-3}(\lambda - a_{n-1}) b_{n-1} b_n)^{-1} > 0$ (< 0), if $\lambda > a_{n-1}$ ($< a_{n-1}$, resp.), then $\lambda_1, \ldots, \lambda_n$ can be ordered as

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \cdots \geq \lambda_{k-1} > \lambda_k > \lambda_{k+1} \geq \lambda_{k+2} > \cdots \,.$$

If k is odd, then $\lambda_1, \ldots, \lambda_n$ can be ordered as

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \cdots > \lambda_{k-1} \geq \lambda_k > \lambda_{k+1} > \lambda_{k+2} \geq \cdots \,.$$

2nd case: $a_{n-1} = \lambda_k$ is an eigenvalue of P.

In this case,

$$\det(\lambda I - P_\rho) = b_1 b_2 \cdots b_{n-3}(\lambda - a_{n-1}) b_{n-1} b_n \left\{ \xi(\lambda) - \left(\rho + \frac{1}{\rho}\right) \right\}$$

where $\xi(\lambda)$ is an polynomial of degree $n-1$, such that the roots of $\xi(\lambda) = 2$ are the eigenvalues of P different from a_{n-1}. Also, the coefficient $(b_1 b_2 \cdots b_{n-3} b_{n-1} b_n)^{-1}$ of λ^{n-1} in $\xi(\lambda)$ is always positive, and

$$\xi(\lambda) = (b_1 \cdots b_{n-3} b_{n-1} b_n)^{-1}((\lambda - \lambda_1) \cdots (\lambda - \lambda_{k-1})(\lambda - \lambda_{k+1}) \cdots (\lambda - \lambda_n) + 2) \,.$$

If k is even, then $\xi(a_{n-1}) < 2$, and therefore $\lambda_1, \ldots, \lambda_n$ can be ordered as

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \cdots \geq \lambda_{k-1} > \lambda_k > \lambda_{k+1} \geq \lambda_{k+2} > \cdots \,.$$

If k is odd, then $\xi(a_{n-1}) > 2$, and therefore $\lambda_1, \ldots, \lambda_n$ can be ordered as

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \cdots > \lambda_{k-1} > \lambda_k > \lambda_{k+1} > \lambda_{k+2} \geq \cdots \,.$$

Therefore, we have the result. ∎

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